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October 5, 2016

Abstract

We consider a family of quantum channels characterized by the fact that certain (in general nonorthogonal) Pure states at the channel entrance are mapped to (tensor) Products of Pure states (PPP, hence "pcubed") at the complementary outputs (the main output and the "environment") of the channel. The pcubed construction, a reformulation of the twisted-diagonal procedure in [1], can be used to produce a large class of degradable quantum channels; degradable channels are of interest because their quantum capacities are easy to calculate. Several known types of degradable channels are either pcubed channels, or subchannels (employing a subspace of the channel entrance), or continuous limits of pcubed channels. The pcubed construction also yields channels which are neither degradable nor antidegradable (i.e., the complement of a degradable channel); a particular example of a qutrit channel of this type is studied in some detail. Determining whether a pcubed channel is degradable or antidegradable or neither is quite straightforward given the pure input and output states that characterize the channel. Conjugate degradable pcubed channels are always degradable.

Contents

1   Introduction

The transmission of quantum information is a central problem in quantum information theory, and a great deal of attention has been devoted to understanding noisy quantum channels, of which the simplest kind is defined in Sec. 2.1. Of particular interest is finding the asymptotic quantum capacity Q of such a channel, a task which is much more difficult than its analog in classical information theory. The asymptotic capacity of a classical channel is equal to the mutual information between the channel's input and output (see Ch. 7 in [2]), maximized over all possible probability distributions for the input. Because the mutual information is a concave function of the input probability distribution, this quantity is easily computed numerically if a closed-form expression is not available. In addition the classical capacity is additive: the capacity of two independent channels used in parallel is just the sum of the individual capacities.

A quantum analog of the classical mutual information is the coherent information, and finding its maximum over all possible input density operators yields the *one-shot quantum capacity* $Q^{(1)}$ (sometimes also called the coherent information) of the channel (see Ch. 13 in [3]). But unlike its classical counterpart, the coherent information need not be a concave function of the input density operator, and $Q^{(1)}$ is in general not additive for two channels placed in parallel. (See Sec. 2.2 for additional remarks.)

However, Devetak and Shor [4] showed that $Q^{(1)}$ is additive for any two (not necessarily identical) *degradable* quantum channels (see the definition in Sec. 2.3) placed in parallel. From this it follows that $Q = Q^{(1)}$ for a degradable channel, and Q for two such channels placed in parallel is the sum of its values for the individual channels. In addition, the coherent information of a degradable channel is a concave function of the input density operator [5], allowing $Q^{(1)}$ to be computed efficiently. In this respect the quantum capacity problem for degradable quantum channels resembles that of classical channels. However, checking whether a given quantum channel is or is not degradable is not altogether straightforward, and while several interesting classes of degradable channels are known [1,6] the full limits of this family of channels have not been established.

Very little is known about the asymptotic capacities of quantum channels which are not degradable. Cases are known for which $Q^{(1)}$ and Q are not additive; one of the most striking [7] is a pair of channels both of which have asymptotic capacities Q equal to zero, but when placed in parallel have a finite $Q^{(1)}$, and thus a nonzero asymptotic Q. Various upper and lower bounds on capacities of nondegradable channels are known—some obtained using a clever construction involving degradable channels [8]—but at present the source of nonadditivity is not understood. One may hope that a better understanding of degradable channels, including what happens when degradability breaks down upon variation of some parameter, may provide useful insights.

In this article we show that a large class of degradable channels can be constructed using isometries which map Pure states to Products of Pure states, thus PPP or *pcubed*. Not all pcubed channels are degradable, but testing for degradability is relatively easy, and degradable families constructed in this way include several examples known previously. Indeed, by including subchannels (which use a subspace of the channel entrance), and continuous limits, as illustrated in some examples in Sec. 4, it seems possible to provide pcubed constructions for many of the degradable channels discussed in [6], although we have not studied every case found in that very useful paper. Not all pcubed channels are degradable, and it is easy to construct examples in which a continuous variation of a parameter leads from a degradable to a nondegradable channel, thus allowing a study of what happens when the pleasant properties associated with degradability are lost. Section 5 is devoted to a very simple example.

The pcubed construction is in effect what Chefles and his collaborators used when considering the problem of quantum channels that map certain sets of pure states to other sets of pure states [9–11]. And it coincides in many cases, see Sec. 3.2, with the *twisted diagonal* construction used by Wolf and Pérez-García [1] in their study of degradable channels. The main way our work differs from previous studies is in focusing on the isometry which generates a channel superoperator, rather than on the superoperator itself. The two are formally equivalent, but sometimes the isometry is easier to understand; in particular when studying degradable channels, but perhaps in other cases as well.

The remainder of this article is structured as follows. Section 2 contains various definitions and relationships which, while not new, should help make the present treatment reasonably self contained. In particular, the relationship between isometries, Kraus operators, and channel superoperators is summarized in Sec. 2.1, coherent information and the quantum capacities $Q^{(1)}$ and Q are defined in Sec. 2.2, and degradable and conjugate degradable [12] channels in Secs. 2.3 and 2.4. Section 3 introduces pcubed isometries and channels. Following the basic definitions in Sec. 3.1, the connection with twisted-diagonal channels is worked out in Sec. 3.2, and degradable pcubed channels are discussed in Secs. 3.3. An argument in Sec. 3.4 shows

that a pcubed channel which is conjugate degradable is also degradable. Specific examples of degradable pcubed channels are discussed in Sec. 4; these include some qubit and qutrit channels, erasure channels, and Hadamard channels (the complements of entanglement-breaking channels). Section 5, with some details in the Appendix, presents results for a particular qutrit channel in which a continuous variation of parameters leads out of a degradable regime, and certain properties associated with a degradable channel break down. Section 6 summarizes the paper.

2 Quantum Isometries and Channels

2.1 Channels and complementary channels

A noisy quantum channel, also called a quantum operation, can be constructed starting with a unitary operator U, representing the time development of a closed quantum system, that carries $\mathcal{H}_a \otimes \mathcal{H}_e$ to $\mathcal{H}_b \otimes \mathcal{H}_c$, where the Hilbert space $\mathcal{H}_a$ represents the *channel entrance*, $\mathcal{H}_e$ the environment, and $\mathcal{H}_b$ and $\mathcal{H}_c$ the *direct* and *complementary* channel outputs. One often equates $\mathcal{H}_b$ with $\mathcal{H}_a$ and $\mathcal{H}_c$ with $\mathcal{H}_e$, i.e., the complementary output is the environment, but greater generality is possible if one allows the dimensions of $\mathcal{H}_a$ and $\mathcal{H}_b$ to be different. If the environment is always in the same initial state $|e_0\rangle$ (a mixed state of the environment can always be purified), its action on the states in $\mathcal{H}_a$ can be represented by an *isometry*

$$J : \mathcal{H}_a \to \mathcal{H}_b \otimes \mathcal{H}_c; \quad J|a\rangle := U|a\rangle \otimes |e_0\rangle. \tag{1}$$

Our discussion is based on this isometry, and assumes the three Hilbert spaces have (finite) dimensions d_a, d_b, and d_c, which can be arbitrary except for the requirement $d_a \leq d_b d_c$, which is necessary for J to be an isometry, i.e., $J^\dagger J = I_a$. The isometry then produces two quantum channels represented by the superoperators

$$\mathcal{B}(O) = \mathrm{Tr}_c(JOJ^\dagger), \quad \mathcal{C}(O) = \mathrm{Tr}_b(JOJ^\dagger) \tag{2}$$

carrying the space $\hat{\mathcal{H}}_a$ of operators on $\mathcal{H}_a$, of which O is a typical example, into the operator spaces $\hat{\mathcal{H}}_b$ and $\hat{\mathcal{H}}_c$, respectively. It is customary to refer to $\mathcal{C}$ as the *complement* of $\mathcal{B}$, with $\mathcal{B}$ the complement of $\mathcal{C}$, so a single isometry produces a complementary pair of channels. The fact that J is an isometry implies that $\mathcal{B}$ and $\mathcal{C}$ are *completely positive and trace preserving* (CPTP); conversely, any CPTP map can be produced in this way starting with a suitable isometry. One way to represent the map $\mathcal{B}$ is to use Kraus operators $L_l : \mathcal{H}_a \to \mathcal{H}_b$ which define the isometry

$$J = \sum_l |c_l\rangle \otimes L_l, \quad \mathcal{B}(O) = \sum_l L_l O L_l^\dagger, \tag{3}$$

where $\{|c_j\rangle\}$ is an orthonormal basis of $\mathcal{H}_c$. Of course a similar representation is possible for $\mathcal{C}$, with Kraus operators mapping $\mathcal{H}_a$ to $\mathcal{H}_c$.

The isometry J maps the channel entrance $\mathcal{H}_a$ onto a subspace of $\mathcal{H}_b \otimes \mathcal{H}_c$, and this *image subspace* determines the properties of the channel pair up to a unitary on $\mathcal{H}_a$; i.e., if an isometry J' has the same image subspace as J, then $J' = JU$ for some unitary U on $\mathcal{H}_a$. If J itself is restricted to a subspace of $\mathcal{H}_a$, then it is (obviously) again an isometry acting on this restriction, and produces a pair of channels which we will refer to as *subchannels* of the original channels; equivalently, these subchannels are associated with a subspace of the original image subspace on $\mathcal{H}_b \otimes \mathcal{H}_c$.

Two quantum channels $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be "in parallel" or "used simultaneously" or "used independently" if together they constitute a single channel $\mathcal{B}$ defined as follows. If $J_1 : \mathcal{H}_{a1} \to \mathcal{H}_{b1} \otimes \mathcal{H}_{c1}$ and $J_2 : \mathcal{H}_{a2} \to \mathcal{H}_{b2} \otimes \mathcal{H}_{c2}$ are the isometries that lead to $\mathcal{B}_1$ and $\mathcal{B}_2$, let $J = J_1 \otimes J_2$ be the map from $(\mathcal{H}_a = \mathcal{H}_{a1} \otimes \mathcal{H}_{a2})$ to $(\mathcal{H}_b = \mathcal{H}_{b1} \otimes \mathcal{H}_{b2}) \otimes (\mathcal{H}_c = \mathcal{H}_{c1} \otimes \mathcal{H}_{c2})$ defined in the obvious way:

$$J(|a1\rangle \otimes |a2\rangle) = (J_1|a1\rangle) \otimes (J_2|a2\rangle). \tag{4}$$

Then $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ as defined in (2) for J in (4) is what we mean by $\mathcal{B}_1$ and $\mathcal{B}_2$ in parallel, and of course its complement $\mathcal{C}$ is $\mathcal{C}_1$ and $\mathcal{C}_2$ in parallel. The definition extends in an obvious way to three or more channels in parallel.

2.2 Coherent information and quantum capacity

The *coherent information* or *entropy bias* of a channel $\mathcal{B}$ with complementary channel $\mathcal{C}$ is defined by the formula

$$\Delta S(\rho) = S(\mathcal{B}(\rho)) - S(\mathcal{C}(\rho)), \tag{5}$$

where ρ is the input density operator, a positive operator with trace equal to 1, and $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy. The *one-shot quantum capacity*, also known (somewhat confusingly) as the *coherent information of the channel* is defined by the formula

$$Q^{(1)}(\mathcal{B}) = \max_{\rho} \Delta S(\rho), \tag{6}$$

where the maximum is taken over all density operators in $\hat{\mathcal{H}}_a$. The one-shot capacity for the $\mathcal{C}$ channel can be written as

$$Q^{(1)}(\mathcal{C}) = -\min_{\rho} \Delta S(\rho). \tag{7}$$

The one shot capacity is superadditive, meaning that for channels $\mathcal{B}_1$ and $\mathcal{B}_2$ in parallel,

$$Q^{(1)}(\mathcal{B}_1 \otimes \mathcal{B}_2) \geq Q^{(1)}(\mathcal{B}_1) + Q^{(1)}(\mathcal{B}_2). \tag{8}$$

The (asymptotic) quantum capacity $Q(\mathcal{B})$ [13, 14] is equal to

$$Q(\mathcal{B}) = \lim_{n \mapsto \infty} \frac{1}{n} Q^{(1)}(\mathcal{B}^{\otimes n}), \tag{9}$$

and thus in light of (8) is bounded below by $Q^{(1)}(\mathcal{B})$.

2.3 Degradable channels

One of the channels in a pair produced by an isometry is said to be *degradable* [4] if its output can be sent through another noisy channel in such a way that the combination is equivalent to the other, complementary, channel in the pair, which is then said to be *antidegradable*. We shall also refer to a *degradable pair* of channels, and in addition use the term *undegradable* to refer to either a single channel or a complementary pair in a situation in which the channel is neither degradable nor antidegradable, and thus the same is true of the complementary channel.

In particular, the $a \to b$ channel of Sec. 2, with superoperator $\mathcal{B}$ defined in (2) is degradable if there is an additional isometry

$$K : \mathcal{H}_b \to \mathcal{H}_c \otimes \mathcal{H}_d \tag{10}$$

giving rise to a superoperator

$$\mathcal{D}(S) = \text{Tr}_d(KSK^\dagger) \tag{11}$$

with the property that, for $\mathcal{C}$ defined in (2),

$$\mathcal{C} = \mathcal{D} \circ \mathcal{B}, \text{ i.e., } \mathcal{C}(O) = \mathcal{D}(\mathcal{B}(O)). \tag{12}$$

The ancillary space $\mathcal{H}_d$ plays no further role in the definition of degradability. One can also think of K as mapping to a new Hilbert space $\mathcal{H}_c'$ isomorphic to $\mathcal{H}_c$, but the simplest statement of the degradability condition uses $\mathcal{H}_c$. Note that it is possible for a channel to be simultaneously degradable and antidegradable, in which case its complement is also both degradable and antidegradable.

If two degradable channels $\mathcal{B}_1$ and $\mathcal{B}_2$ are placed in parallel, the combined channel $\mathcal{B}_1 \otimes \mathcal{B}_2$ is again degradable, and in addition the inequality (8) becomes an equality [4]. This has the consequence that for any degradable channel $\mathcal{B}$ the asymptotic capacity $Q(\mathcal{B})$ is equal to $Q^{(1)}(\mathcal{B})$, and if both $\mathcal{B}_1$ and $\mathcal{B}_2$ are degradable, $Q(\mathcal{B}_1 \otimes \mathcal{B}_2)$ is equal to $Q(\mathcal{B}_1) + Q(\mathcal{B}_2)$. In addition $\Delta S(\rho)$, (5), is a concave function of ρ when $\mathcal{B}$ is degradable [5], so its maximum $Q^{(1)}$ is relatively easy to calculate. As a consequence, quantum capacities of degradable channels inherit some of the nice properties of Shannon capacities of classical noisy channels.

2.4 Conjugate degradable channels

A quantum channel $\mathcal{B}$ with complement $\mathcal{C}$ is said to be *conjugate degradable* [12] provided there is a CPTP map $\bar{\mathcal{D}}$ from $\hat{\mathcal{H}}_b$ to $\hat{\mathcal{H}}_c$ such that

$$\mathcal{T} \circ \mathcal{C} = \bar{\mathcal{D}} \circ \mathcal{B}, \tag{13}$$

where $\mathcal{T}$ is a superoperator defined by choosing a particular orthonormal basis $\{|c_k\rangle\}$ of $\mathcal{H}_c$, and for any operator $M \in \hat{\mathcal{H}}_c$ defining $M^{\mathrm{T}} = \mathcal{T}(M)$ as the operator whose matrix is related to that of M by a transpose:

$$\langle c_l|M^{\mathrm{T}}|c_k\rangle = \langle c_k|M|c_l\rangle. \tag{14}$$

A channel can be both degradable and conjugate degradable if both (12) and (13) are satisfied, which will in general require distinct maps $\mathcal{D}$ and $\bar{\mathcal{D}}$.

One can also define conjugate degradability by replacing $\mathcal{T}$ in (13) with a complex conjugation operator $\mathcal{K}$, defining $M^{\mathrm{C}} = \mathcal{K}(M)$ to be the operator such that

$$\langle c_l|M^{\mathrm{C}}|c_k\rangle = \langle c_l|M|c_k\rangle^*. \tag{15}$$

Note that $\mathcal{K}$ is an antilinear operator, and for this reason (13) with $\mathcal{K}$ in place of $\mathcal{T}$ is only required to hold for Hermitian operators $O = O^\dagger$, which of course includes density operators. Because $\mathcal{T}$ is linear, if (13) holds for Hermitian operators it also holds for all operators in $\hat{\mathcal{H}}_a$.

Both $\mathcal{T}$ and $\mathcal{K}$ are defined for a particular orthonormal basis of $\mathcal{H}_c$, but the basis dependence is of no importance for the present discussion; if a different basis is used, one only needs to adjust $\mathcal{C}$ and $\bar{\mathcal{D}}$ by applying a suitable unitary (i.e., $\rho \to U\rho U^\dagger$) to their outputs, and (13) will again be satisfied.

At present it is not known if there are channels which are conjugate degradable but not degradable. If they exist they are *not* of the pcubed type, see Sec. 3.4.

3 Pcubed Isometries and Channels

3.1 Definitions

A pure state to product of pure states, or pcubed, isometry J is defined as follows. Let $\{|\alpha_j\rangle\}$ for $1 \le j \le d$ be collection of normalized kets that span $\mathcal{H}_a$, and $\{|\beta_j\rangle\}$ and $\{|\gamma_j\rangle\}$ be collections of normalized kets in $\mathcal{H}_b$ and $\mathcal{H}_c$ such that

$$J|\alpha_j\rangle = |\beta_j\rangle \otimes |\gamma_j\rangle \text{ for } 1 \le j \le d. \tag{16}$$

The necessary and sufficient condition for J to be an isometry is that it preserves inner products, which is to say:

$$A_{jk} = \langle \alpha_j|\alpha_k\rangle = \langle \beta_j|\beta_k\rangle\langle \gamma_j|\gamma_k\rangle = B_{jk}C_{jk}, \text{ or } A = B * C, \tag{17}$$

where we use $*$ to denote the element-wise or Hadamard product of the Gram matrices A, B, and C constructed from the three collections of kets. Note that the diagonal elements of these matrices are all equal to 1 by the normalization condition, and since they are Hermitian, (17) constitutes a set of $d(d-1)/2$ independent equations for the (in general complex) off-diagonal elements.

The kets in $\{|\alpha_j\rangle\}$ are *not* assumed to be orthogonal, and they need not be linearly independent as long as their span is equal to $\mathcal{H}_a$, so that the isometry in (16) is well defined. The same is true of the collection $\{|\beta_j\rangle\}$, except that it need not span $\mathcal{H}_b$, and similarly the $\{|\gamma_j\rangle\}$ need not span $\mathcal{H}_c$. In applications we generally assume that the $\{|\alpha_j\rangle\}$ are linearly independent, and thus form a (in general not orthonormal) basis of $\mathcal{H}_a$, so $d = d_a$; and also that $d_b = d_c = d$, though the $\{|\beta_j\rangle\}$ and $\{|\gamma_j\rangle\}$ need not span $\mathcal{H}_b$ and $\mathcal{H}_c$. The isometry J in (16) is not changed if for each j, $|\alpha_j\rangle$, $|\beta_j\rangle$ and $|\gamma_j\rangle$ are multiplied by phases $e^{i\lambda_j}$, $e^{i\mu_j}$ and $e^{i\nu_j}$, with $\lambda_j - \mu_j - \nu_j$ a multiple of 2π. However, this will in general change the off-diagonal elements of A, B, and C, though (17) will still be satisfied. Conversely, a set of A, B, and C matrices satisfying (17) determine J only up to local unitaries, since a Gram matrix only determines the corresponding kets up to a unitary transformation.

As A, B and C are Gram matrices, they are positive semi-definite, and if the $|\alpha_j\rangle$ are linearly independent, A will be positive definite, with all its eigenvalues strictly positive. The Hadamard product of two positive definite matrices is positive definite (p. 458 of [15]), whereas if only one of the matrices is positive definite and the other is positive semidefinite, or both are positive semidefinite, the product will certainly be positive semidefinite, but may or may not be positive definite. Since any positive semidefinite matrix is a Gram

matrix for a suitable (not unique) collection of kets, what we call a pcubed isometry can be obtained by choosing any two positive (semi)definite matrices B and C, with 1's on the diagonal, and taking the product (17). Or one can start with a positive (semi)definite A and a positive (semi)definite B and define $C = A/*B$ by element-wise division; if C is positive semidefinite, the three matrices define a pcubed isometry. However, if for some j and k both A_{jk} and B_{jk} are zero, the quotient is not well defined, and there may or may not be values of C_{jk} for which C has the desired positivity property. (In our discussion of degradable channels in Sec. 3.3 we will assume that both the $|\alpha_j\rangle$ and the $|\beta_j\rangle$ are linearly independent bases of $\mathcal{H}_a$ and $\mathcal{H}_b$.)

The image subspace (see the definition in Sec. 2) of a pcubed isometry is spanned by products of pure states $|\beta_j\rangle \otimes |\gamma_j\rangle$, and such subspaces are rather special. On the other hand, since the entire space $\mathcal{H}_b \otimes \mathcal{H}_c$ is spanned by such products, any subspace is always a subspace of one of these "special" types of spaces—which means that only in exceptional circumstances will subchannels of pcubed channels be of interest. One of these exceptional circumstances is that in which the $a \to b$ channel $\mathcal{B}$ is degradable, in which case the corresponding subchannel is also degradable.

3.2 Twisted diagonal channels

A twisted-diagonal $a \to b$ channel is defined in [1] for the case $d_a = d_b = d$ through the requirement that the Kraus operators $L_l : \mathcal{H}_a \to \mathcal{H}_b$, $1 \le l \le d$, be simultaneously diagonalizable in the sense that given orthonormal bases $\{|a_j\rangle\}$ of $\mathcal{H}_a$ and $\{|b_k\rangle\}$ of $\mathcal{H}_b$, there are invertible operators $X \in \hat{\mathcal{H}}_a$ and $Y \in \hat{\mathcal{H}}_b$ such that these make all the Kraus operators diagonal:

$$\langle b_k | Y L_l X | a_j \rangle = \lambda_{jl} \delta_{jk}. \tag{18}$$

This means there is a collection of d linearly independent (X is nonsingular) kets $|\bar{a}_j\rangle = X|a_j\rangle$ in $\mathcal{H}_a$ and a corresponding linearly independent collection of kets $|\bar{b}_j\rangle$ in $\mathcal{H}_b$ such that every Kraus operator maps every $|\bar{a}_j\rangle$ to a (possibly zero) multiple of $|\bar{b}_j\rangle$. The formal argument showing that the corresponding isometry is pcubed begins by noting that (18) implies that

$$L_l = \sum_j \lambda_{jl} \bar{Y} |b_j\rangle \langle a_j| \bar{X}, \tag{19}$$

where $\bar{X} = X^{-1}$ and $\bar{Y} = Y^{-1}$, and hence the isometry (3) may be written in the form

$$J = \sum_{jl} \lambda_{jl} |c_l\rangle \otimes \bar{Y} |b_j\rangle \langle a_j| \bar{X} = \sum_j (|\beta_j\rangle \otimes |\gamma_j\rangle) \langle \bar{\alpha}_j|, \tag{20}$$

where

$$|\beta_j\rangle = \nu_j \bar{Y} |b_j\rangle, \quad \langle \bar{\alpha}_j| = \mu_j \langle a_j| \bar{X}, \quad |\gamma_j\rangle = \sum_l (\lambda_{jl}/\mu_j \nu_j) |c_l\rangle, \tag{21}$$

where the μ_j and ν_j are positive coefficients. Since the $\{\langle \bar{\alpha}_j|\}$ are linearly-independent there is a dual basis $\{|\alpha_j\rangle\}$ of $\mathcal{H}_a$ such that

$$\langle \bar{\alpha}_j | \alpha_k \rangle = \delta_{jk}, \tag{22}$$

and we assume the positive μ_j coefficients in (21) are chosen so that the $|\alpha_j\rangle$ (not the $|\bar{\alpha}_j\rangle$) are normalized, $\langle \alpha_j | \alpha_j \rangle = 1$. The positive ν_j coefficients are chosen so that $\langle \beta_j | \beta_j \rangle = 1$. The right side of (20), together with (22) shows that the isometry J obtained using twisted-diagonal Kraus operators is of the pcubed form (3). That the $|\gamma_j\rangle$ in (21) are normalized follows from the fact that J is an isometry, though it can also be checked using the usual closure condition $\sum_l L_l^\dagger L_l = I_a$ and some algebra.

Running the preceding argument in reverse shows that as long as the $|\alpha_j\rangle$ and the $|\beta_j\rangle$ (or $|\gamma_j\rangle$ if one thinks of the Kraus operators as mapping $\mathcal{H}_a$ to $\mathcal{H}_c$) are linearly independent, i.e., they form bases of $\mathcal{H}_a$ and $\mathcal{H}_b$ with $d_a = d_b$, pcubed channels are twisted diagonal channels. However, if the $|\alpha_j\rangle$ are linearly independent but the $|\beta_j\rangle$ are not, the corresponding pcubed isometry is not of the twisted-diagonal form; for example

$$J|1\rangle = |11\rangle, \quad J|2\rangle = |12\rangle, \quad J|3\rangle = |21\rangle, \tag{23}$$

where the kets $|j\rangle$ are orthonormal. In the notation of (16), $|\beta_1\rangle = |\beta_2\rangle = |1\rangle$ and $|\gamma_1\rangle = |\gamma_3\rangle = |1\rangle$, so both the $|\beta_j\rangle$ and the $|\gamma_j\rangle$ are linearly dependent, and neither the $a \to b$ nor the $a \to c$ channel is twisted diagonal. When discussing degradable $a \to b$ pcubed channels we will assume that both the $\{|\alpha_j\rangle\}$ and the $\{|\beta_j\rangle\}$ are linearly independent, and thus this kind of channel is of the twisted-diagonal form.

3.3 Degradable pcubed channels

We assume that the isometry is of the form (16) with both sets $\{|\alpha_j\rangle\}$ and $\{|\beta_j\rangle\}$ linearly independent. In this case the degrading isometry K in (10) must necessarily carry each $|\beta_j\rangle$ to a product state $|\gamma_j\rangle \otimes |\delta_j\rangle$. The reason is that $\mathcal{C}([\alpha_j])$, where we use the abbreviation $[\psi] = |\psi\rangle\langle\psi|$, will be the pure state $[\gamma_j]$, and this means that $\mathcal{D}([\beta_j])$ must also be the pure state $[\gamma_j]$. This is only possible if K maps $|\beta_j\rangle$ onto a product of pure states, rather than onto some entangled state, which is to say $|\gamma_j\rangle \otimes |\delta_j\rangle$ for a suitable choice of $|\delta_j\rangle$ in $\mathcal{H}_d$. But as the $|\beta_j\rangle$ are assumed to be linearly independent, K is the pcubed isometry

$$K|\beta_j\rangle = |\gamma_j\rangle \otimes |\delta_j\rangle, \tag{24}$$

and the analog of (17) is

$$B_{jk} = C_{jk}D_{jk}, \text{ or } B = C * D, \text{ with } D_{jk} = \langle\delta_j|\delta_k\rangle. \tag{25}$$

This provides an easy prescription for constructing a large family of degradable channels: Choose any two positive definite $d \times d$ matrices C and D, and use (25) and (17) to define B and then A. The $a \to b$ channel with superoperator $\mathcal{B}$, (3) will then be a degradable channel. Or one can choose a positive definite B, and C positive (semi)definite such that their Hadamard product A is positive definite. Then if $D = C/*B$, to use a fairly obvious notation for element-wise division, is positive semidefinite the $a \to b$ channel will be degradable, and otherwise it will not be degradable. Checking the positivity of D is equivalent to showing that the degrading superoperator $\mathcal{D}$ is CPTP. The H matrix whose positivity was used as a condition for degradability in [1] is the same as our D matrix, though showing the equivalence involves some algebra; thus our condition is equivalent to theirs.

A large number of examples of channels known to be degradable either belong to the pcubed family, or are subchannels of degradable pcubed channels, or can be obtained as limits of pcubed channels by varying a parameter. Some specific cases are discussed below in Sec. 4. However, it seems unlikely that all degradable channels belong to the pcubed family even when that is extended by the two methods just discussed.

3.4 Conjugate degradable pcubed channels

We look for a channel $\mathcal{B}$ generated by a pcubed isometry (16), assuming that both $\{|\alpha_j\rangle\}$ and $\{|\beta_j\rangle\}$ are linearly independent (with no similar requirement on the $\{|\gamma_j\rangle\}$), such that there exists a CPTP map $\bar{\mathcal{D}}$ satisfying (13). We begin by noting that for the channel pair $\mathcal{B}, \mathcal{C}$ as defined by inserting (16) in (2),

$$\mathcal{B}(|\alpha_j\rangle\langle\alpha_k|) = C_{kj}|\beta_j\rangle\langle\beta_k|, \quad \mathcal{C}(|\alpha_j\rangle\langle\alpha_k|) = B_{kj}|\gamma_j\rangle\langle\gamma_k|, \tag{26}$$

and, in addition,

$$\mathcal{T}(|\gamma_j\rangle\langle\gamma_k|) = |\bar{\gamma}_k\rangle\langle\bar{\gamma}_j|, \tag{27}$$

where the $\{|\bar{\gamma}_j\rangle\}$ are defined by their coefficients

$$\langle c_l|\bar{\gamma}_j\rangle = \langle\gamma_j|c_l\rangle = \langle c_l|\gamma_j\rangle*, \tag{28}$$

in the orthonormal basis $\{|c_l\rangle\}$ used in (14) to define $\mathcal{T}$. Note that j and k are in the *opposite order* on the two sides of (27).

Since the $|\alpha_j\rangle\langle\alpha_k|$ span the operator space $\hat{\mathcal{H}}_a, \mathcal{T} \circ \mathcal{C} = \bar{\mathcal{D}} \circ \mathcal{B}$, (13), is equivalent to the requirement that

$$B_{kj}|\bar{\gamma}_k\rangle\langle\bar{\gamma}_j| = C_{kj}\bar{\mathcal{D}}(|\beta_j\rangle\langle\beta_k|) \tag{29}$$

be satisfied for every j and k. For $j = k$ (recall that $B_{jj} = 1 = C_{jj}$) this means that

$$\bar{\mathcal{D}}([\beta_j]) = [\bar{\gamma}_j], \tag{30}$$

where we remind the reader that $[\psi]$ is our notation for the projector $|\psi\rangle\langle\psi|$. Thus, using the same argument as in Sec. 3.3, we see that $\bar{\mathcal{D}}$ must be a channel generated by a pcubed isometry of the form

$$\bar{K}|\beta_j\rangle = |\bar{\gamma}_j\rangle \otimes |\delta_j\rangle, \tag{31}$$

and hence

$$\bar{\mathcal{D}}(|\beta_j\rangle\langle\beta_k|) = D_{kj}|\bar{\gamma}_j\rangle\langle\bar{\gamma}_k|, \quad D_{kj} := \langle\delta_k|\delta_j\rangle. \tag{32}$$

Inserting (32) in (29) yields

$$B_{kj}|\bar{\gamma}_k\rangle\langle\bar{\gamma}_j| = C_{kj}D_{kj}|\bar{\gamma}_j\rangle\langle\bar{\gamma}_k|, \tag{33}$$

or, upon applying $\mathcal{T}$ to both sides (note that $\mathcal{T} \circ \mathcal{T}$ is the identity)

$$B_{kj}|\gamma_j\rangle\langle\gamma_k| = C_{kj}D_{kj}|\gamma_k\rangle\langle\gamma_j|. \tag{34}$$

For $j = k$ (34) is automatically satisfied, since the B, C, and D matrices have 1's on the diagonals, while for $j \neq k$ there are two distinct situations:

Case I: $|\langle\gamma_j|\gamma_k\rangle| = 1$, so $|\gamma_j\rangle$ and $|\gamma_k\rangle$ are identical up to a phase. Consequently, (34) will be satisfied provided (take the trace of both sides)

$$B_{kj} = C_{jk}D_{kj}. \tag{35}$$

Case II. $|\langle\gamma_j|\gamma_k\rangle| < 1$, so $|\gamma_j\rangle$ and $|\gamma_k\rangle$ are not multiples of each other, which means the operators $|\gamma_k\rangle\langle\gamma_j|$ and $|\gamma_j\rangle\langle\gamma_k|$, which do not commute, are also not multiples of each other. Hence the coefficients on both sides of (34) must vanish:

$$B_{kj} = 0 = C_{kj}D_{kj}. \tag{36}$$

Both cases may be present for different $j \neq k$ pairs. Hence it is convenient to segregate the indices j into collections $\mathcal{J}_m$, such that j and k are in the same collection if $|\gamma_j\rangle$ and $|\gamma_k\rangle$ are equal up to a phase, Case I, and j and k belong to different collections when $|\gamma_j\rangle$ and $|\gamma_k\rangle$ are not multiples of each other, Case II. Let us add a superscript to each ket to indicate the corresponding collection: $|\alpha_j^m\rangle = |\alpha_j\rangle$, $|\beta_j^m\rangle = |\beta_j\rangle$, etc., when j is a member of $\mathcal{J}_m$. With this notation the isometry J, (16), can be written as

$$J|\alpha_j^m\rangle = |\beta_j^m\rangle \otimes |\gamma^m\rangle, \tag{37}$$

where $|\gamma^m\rangle$ is $|\gamma_j^m\rangle$ for some $j \in \mathcal{J}_m$, and the phases of the others have been absorbed in the definitions of the $|\beta_j^m\rangle$, which involves no loss of generality. Then (26) can be rewritten in the form:

$$\mathcal{B}(|\alpha_j^m\rangle\langle\alpha_k^n|) = C^{nm}|\beta_j^m\rangle\langle\beta_k^n|, \ \ C^{nm} := \langle\gamma^n|\gamma^m\rangle; \ \ \ \mathcal{C}(|\alpha_j^m\rangle\langle\alpha_k^n|) = B_{kj}\delta_{mn}[\gamma^m], \tag{38}$$

where for $m \neq n$, (37) implies that $B_{kj} = \langle\beta_c^n|\beta_j^m\rangle = 0$. The isometry $\bar{K}$ in (31) has the form:

$$\bar{K}|\beta_j^m\rangle = |\bar{\gamma}^m\rangle \otimes |\delta_j^m\rangle, \tag{39}$$

and generates the channel $\bar{D}$, (32) required to show that $\mathcal{B}$ is conjugate degradable.

However, if we use the isometry

$$K|\beta_j^m\rangle = |\gamma^m\rangle \otimes |\delta_j^m\rangle, \tag{40}$$

in which $|\bar{\gamma}^m\rangle$ in (39) has been replaced by $|\gamma^m\rangle$, the corresponding $\mathcal{D}$, (11), satisfies $\mathcal{C} = \mathcal{D} \circ \mathcal{B}$, (12), and hence $\mathcal{B}$ is both degradable and conjugate degradable. The preceding argument employs special features of a pcubed approach, so it leaves open the possibility that there may exist conjugate degradable channels that are *not* degradable. But perhaps it provides some insight into which channels one should study.

4 Examples

4.1 Qubit to qubits

For $d_a = d_b = d_c = 2$ both channels of the complementary pair map a qubit input to a qubit output. It is known that all such pairs are degradable [1], so any channel of this type is either degradable or antidegradable (or both). The pcubed approach leads to the same result from a slightly different perspective, so going through it is a useful exercise. Instead of 1 and 2 we use 0 and 1 to label the kets in (16), and the rows and columns of the corresponding matrices. The four 2×2 matrices A, B, C, and D have 1's on the diagonal, and the two off-diagonal elements are complex conjugates of each other. Equations (17) and (25) will be satisfied provided

$$A_{01} = B_{01}C_{01}, \quad B_{01} = C_{01}D_{01} \tag{41}$$

Each matrix is positive (semi)definite if the magnitude of the off-diagonal element is less than (or equal to) 1. Thus the condition that the $a \to b$ channel with superoperator $\mathcal{B}$ be degradable corresponds to the requirement that $|B_{01}| \leq |C_{01}| \leq 1$.

A degradable channel pair of this type has a simple geometrical interpretation. The map $\mathcal{B}$ carries density operators forming a Bloch sphere in the channel input a, into an ellipsoid lying inside the Bloch sphere of b, but touching it at the two points $[\beta_0] = |\beta_0\rangle\langle\beta_0|$ and $[\beta_1] = |\beta_1\rangle\langle\beta_1|$ corresponding to pure states. The fact that $|A_{01}|$ is less than $|B_{01}|$ means that these points are closer together, less distinguishable, than the corresponding pure states $[\alpha_0]$ and $[\alpha_1]$ at the channel entrance. The channel is degradable when $|B_{01}| \leq |C_{01}|$, so the distance on the Bloch sphere between $[\beta_0]$ and $[\beta_1]$ is greater than that between $[\gamma_0]$ and $[\gamma_1]$ for the complementary channel. When $|B_{01}| \geq |C_{01}|$ the channel $\mathcal{C}$ is degradable and $\mathcal{B}$ is antidegradable, as the distance between $[\gamma_0]$ and $[\gamma_1]$ is greater than that between $[\beta_0]$ and $[\beta_1]$, while for $|B_{01}| = |C_{01}|$ $\mathcal{B}$ and $\mathcal{C}$ are both degradable and antidegradable.

It is convenient to write

$$
\begin{aligned}
|\alpha_0\rangle &= a_0|0\rangle + a_1|1\rangle, & |\alpha_1\rangle &= a_0|0\rangle - a_1|1\rangle, \\
|\beta_0\rangle &= b_0|0\rangle + b_1|1\rangle, & |\beta_1\rangle &= b_0|0\rangle - b_1|1\rangle, \\
|\gamma_0\rangle &= c_0|0\rangle + c_1|1\rangle, & |\gamma_1\rangle &= c_0|0\rangle - c_1|1\rangle
\end{aligned}
\tag{42}
$$

using the standard orthonormal basis $|0\rangle$ and $|1\rangle$, with a_0, a_1, b_0, etc. real numbers between 0 and 1 chosen so that the kets are normalized: $a_0^2 + a_1^2 = 1$, etc. The off-diagonal matrix element $A_{01} = A_{10}$ is $a_0^2 - a_1^2$, and B_{01} and C_{01} are given by analogous expressions. The condition $|B_{01}| \leq |C_{01}|$ for the $a \to b$ channel to be degradable is equivalent to $b_1 \geq c_1$.

The isometry $J|\alpha_j\rangle = |\beta_j\rangle \otimes |\gamma_j\rangle$ using the definitions in (42) takes the form

$$
J|0\rangle = f_0|00\rangle + f_1|11\rangle, \quad J|1\rangle = g_0|01\rangle + g_1|10\rangle,
\tag{43}
$$

in the standard basis, where again the f_j and g_j are real numbers between 0 and 1, $f_0^2 + f_1^2 = g_0^2 + g_1^2 = 1$, and

$$
f_0 = b_0 c_0 / a_0, \quad f_1 = b_1 c_1 / a_0, \quad g_0 = b_0 c_1 / a_1, \quad g_1 = b_1 c_0 / a_1.
\tag{44}
$$

expresses them in terms of the parameters in (42).

Any qubit to qubits isometry can be written, up to local unitaries, in the form (43) with a suitable choice of the f_j and g_j, since a two-dimensional subspace of two qubits always possesses an orthonormal basis of the form shown on the right side [16]. Therefore such an isometry is of the pcubed form provided there are values of a_j, b_j, and c_j such that (44) is satisfied. This is easily seen to be the case unless one of the f_j or g_j is 0. If, for example $f_1 = 0$, either b_1 or c_1 must be 0, so either g_0 or g_1 will vanish. Hence the isometry (43) with

$$
f_0 = 1, \quad f_1 = 0, \quad g_0 = \sqrt{p}, \quad g_1 = \sqrt{1-p},
\tag{45}
$$

with $0 < p < 1$, so both g_0 and g_1 are positive, corresponding to the amplitude damping channel, illustrates the exceptional situation in which (43) is *not* pcubed. From a geometrical point of view this situation corresponds to a limiting case in which the contact points between the ellipsoid and the Bloch sphere, for both the $\mathcal{B}$ and $\mathcal{C}$ channels, coalesce into a single point, which suggests that the isometry corresponding to (45) is the limit of pcubed isometries. To see that this is indeed the case, let a_1, b_1, and c_1 be small positive quantities chosen so that

$$
c_1 / a_1 = \sqrt{p}, \quad b_1 / a_1 = \sqrt{1-p},
\tag{46}
$$

and take the limit as a_1 tends to zero, with p fixed. The result is that the f_j and g_j parameters tend continuously to those in (45). The condition for the $a \to b$ channel $\mathcal{B}$ to be degradable, $b_1 \geq c_1$, is fulfilled for $p \leq 1/2$, and one can check that the pcubed isometry (24) for the degrading map exists and tends to the appropriate limit as long as $p \leq 1/2$. Thus all qubit-to-qubits channel pairs are given either directly or as a limit of pcubed isometries whose degradability is determined by whether the $D = B/*C$ matrix, (25) is positive (semi)definite.

4.2 Erasure channel

A one qubit erasure channel $a \to b$ transmits a linear combination of $|0\rangle$ and $|1\rangle$ unchanged to the output space $\mathcal{H}_b$ with a probability $1 - p$, but with probability p replaces it with the error flag $|e\rangle$. It can be represented by the $\mathcal{H}_a$ to $\mathcal{H}_b \otimes \mathcal{H}_c$ isometry

$$
J|0\rangle = \sqrt{1-p}\,|0e\rangle + \sqrt{p}\,|e0\rangle, \quad J|1\rangle = \sqrt{1-q}\,|1e\rangle + \sqrt{1-q}\,|e1\rangle, \quad J|e\rangle = |ee\rangle
\tag{47}
$$

where we have added a state $|e\rangle$ to the input space so that $d_a = d_b = d_c = 3$, and introduced an additional probability q which when set equal to p yields the usual erasure channel. The kets $|0\rangle$, $|1\rangle$, and $|e\rangle$ form an orthonormal basis for each of the Hilbert spaces. The usual erasure channel (for $q = p$) with input the subspace of $\mathcal{H}_a$ spanned by $|0\rangle$ and $|1\rangle$ is a subchannel of the channel defined by (47),

The isometry (47) is the limit of a pcubed isometry of the form (16) in which

$$|\beta_0\rangle = (|e\rangle + \zeta\sqrt{1-p}\,|0\rangle)/b_0, \quad |\beta_1\rangle = (|e\rangle + \eta\sqrt{1-q}\,|1\rangle)/b_1, \quad |\beta_e\rangle = |e\rangle,$$
$$|\gamma_0\rangle = (|e\rangle + \zeta\sqrt{p}\,|0\rangle)/c_0, \qquad |\gamma_1\rangle = (|e\rangle + \eta\sqrt{q}\,|1\rangle)/c_1, \qquad |\gamma_e\rangle = |e\rangle, \tag{48}$$

where the normalization factors are

$$b_0 = \sqrt{1+(1-p)\zeta^2}, \quad b_1 = \sqrt{1+(1-q)\eta^2}, \quad c_0 = \sqrt{1+p\,\zeta^2}, \quad c_1 = \sqrt{1+q\,\eta^2}, \tag{49}$$

where ζ and η are small positive numbers which tend to zero. To see that the isometry so defined converges to (47), note that the subspace of $\mathcal{H}_b \otimes \mathcal{H}_c$ that contains $|\beta_0\rangle \otimes |\gamma_0\rangle$, $|\beta_1\rangle \otimes |\gamma_1\rangle$, and $|\beta_e\rangle \otimes |\gamma_e\rangle$ is spanned by $|ee\rangle$ and two other kets which differ from their counterparts on the right side of (47) by small corrections of order ζ and η.

The 3×3 B and C matrices constructed from (48) (see (17)) have 1's on the diagonal and off-diagonal elements (note that $B_{e0} = B_{0e}$, etc.)

$$B_{0e} = 1/b_0, \quad B_{1e} = 1/b_1, \quad B_{01} = 1/b_0 b_1, \quad C_{0e} = 1/c_0, \quad C_{1e} = 1/c_1, \quad C_{01} = 1/c_0 c_1. \tag{50}$$

As these are real and positive, the same will be true of the off-diagonal elements of the matrix $D = B/*C$, (27), so D will satisfy the positivity conditions given in (48), and the full $a \to b$ channel defined by (47), thus also the subchannel with inputs spanned by $|0\rangle$ and $|1\rangle$, will be degradable, as long as $B_{jk} \leq C_{jk}$, or

$$0 \leq p \leq 1/2, \quad 0 \leq q \leq 1/2. \tag{51}$$

The same argument works for a d-dimensional erasure channel with input space spanned by $|0\rangle$, $|1\rangle$, ... $|d-1\rangle$ and erasure probabilities $p_0, p_1, \ldots$ playing the role of p and q in (47). By making this a subchannel of the $d+1$-dimensional channel obtained by adding $|e\rangle$ to the input space and using the obvious analogs of (48) and can demonstrate degradability provided all the p_j are no greater than $1/2$.

4.3 Entanglement breaking and Hadamard

An entanglement-breaking channel [17] can be defined in various ways, of which the most convenient for our purposes is that it is generated by Kraus operators of rank 1. The complement of an entanglement-breaking channel is a Hadamard channel, and is known to be degradable [18]. We will show that a Hadamard channel is a subchannel of a pcubed channel.

If the $a \to c$ channel is entanglement breaking, then it and the complementary Hadamard channel are generated by an isometry of the form

$$\hat{J} = \sum_j |b_j\rangle \otimes |\gamma_j\rangle\langle\hat{\alpha}_j|, \tag{52}$$

where the $|b_j\rangle$ form an orthonormal basis of $\mathcal{H}_b$ and the $L_j = |\gamma_j\rangle\langle\hat{\alpha}_j|$ are Kraus operators mapping $\mathcal{H}_a$ to $\mathcal{H}_c$. There is no assumption that the collections $|\gamma_j\rangle$ and $|\hat{\alpha}_j\rangle$ are orthogonal. For convenience we assume the $|\gamma_j\rangle$ are normalized (as can always be arranged using a suitable normalization for the $|\hat{\alpha}_j\rangle$), which allows the completeness condition for the Kraus operators to be written in the form

$$\sum_j L_j^\dagger L_j = \sum_j |\hat{\alpha}_j\rangle\langle\hat{\alpha}_j| = I_a. \tag{53}$$

Thus the d_b positive operators $|\hat{\alpha}_j\rangle\langle\hat{\alpha}_j|$ form a POVM. If d_b is larger than d_a the Naimark extension (or dilation) theorem [19] tells us there exists a Hilbert space $\mathcal{H}_A$ with an orthonormal basis $|a_j\rangle$ which contains $\mathcal{H}_a$ as a subspace, such that

$$|a_j\rangle = P|\alpha_j\rangle, \tag{54}$$

where P is the projector in $\hat{\mathcal{H}}_A$ which projects onto $\mathcal{H}_a$ regarded as a subspace of $\mathcal{H}_A$. (If $d_b = d_a$ then $\mathcal{H}_A$ is the same as $\mathcal{H}_a$, and $|\alpha_j\rangle = |\hat{\alpha}_j\rangle$.) We now define an isometry $J : \mathcal{H}_A \to \mathcal{H}_b \otimes \mathcal{H}_c$ by the formula

$$J|a_j\rangle = |b_j\rangle \otimes |\gamma_j\rangle, \tag{55}$$

10

which is (16) with $|\alpha_j\rangle = |a_j\rangle$ and $|\beta_j\rangle = |b_j\rangle$, and thus pcubed. That it is an isometry follows from the fact that the $|a_j\rangle$ and $|b_j\rangle$ are orthonormal bases, and the $|\gamma_j\rangle$ are normalized. Thus A and B in (17) are the identity matrices, and making D the identity matrix means (54) will also be satisfied. Hence J defines a degradable $A \to b$ channel. But since any subchannel of a degradable channel, obtained by restricting the channel entrance to a subspace while leaving the exit spaces $\mathcal{H}_b$ and $\mathcal{H}_c$ unchanged, is (obviously) also degradable, the Hadamard channel induced by $\hat{J} = JP$ is also degradable.

5 Qutrit to qutrits

In the case of qutrits, $d_a = d_b = d_c = 3$, the 3×3 matrices of Sec. 3.1 are of the form

$$M = \begin{pmatrix} 1 & m_1 & m_2 \\ m_1^* & 1 & m_3 \\ m_2^* & m_3^* & 1 \end{pmatrix}, \tag{56}$$

where the m_j are complex numbers, equal to a_j, b_j, and c_j for the matrices A, B and C, with $a_j = b_j c_j$. The necessary and sufficient conditions for M to be positive semidefinite are:

$$|m_1| \leq 1, \ |m_2| \leq 1, \ |m_3| \leq 1, \ |m_1|^2 + |m_2|^2 + |m_3|^2 \leq 1 + 2\,\mathrm{Re}(m_1 m_2^* m_3), \tag{57}$$

and it will be positive definite if all the inequalities are strict.

5.1 Equal off-diagonal elements

A very simple situation is that in which the off-diagonal elements are equal real numbers

$$b_j = b, \ c_j = c, \ a_j = a = bc. \tag{58}$$

The inequalities (57) will be satisfied provided $-1/2 \leq b, c \leq 1$, which ensures that $a = bc$ falls in the same range. The $\mathcal{B}$ channel is degradable and the $\mathcal{C}$ channel anti-degradable when $D = B/ * C$, (25) is positive semi-definite, which is to say,

$$-1/2 \leq b/c \leq 1, \tag{59}$$

whereas $\mathcal{C}$ is degradable and $\mathcal{B}$ anti-degradable for c/b between $-1/2$ and 1.

Figure 1 shows the regions in the b, c plane where $\mathcal{B}$ is degradable ($\mathcal{C}$ anti-degradable) or anti-degradable ($\mathcal{C}$ degradable), which are mapped into each other by symmetry when b and c are interchanged. This symmetry also implies that both one-shot capacities $Q^{(1)}(\mathcal{B})$ and $Q^{(1)}(\mathcal{C})$, (6) and (7), vanish along the line $b = c$. In addition, in the regions labeled I and II the $\mathcal{B}$ and $\mathcal{C}$ channels are *undegradable* as defined in Sec. 2.3: each is neither degradable nor is it antidegrable: neither b/c nor c/b lies between $-1/2$ and 1.

Figure 2 shows the $b \geq 0$ and $c \leq 0$ rectangle that contains wedge I of Fig. 1, with additional information from numerical studies of the input density operators ρ on $\mathcal{H}_a$ which maximize and minimize $\Delta S(\rho)$, (5), and thus determine $Q^{(1)}(\mathcal{B})$ and $Q^{(1)}(\mathcal{C})$. The one shot capacity $Q^{(1)}(\mathcal{B})$ is positive everywhere to the left of curve 3, where it goes continuously to zero, and remains zero for larger b. Similarly, $Q^{(1)}(\mathcal{C})$ is positive everywhere to the right of curve 2 where it rises continuously from zero, and zero everywhere to the left of this curve. Hence there is a region between curves 2 and 3 where $Q^{(1)}(\mathcal{B})$ and $Q^{(1)}(\mathcal{C})$ are positive, something impossible for a degradable channel pair. The other curves in the figure indicate boundaries of concavity or convexity of $\Delta S(\rho)$, (5), and points at which the optimizing density operator changes its character, as discussed next.

5.2 Optimizing density operators

Understanding what happens in the wedge between lines 1 and 4 in Fig. 2 (region I in Fig. 1) requires paying some attention to the symmetries of the optimizing density operators. The fact that the off-diagonal elements of the A matrix are equal makes it invariant under any permutation of the three kets $|\alpha_j\rangle$ for $j = 1, 2, 3$, the symmetry group S_3, or D_2 for an equilateral triangle, and likewise B and C upon permuting the $|\beta_j\rangle$ and of the $|\gamma_j\rangle$. A convenient orthonormal or *standard* basis $\{|0\rangle, |1\rangle, |2\rangle\}$ of $\mathcal{H}_a$ is one in which the
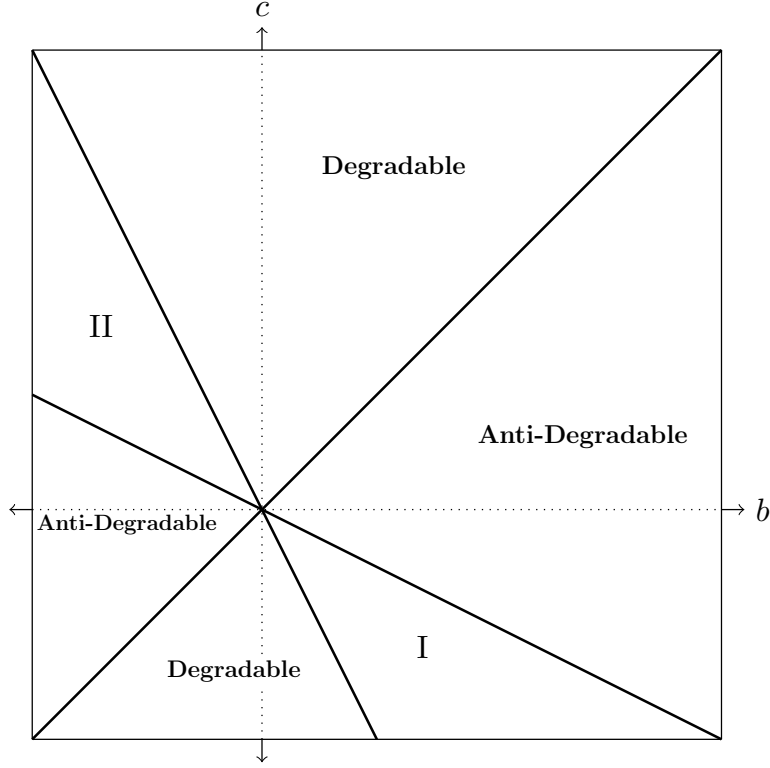
Figure 1: The (b, c) plane phase diagram with b and c between $-1/2$ and 1. The $\mathcal{B}$ channel is degradable ($\mathcal{C}$ anti-degradable) or anti-degradable ($\mathcal{C}$ degradable) in the wedges carrying these labels, while in I and II the channel pair is undegradable (neither $\mathcal{B}$ nor $\mathcal{C}$ is either degradable or antidegradable).
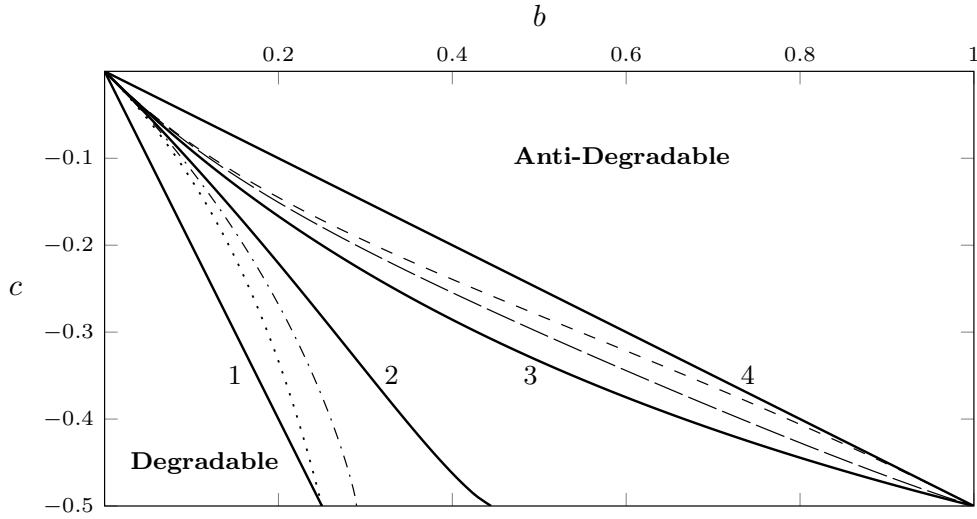

Figure 2: The region $b \geq 0$ and $c \leq 0$ showing details of region I in Fig. 1, which lies between the straight lines 1 and 4. See text for the significance of the different curves.

nonorthogonal kets $|\alpha_j\rangle$ take the form:

$$|\alpha_1\rangle = \sqrt{\frac{1+2a}{3}}|0\rangle - \sqrt{\frac{1-a}{2}}|1\rangle - \sqrt{\frac{1-a}{6}}|2\rangle,$$

$$|\alpha_2\rangle = \sqrt{\frac{1+2a}{3}}|0\rangle + \sqrt{\frac{1-a}{2}}|1\rangle - \sqrt{\frac{1-a}{6}}|2\rangle,$$

$$|\alpha_3\rangle = \sqrt{\frac{1+2a}{3}}|0\rangle + \sqrt{\frac{2(1-a)}{3}}|2\rangle, \tag{60}$$

where on can visualize the components of the $|\alpha_j\rangle$ in the $|1\rangle$, $|2\rangle$ plane as vertices of an equilateral triangle. We use similar standard bases of $\mathcal{H}_b$ and $\mathcal{H}_c$ in which the $\{|\beta_j\rangle\}$ and $\{|\gamma_j\rangle\}$ have the form shown in (60), but with a replaced by b and by c.

A representation of the D_2 (S_3) symmetry is provided by powers and products of the orthogonal (hence unitary) matrices

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{61}$$

using the standard basis in (60), or its counterparts for $\mathcal{H}_b$ and $\mathcal{H}_c$. Here F is a "reflection" that interchanges $|\alpha_1\rangle$ and $|\alpha_2\rangle$, whereas R is a "rotation" $|\alpha_1\rangle \to |\alpha_2\rangle \to |\alpha_3\rangle \to |\alpha_1\rangle$. The pcubed isometry J in (16) is symmetrical in the sense that

$$JG_a = (G_b \otimes G_c)J, \tag{62}$$

where G is any one of the matrices of the group representation, and the subscript indicates which space, $\mathcal{H}_a$ or $\mathcal{H}_b$ or $\mathcal{H}_c$, it is acting on. In light of (62) the superoperators defined in (2) have the property that

$$\mathcal{B}(G_a \, \rho \, G_a^\dagger) = G_b \, \mathcal{B}(\rho) \, G_b^\dagger, \quad \mathcal{C}(G_a \, \rho \, G_a^\dagger) = G_c \, \mathcal{C}(\rho) \, G_c^\dagger, \tag{63}$$

where ρ is an input density operator. In addition the fact that the parameters b and c are real numbers means that the full symmetry group for the superoperators includes complex conjugation in the standard basis, an antiunitary operation that commutes with the other group elements; for this case think of $G\rho G^\dagger$ in (63) as the complex conjugate (or transpose) of the density operator in the standard basis.

Our numerical studies indicate that the optimizing density operator ρ on $\mathcal{H}_a$ that yields the maximum, $Q^{(1)}(\mathcal{B})$, or minimum, $-Q^{(1)}(\mathcal{C})$, of $\Delta S(\rho)$ (see (5)) is always of the form

$$\rho_R = \begin{pmatrix} 1-2s & 0 & 0 \\ 0 & s & it \\ 0 & -it & s \end{pmatrix}, \tag{64}$$

for some choice of the two real parameters $0 \le s \le 1/2$ and $0 \le t \le s$, where the subscript R indicates that ρ_R is invariant under the ("rotation") subgroup generated by the matrix R in (61). Indeed, we find that the optimum input ρ is always one of three special cases:

$$\rho_0 = \begin{pmatrix} 1-2s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1-2s & 0 & 0 \\ 0 & s & is \\ 0 & -is & s \end{pmatrix}. \tag{65}$$

obtained from ρ_R by setting $t = 0$, or $t = 0$ and $s = 1/2$, or $t = s$. Note that both ρ_1 and ρ_2 are rank 2 (one eigenvalue equal to zero), and their supports are two-dimensional subspaces that do not intersect (except for the origin) for $s < 1/2$. The diagonal ρ_0 is the most general matrix invariant under the full symmetry group, which means that its images under $\mathcal{B}$ and $\mathcal{C}$ will also have this form (but of course with different values of the parameter s). The images of ρ_1 will be of the form ρ_0, but with s not equal to $1/2$ in general, while those of ρ_2 will be of the form ρ_R, with, in general, $t < s$ rather than $t = s$.

For values of (b, c) falling within the degradable (or antidegradable) regions in Fig. 1 the input density operator that maximizes (or minimizes) $\Delta S(\rho)$ is always of the form ρ_0, a plausible consequence of the fact that degradability implies the concavity of $\Delta S(\rho)$ as a function of ρ, so one expects a unique maximum. Since when a maximum is achieved at some ρ, it is also, by symmetry, achieved at $G_a\rho G_a^\dagger$, these two must be identical if the maximum is unique. When $\mathcal{C}$ is degradable and $\mathcal{B}$ antidegradable the same argument applies to the minimum of the convex function $\Delta S(\rho)$.

In the region between lines 1 and 4 in Fig. 2 where the channel pair is undegradable, there is no general reason to expect that $\Delta S(\rho)$ will be either concave or convex. Our numerical studies indicate that $\Delta S(\rho)$, which is necessarily concave in the degradable region to the left of line 1, continues to be concave up to the dotted line. Here the breakdown of concavity begins at the pure state ρ_0 with $s = 0$ and, as b increases, manifests itself in the fact that the second derivative of $\Delta S(\rho_R)$ with respect to t evaluated at $t = 0$ is positive (contrary to concavity) for s lying in an interval $(0, s_0)$. As b increases s_0 increases until it arrives at the value of s that maximizes $\Delta S(\rho_0)$, which occurs at the dot-dash line in the figure. At this point the optimizing density operator changes discontinuously from ρ_0 to ρ_2 with increasing b, though $Q^{(1)}(\mathcal{B})$ is continuous. For larger values of b the optimizing density operator continues to have the form ρ_2, with s depending on b and c, until $Q^{(1)}(\mathcal{B})$ goes to zero continuously at line 3.

Analogous changes occur as b decreases across line 4 in Fig. 2, the boundary of the degradable region for $\mathcal{C}$, where $\Delta S(\rho)$ must be convex. Our numerical studies indicate that convexity continues to hold to the right of the dashed line with smaller dashes just to the left of line 4, and it begins to break down at the pure state ρ_0 with $s = 0$, but this time the signature is a change in the sign of $\partial^2 \Delta S / \partial s^2$. As b continues to decrease the value of s that minimizes ΔS increases until it reaches its maximum value of $1/2$, at the dashed line with the larger dashes. At still smaller values of b the optimizing (minimizing) density operator is ρ_1, all the way until $Q^{(1)}(\mathcal{C})$ goes to zero at curve 2.

Some additional details including formulas for the curves in Fig. 2 are given in the Appendix.

5.3 Summary

Let us summarize what seem to be the most significant features found in the wedge region between solid lines 1 and 4 in Fig. 2 (I in Fig. 1) where the channel pair is undegradable. First, in the region between solid lines 2 and 3, both $Q^{(1)}(\mathcal{B})$ and $Q^{(1)}(\mathcal{C})$, the one-shot capacities of the two complementary channels, are positive, and in this overlap region the two optimizing density operators are both of rank 2, supported on nonintersecting subspaces of $\mathcal{H}_a$. The regions where $\Delta S(\rho)$ is either concave or convex extend slightly outside the regions where these properties are guaranteed by the degradability of $\mathcal{B}$ and of $\mathcal{C}$, respectively, and the breakdown of concavity or convexity begins at a pure state on the boundary of the convex set of input density operators. There are in addition some qualitative changes in the character of the optimizing density operators. For $Q^{(1)}(\mathcal{B})$ this is connected to a breakdown of local concavity of $\Delta S(\rho)$ at its maximum, resulting in a change in the optimizing density operator from ρ_0 to the less symmetric ρ_2 at the dot-dash line. In the case of $Q^{(1)}(\mathcal{C})$ it arises when s in ρ_0 at the minimum of $\Delta S(\rho)$ arrives at its maximum value of $1/2$.

6 Conclusion

We have shown that pcubed, "pure to product of pure states", isometries are a useful tool for constructing examples of degradable quantum channels, since degradability is relatively easy to confirm by checking the positivity of the D matrix defined in Sec. 3.3. As this is a $d \times d$ matrix, checking its positivity is generally simpler than that of the corresponding $d^2 \times d^2$ Choi matrix of a comparable channel. Furthermore, by simply choosing positive C and D matrices one obtains, via the Hadamard (entry-wise) products $B = C * D$ and $A = C * C * D$, an $a \to b$ channel which is guaranteed to be degradable, and whose $Q = Q^{(1)}$ quantum capacity is therefore relatively easy to calculate. Additional examples of degradable channels can be constructed using subchannels of degradable pcubed channels, or the limit of a family of such channels depending on a parameter, or a combination of these two procedures, as illustrated by the examples in Sec. 4. It is even conceivable, though it seems unlikely, that all degradable channels can be obtained from degradable pcubed channels in this manner. In any case the connection between pcubed and degradability could be a fruitful area for further study.

The family of pcubed channels includes much more than just degradable channels. Indeed, since an isometry corresponds (ignoring unitaries on $\mathcal{H}_a$) to a particular subspace of $\mathcal{H}_b \otimes \mathcal{H}_c$, and as it is easy to construct (in many different ways) a basis of $\mathcal{H}_b \otimes \mathcal{H}_c$ using product states, *any* quantum channel is a subchannel of *some* pcubed channel. By itself this observation is of no great significance. However, as shown by the specific example in Sec. 5 of a qutrit mapped to a product of qutrits, the pcubed construction allows the construction of simple examples of channel pairs which are neither degradable nor antidegradable, but nonetheless can have nonzero one-shot capacities, and by varying a parameter one can move continuously from a degradable to a undegradable channel pair and ask what happens to the one-shot capacity, and to

the concavity/convexity properties of the coherent information as a function of the input density operator. The interesting behavior in the very simple case of equal off-diagonal elements of the A, B, and C matrices, Secs. 5.1 and 5.2, is not something we would have anticipated in advance of numerical calculations. These features may or may not prove useful in future studies of quantum capacity and its additivity properties, they at least seem worth thinking about.

From a somewhat broader perspective, our work suggests it may be useful to think about properties of quantum channels in terms of the corresponding isometries, even when these are not of the pcubed type, rather than placing the entire emphasis on channel superoperators, as in much of the published literature. To be sure, there is a one-to-one correspondence, up to local unitaries, between isometries and completely positive trace preserving maps. Nevertheless, relationships and intuitive insight can sometimes emerge more readily from one perspective than from the other, and thus both need to be kept in mind.

Acknowledgments

A   Appendix. Details of Fig. 2

Numerical studies were carried out at several thousands of points in the part of the b, c plane shown in Fig. 2 to find the maximum and minimum values of $\Delta S(\rho)$, setting $\rho = R^\dagger R$ with R an upper triangular matrix depending on 3 real parameters for the diagonal elements and 6 real parameters representing the real and imaginary parts of the off diagonal elements, with the sum of the squares equal 1. We used standard numerical optimization techniques (Mathematica). Concavity and convexity were checked by evaluating the Hessian of $\Delta S(\rho)$ for several thousand randomly-chosen density operators, at a large number of b, c points. Once the general character of the optimizing density operators emerged, as discussed in Sec. 5.3, we obtained expressions determining the different curves in Fig. 2 as discussed below, and checked that these were in good agreement with our earlier numerical studies.

The density operator ρ_R in (64) depends on two parameters s and t, and by symmetry $\mathcal{B}(\rho_R)$ and $\mathcal{C}(\rho_R)$ are of the same form but with different values of the parameters, which we denote by s_b, t_b and s_c, t_c. Thus

$$s_b = \frac{1-b}{1+2bc}\left[\frac{1-c}{3} + \frac{(1+2b)c}{1-bc}s\right], \quad t_b = \frac{1-b}{1-bc}t; \tag{66}$$

the formulas for s_c and t_c are obtained by interchanging b and c. A density operator of the form ρ_R has eigenvalues $1-2s, s+t, s-t$, and those of its image under $\mathcal{B}$ are the same with s and t replaced by s_b and t_b, and similarly for $\mathcal{C}$. This leads to comparatively simple formulas for the von Neumann entropy and thus $\Delta S(\rho)$, so much of the analysis can be carried out with explicit formulas.

As b increases out of the degradable region to the left of line 1 in Fig. 2 the breakdown of the concavity of $\Delta S(\rho)$ is connected with a change in sign of $[\partial^2 \Delta S(\rho_R)/\partial t^2]_{t=0}$. Equating this quantity to zero yields a formula

$$s = \frac{(1-bc)(b+c-2bc)}{(-3bc)(1+2b+2c-2bc)} \tag{67}$$

for s as a function of b and c. Setting $s = 0$ gives the formula

$$c = -b/(1-2b) \tag{68}$$

for the dotted line, and $\Delta S(\rho)$ is concave everywhere to the left of this line. As b increases further the value of s given by (67) increases until it reaches the value at which $\Delta S(\rho_0)$ is maximum. Setting $d\Delta S(\rho_0)/ds = 0$ yields a formula

$$(1-b)(1+2b)\,c\,\log(-2+1/s_b) = (1-c)(1+2c)\,b\,\log(-2+1/s_c) \tag{69}$$

in which s_b and s_c depend on s through (66) and its counterpart with b and c interchanged. Eliminating s between (67) and (69) gives a formula relating b and c which can be solved numerically to give the dot-dash

curve in the figure. Line 3 in the figure is where the maximum of $\Delta S(\rho_2)$ goes to zero, terminating the region where $\mathcal{Q}^{(1)}(\mathcal{B}) > 0$. This occurs when the eigenvalues of $\mathcal{B}(\rho_2)$ as functions of s become degenerate with those of $\mathcal{C}(\rho_2)$ along the line

$$c = -b/(1+b). \tag{70}$$

The remaining curves in Fig. 2 are associated with the complementary channel $\mathcal{C}$, with $\mathcal{Q}^{(1)}(\mathcal{C})$ given by the minimum rather than the maximum of $\Delta S(\rho)$. To the right of line 4, $\mathcal{C}$ is degradable ($\mathcal{B}$ antidegradable) and $\Delta S(\rho)$ is a convex function. Our numerical studies indicate that this convexity persists for slightly smaller value of b, and first breaks down at $s = 0$ when $d^2\Delta S(\rho_0)/ds^2$ changes sign. Equating this to zero at $s = 0$ produces the relatively simple expression

$$b\,(1-c)(1+2c) = (-c)(1-b)(1+2b) \tag{71}$$

and thus a formula

$$c = \left(1 + 2b - 2b^2 - \sqrt{1 + 4(b + 2b^2 - 2b^3 + b^4)}\right)/4b \tag{72}$$

for the curve with narrow dashes lying just to the left of line 4.

As b continues to decrease $\Delta S(\rho)$ is minimized by $\rho = \rho_0$, but with s increasing until it reaches its maximum value of $1/2$, which first occurs when $d\Delta S(\rho_0)/ds = 0$ at $s = 1/2$. The transition line is given by (69) with s_b and s_c set equal to their values when $s = 1/2$. Solving it numerically for c as a function of b yields the curve with long dashes in Fig. 2. The optimizing (minimizing) density operator for yet smaller values of b continues to be ρ_1, and $\mathcal{Q}^{(1)}(\mathcal{C})$ finally reaches zero at curve 2, where $S(\mathcal{B}(\rho_1)) = S(\mathcal{C}(\rho_1))$, or

$$
\begin{aligned}
(1 + 2b)(1 - c)\log[2(1 + 2b)(1 - c)] + (1 - b)(2 + c)\log[(1 - b)(2 + c)] = \\
(1 + 2c)(1 - b)\log[2(1 + 2c)(1 - b)] + (1 - c)(2 + b)\log[(1 - c)(2 + b)].
\end{aligned} \tag{73}
$$

A numerical solution gives curve 2 in Fig. 2. (The slight hook visible at the foot of the curve is genuine, and reflects the $(1 + 2c)\log(1 + 2c)$ term in (73).)

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