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How to quantify coherence: distinguishing speakable and unspeakable notions

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Quantum coherence is a critical resource for many operational tasks. Understanding how to quantify and manipulate it also promises to have applications for a diverse set of problems in theoretical physics. For certain applications, however, one requires coherence between the eigenspaces of specific physical observables, such as energy, angular momentum, or photon number, and it makes a difference *which* eigenspaces appear in the superposition. For others, there is a preferred set of subspaces relative to which coherence is deemed a resource, but it is irrelevant which of the subspaces appear in the superposition. We term these two types of coherence unspeakable and speakable respectively. We argue that a useful approach to quantifying and characterizing *unspeakable* coherence is provided by the resource theory of asymmetry when the symmetry group is a group of translations, and we translate a number of prior results on asymmetry into the language of coherence. We also highlight some of the applications of this approach, for instance, in the context of quantum metrology, quantum speed limits, quantum thermodynamics, and NMR. The question of how best to treat *speakable* coherence as a resource is also considered. We review a popular approach in terms of operations that preserve the set of incoherent states, propose an alternative approach in terms of operations that are covariant under dephasing, and we outline the challenge of providing a physical justification for either approach. Finally, we note some mathematical connections that hold among the different approaches to quantifying coherence.

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I. INTRODUCTION AND SUMMARY

Many properties of quantum states can be better understood by considering them as constituting a resource [1]. The properties of entanglement [2, 3], asymmetry [4–13], and athermality [14–19] are good examples. We are here concerned with the property of having coherence relative to some decomposition of the Hilbert space. This property appears to be necessary for certain types of tasks, and as such it is natural to attempt to understand coherence from the resource-theoretic perspective. This has led to some proposals for how to define a resource theory of coherence and in particular how to quantify coherence [10, 20–23].

The following is a list of operational tasks for which quantum coherence seems to be a resource.

- Quantum metrology. An example is the task of estimating the phase shift on a field mode, used in quantum accelerometers and gravitometers. Here one requires the ability to prepare and measure coherent superpositions of different occupation numbers of the mode. Another example is estimating the rotation of a quantum gyroscope about some axis, where one must be able to prepare and measure coherent superpositions of eigenstates of the angular momentum operator along that axis, a problem that is relevant for developing high precision measurements of magnetic field strength [24]. A third example is building high precision clocks, where one must be able to prepare and measure coherent superpositions of energy eigenstates [25–28].

- Reference frame alignment. Examples include aligning distant gyroscopes, synchronizing distant clocks, and phase-locking distant phase references. Each example requires communicating quantum states that carry the appropriate sort of information (orientation, time, and phase, for instance) and therefore, like the metrology examples, requires coherence relative to the appropriate eigenspaces [29].

- Thermodynamic tasks. An example is the task of extracting as much work as possible from a given quantum state given a bath at some fixed background temperature. This require states that are not in thermal equilibrium at the background temperature. Resources here include not only those states having a nonthermal distribution of energy eigenstates, but also those that have coherence between the energy eigenspaces [30–33].

- Computational, cryptographic and communication tasks. For these sorts of tasks, it is well known that having access only to preparations and measurements that are all diagonal in some basis, hence incoherent, is not sufficient for achieving any quantum advantage. So it is natural to seek to study such coherence as a resource.

For many resource theories, there are also applications to problems in theoretical physics. For example, while entanglement theory was originally developed through its role as a resource in operational tasks such as quantum teleportation [34] and dense coding [35], the possibility of quantifying entanglement has since found applications in diverse problems, including the study of phase transitions, characterizing the ground states of many-body systems [36–39], holography in quantum field theories [40], and the black hole information-loss paradox [41–43]. Similarly, the possibility of quantifying coherence is expected to shed light on various problems in theoretical physics. The following are a few examples.

- Quantum speed limits. The Mandelstam-Tamm bound [44] and the Margolus-Levitin bound [45] are upper bounds on the minimum time it takes for a system in some state to evolve to a (partially) distinguishable state. This time is clearly related to the amount of coherence between energy eigenstates and therefore quantifying this coherence can shed light on quantum speed limits [21, 46].

- Magnetic resonance techniques. For such techniques, in particular in NMR, if the system one is probing consists of many spins, then the large dimension of the Hilbert space together with constraints on the measurements are such that full tomography is not possible. Still, one can obtain much useful information about the state by measuring the degree of coherence relative to the quantization axis [47]. If the system is quantized along the $\hat{z}$ axis, then *coherence of order* q of the state ρ is defined as the norm of the sum of the off-diagonal terms $\rho_{m_1 m_2}|m_1\rangle\langle m_2|$ with $m_2 - m_1 = q$, where $|m\rangle$ is the eigenstate with eigenvalue m of $J_{\hat{z}}$, the magnetic moment in the $\hat{z}$ direction [47][1]. Measuring the quantum coherence of different orders is relatively straightforward and has been useful in many NMR experiments, in particular, in the context of quantum information processing, as well as in simulations of many-body dynamics (See e.g. [47–49]).

- Coherence lengths. The spatial extent over which a quantum state is coherent is an important concept in many-body physics [50], for instance, in the onset of Bose-Einstein condensation [51], and in quantum biology, for instance, in excitation transport in photosynthetic complexes [52–54].

- Order parameters. Quantum phase transitions in the ground states of quantum many-body systems, such as a spin chain, can be studied in terms of the degree of coherence contained in local reductions of the state, such as single-spin or two-spin, density operators [55, 56].

- Decoherence theory. It is well known that interaction of a system with its environment can lead to the loss of coherence relative to preferred subspaces that depend on the nature of the interaction [57]. For instance, if an environment couples to the spatial degree of freedom of a system, then it will reduce the spatial extent over which the system exhibits coherence [57]. Such decoherence plays a significant role in many accounts of the emergence of classicality. Measures of coherence, therefore, can be used as a tool for studying such emergence.

It is critical, however, to distinguish two types of coherence that arise in these various applications. The distinction can be explained as follows. Consider the states

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |\phi\rangle = \frac{|0\rangle + |2\rangle}{\sqrt{2}} \ . \tag{1.1}$$

If we are interested in quantum computation using qutrits and the elements of the set $\{|l\rangle\}_{l\in\{0,1,2\}}$ are the computational basis states, then we would expect the states $|\psi\rangle$ and $|\phi\rangle$ to be equivalent resources because the particular identities of the computational basis states appearing in the superposition are not relevant for any computational task. In this case, l is simply an arbitrary label or *flag* for different distinguishable pure states. If, on the other hand, we are considering a phase estimation task and the elements of the set $\{|l\rangle\}_{l\in\{0,1,2\}}$ are eigenstates of the number operator, then there is a significant difference between the states $|\psi\rangle$ and $|\phi\rangle$. For instance, $|\psi\rangle$ can detect a phase shift of π while $|\phi\rangle$ cannot. Conversely, if one's task is to estimate a very small phase shift, then $|\phi\rangle$ is a better resource than $|\psi\rangle$, because the former becomes more orthogonal to itself than the latter under a small phase shift. Similarly, starting from the incoherent state $|0\rangle$, to prepare a state close to $|\phi\rangle$ one needs to have access to a phase reference with higher precision than the phase reference required to prepare a state close to $|\psi\rangle$ [8]. Here, l is not an arbitrary label, but an eigenvalue of the number operator, and its value is relevant for the task of phase estimation.

These two types of coherence pertain to two types of information which have been termed *speakable* and *unspeakable* [29, 58]. Speakable information is information for which the means of encoding is irrelevant. This is exemplified by the fact that if one seeks to transmit a bit-string, it is irrelevant what degree of freedom one uses to encode the bits. Unspeakable information is information which can only be encoded in certain degrees of freedom. Information about orientation, for instance, is unspeakable because it can only be transmitted using a system

[1] This concept is closely related to the idea of decomposition into modes of asymmetry, introduced in Refs. [8, 9] (See Sec.III A 5 for a short review). In particular, the *qth-quantum coherence component* in the language of [47] is the same as the *mode q component* in the language of [8, 9]. Furthermore, the Frobenius norm of the *qth-quantum coherence component*, which can be measured in NMR experiments, provides lower and upper bounds on a measure of coherence studied in [8, 9], which quantifies the asymmetry of a state in mode q (See Eq.(6.23)).

that transforms nontrivially under rotations. Information about time is also unspeakable because it can only be transmitted by a system that transforms nontrivially under time-translations. We shall therefore refer to the two types of coherence we have outlined above as speakable and unspeakable respectively.

For the list of operational tasks we have provided above, the relevant notion of coherence is the unspeakable one in all cases except for the last item. This is also the case for most of the physical applications listed above. This is because from the point of view of speakable coherence, the eigenvalues of the observable that defines the preferred subspaces are not relevant: the set of preferred subspaces is a set without any order. However, for the examples of quantum speed limits, coherence lengths and magnetic resonance, for instance, it is clear that the eigenvalues of the relevant observable, the Hamiltonian, position, and magnetic moment observables respectively, has important physical meaning, and there is a natural order defined on the preferred subspaces. Therefore, the notion of unspeakable coherence seems to be the more appropriate one in these cases.

Most recent work on coherence as a resource, however, considers only speakable coherence. One might think, therefore, that there is work to be done in defining a resource theory of *unspeakable* coherence. In fact, however, such a resource theory *already exists.* It simply goes by another name: the resource theory of *asymmetry.*[2] To be precise, the resource of unspeakable coherence is nothing more than the resource of asymmetry relative to a group of translations.

A. Resource-Theoretic approach to unspeakable coherence

Consider the task of phase estimation as an example. A state of some field mode has coherence relative to the eigenspaces of the number operator N if and only if it is asymmetric (i.e., symmetry-breaking) relative to the group of phase-shifts generated by N, where such translations of the phase are represented by the group of unitaries $\{e^{-iN\theta}:\ \theta\in(0,2\pi]\}$. This follows from the fact that if a state ρ is symmetric under phase-shifts, that is, $\forall\theta\in(0,2\pi]:\ e^{-iN\theta}\rho e^{iN\theta}=\rho$, then it must be block-diagonal with respect to the eigenspaces of N, while if it is not symmetric under phase-shifts, then it cannot have this form.

Other examples are treated in a similar fashion[3]: coherence relative to the eigenspaces of a Hamiltonian H is simply asymmetry relative to the group of time-translations generated by this Hamiltonian, $\{e^{-iHt}:\ t\in\mathbb{R}\}$; coherence relative to the eigenspaces of the momentum operator P is simply asymmetry relative to the group of spatial translations, $\{e^{-iPx}:\ x\in\mathbb{R}\}$; coherence relative to the eigenspaces of the angular momentum operator J_z is simply asymmetry relative to the group of rotations around $\hat{z}$, $\{e^{-iJ_z\theta}:\ \theta\in(0,2\pi]\}$.

Any resource theory must not only partition the states into those that are resources and those that can be freely prepared at no cost, it must also partition the *operations* into those that are resources and those that can be freely implemented at no cost. The free set of operations is required to be closed under composition and convex combination [1]. In entanglement theory, for instance, not only are the states partitioned into those that are unentangled, hence free, and those that are entangled, hence resources, but the operations are also partitioned into those that can be achieved by Local Operations and Classical Communications (LOCC), which are deemed to be free, and those which cannot, which are deemed to be resources.

If one considers each of the tasks for which unspeakable coherence is a resource, one sees that the freely-implementable operations are those that are covariant under translations, that is, for which first translating and then implementing the operation is equivalent to first implementing the operation and then translating (See Def. 2). For instance, in the task of reference frame alignment, the set-up of the problem is that there are two parties, each of which has a local reference frame (e.g. a gyroscope, a clock, a phase reference), but the group element that relates these two frames (e.g. the rotation, the time translation, the phase shift) is unknown. It is not difficult to show that the operations that one party can implement relative to the other party's reference frame are precisely those that are covariant under the group action [9, 29, 59]. In fact, there are *many* ways of providing a physical justification of the translationally-covariant operations, and we shall review these at length further on.

These considerations imply that the problem of quantifying and classifying unspeakable coherence can be considered a special case of the resource theory of asymmetry where the group under consideration describes a translational symmetry. (The resource theory of asymmetry is more general than this, however, because it is also capable of dealing with non-Abelian groups where asymmetry does not simply correspond to the existence of coherence between some preferred set of subspaces.)

The notion that the resource of coherence should be understood as asymmetry relative to the action of a translational symmetry and that the free operations defining the resource theory are the translationally-covariant ones was first proposed in Ref. [10] and developed in Appendix A of Ref. [8] and in Ref. [21] (See also [23]).

This connection implies that most questions about unspeakable coherence as a resource find their answers in

[2] In early work on the topic, which emphasized the role of asymmetric quantum states in defining quantum reference frames, the resource theory of asymmetry was termed the resource theory of *frameness* [4].

[3] Throughout this article, we use units where $\hbar=1$.

prior work on the resource theory of asymmetry. It suffices to specialize known results to the particular translational symmetry of interest. One of the goals of this article is to explicitly translate some of these known results from the language of asymmetry to the language of coherence, to describe the measures of coherence that result, and to review some of the applications.

This approach to coherence has already been applied to shed light on the various applications of unspeakable coherence outlined above: quantum metrology [8, 10], aligning reference frames [6, 29], quantum thermodynamics [30, 32, 33, 60, 61], quantum speed limits [21, 46]. Furthermore, it was shown in Ref. [10] that for symmetic open-system dynamics, measures of asymmetry are monotonically nonincreasing, thereby yielding a significant generalization of Noether's theorem. Translated into the language of coherence, this result states that for open system dynamics that is translationally-covariant, every measure of coherence which is derived within the translational-covariance approach to coherence provides a monotone of the dynamics. Such measures, therefore, provide a powerful new tool for studying decoherence.

B. Resource-Theoretic approaches to speakable coherence

The second topic we address in this work is whether and how one can develop a resource theory of *speakable* coherence.

When one considers recent work on quantifying coherence from the perspective of the speakable/unspeakable distinction, it is clear that it concerns itself only with the speakable notion (See e.g. [20, 22, 62–68]). Most of this work builds on a proposal by Baumgratz, Cramer and Plenio (BCP) [20]. The set of free operations in the BCP approach, called *incoherent operations*, is defined based on the Kraus decomposition of quantum operations, and is closely related to another set which is called *incoherence-perserving*. These are operations which take every incoherent state to an incoherent state.

Because it concerns speakable coherence, this approach is only appropriate for tasks concerning speakable information. Nonetheless, if one is content to accept that a resource theory of speakable coherence has a more limited scope of applications than one might have naïvely expected, the question arises of whether the BCP approach is the right way to define the resource theory of speakable coherence.

C. Criticism of resource-theoretic approaches to speakable coherence

As we noted earlier, to take a resource-theoretic approach to any given property one must first of all make a proposal for which set of operations can be freely implemented. But a given proposal for how to do so is only expected to have physical relevance if it can be provided with a *physical justification*, that is, if one can provide a restriction on *experimental capabilities* that yields all and only the operations in the free set that is proposed. For instance, in entanglement theory, the restriction on experimental capabilities that yields all and only the LOCC operations between two parties is the absence of any quantum channel between the two parties.

Despite the amount of attention that the BCP approach has received, no one has yet described a physical justification for the set of incoherent operations or the set of incoherence-preserving operations. The property of taking incoherent states to incoherent states is certainly a mathematically well-defined constraint; whether there is an *experimental* constraint that corresponds to this property is the question of interest here. Of course one can imagine physical scenarios in which preparing coherent states is hard, for instance, because of the challenge of isolating one's systems from environmental decoherence. But this does not justify the claim that the set of incoherent operations or the set of incoherence-preserving operations is the natural one to study; to do so one needs to argue that *all* of the operations in a given set can be easily implemented in that physical scenario. However, it is not clear whether such a justification can be found.

For one, because the free states in the resource theory of coherence should be restricted to those that have no coherence between the preferred subspaces, one would expect that the free measurements in a resource theory of coherence should be similarly restricted. However, we show that the BCP proposal places *no constraint* on the sorts of measurements that can be implemented, in the following sense: for *any* POVM, it is possible to find a measurement that realizes it which is considered free in the BCP approach. Therefore, to justify the BCP approach to the resource theory of speakable coherence, one needs to argue that there are physical or experimental constraints which lead to a significant restriction on state-preparations and transformations, but no restriction on the possibilities for discriminating states.

For another, it turns out that even if one finds physical scenarios in which the set of free unitaries is the set of incoherent unitaries (as defined by BCP), this still does not justify the set of incoherent or incoherence-preserving operations as the set of free operations for a resource theory of speakable coherence. As we will show, a general incoherence-preserving (incoherent) operation cannot be implemented using only incoherent states, incoherent unitaries and incoherent measurements. Using a more technical language, this means that incoherence-preserving (incoherent) operations do not admit dilation using only incoherent resources (at least, not in a straightforward way, when we treat all systems evenhandedly).

This lack of dilation for the set of free operations is not necessarily a problem in its own right [4]. One can imagine physical scenarios where the set of operations that we can implement in a *controlled* fashion using the free unitaries and free states, are smaller than the set of all free operations we can implement on the systems of interest, because the latter operations might result from *uncontrolled* interaction of these systems with an environment over which the experimenter has limited control, such as a thermal bath. The lack of a dilation that is even-handed in its treatment of systems implies that this is the only sort of avenue open for providing a physical justification of the incoherence-preserving or incoherent operations. Furthermore, we show that there are other natural proposals for the set of free operations for the resource theory of speakable coherence that share precisely the same set of free unitaries, namely, the incoherent unitaries. Therefore, even if one accepts that the set of incoherent unitaries is the appropriate set of free unitaries for a resource theory of speakable coherence, this does not resolve the question of which of the many sets of free operations consistent with this choice one should use to define the resource theory.

Our concerns about the suitability of the incoherence-preserving operations in a resource theory of coherence are bolstered by comparing them to the *non-entangling* operations in entanglement theory[5]. The non-entangling operations are those that map unentangled states to unentangled states [69]. Like the incoherence-preserving operations, therefore, they are the largest set that maps the free states to the free states. The non-entangling operations are a strictly larger set than the LOCC operations because they include nonlocal operations such as swapping systems between the two parties, and because they allow the implementation of arbitrary POVM measurements on the bipartite system. It is difficult to imagine any restriction on experimental capabilities that yields all and only the non-entangling operations. Indeed, it is widely acknowledged that the LOCC operations—which *do* arise from a natural restriction, having classical channels but not quantum channels—is the physically interesting set, while the non-entangling operations are studied primarily as a mathematical technique for making inferences about LOCC. Incoherence-preserving and incoherent operations may ultimately have a similarly subservient role to play in the resource theory of coherence.

D. A new proposal for the resource theory of speakable coherence

In the absence of a physical justification for the BCP approach, the question arises of whether an alternative choice of the set of free operations might be more suited to a resource-theoretic treatment of *speakable coherence*. Once the question is raised, a natural alternative for the set of free operations immediately suggests itself, namely, those that are *covariant under dephasing*, that is, those that commute with the operation that achieves complete dephasing relative to the preferred subspaces. We call this the *dephasing-covariance approach to coherence*. A variant of this proposal has recently been considered in Ref. [70][6].

This proposal does not have one of the counter-intuitive features of the BCP approach that was outlined above: the set of free measurements includes only the POVMs whose elements are incoherent, as one would expect. Nonetheless, it is still not clear whether the dephasing-covariance approach to coherence has much physical relevance because it is still unclear whether there is any restriction on experimental capabilities that picks out all and only the dephasing-covariant operations. (In particular, it is unclear whether every dephasing-covariant operation admits of a dilation in terms of incoherent states, incoherent measurements and dephasing-covariant unitaries.) We do not settle the issue here.

E. Relation between different approaches

In addition to providing a characterization and assessment of both the dephasing-covariance approach and the BCP approach for treating speakable coherence as a resource, we explore the mathematical relation between the free set of operations that each adopt. In particular, we show that the dephasing-covariant operations relative to a choice of preferred subspaces are a strict subset of the incoherent (incoherence-preserving) operations relative to the same choice. This implies that any measure of coherence in the incoherent (incoherence-preserving) approach is also a measure of coherence in the dephasing-covariance approach.

We also compare the translational-covariance approach to coherence with the dephasing-covariance approach

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[4] In entanglement theory, the free operations are the LOCC operations, which do not all admit of a dilation in terms of the free unitaries (because the only unitaries in the LOCC set are tensor products of local unitaries, which do not support any sort of communication, classical or quantum, between the two parties), and yet this is not considered a problem with entanglement theory because there is a natural physical restriction that yields LOCC as the set of free operations, namely, the fact that classical channels between parties are technologically easy to implement, while quantum channels are not.

[5] One can also define a set of operations that plays a role in entanglement theory which is parallel to the role played by *incoherent* operations in the resource theory of coherence, namely, those bipartite operations for which there is a Kraus decomposition where each term is non-entangling.

[6] Ref. [70] came to our attention as we were preparing this manuscript.

(and, via the connection noted above, with the incoherent and incoherence-perserving approaches).

Any given translational symmetry defines a decomposition of the Hilbert space via the joint eigenspaces of the generators of this symmetry. Thus, for any given translational symmetry, one can consider the sets of dephasing-covariant, incoherent, and incoherence-preserving operations defined relative to this decomposition of the Hilbert space. For instance, the incoherence-preserving operations defined by a given translational symmetry is the set of operations under which any state which is incoherent with respect to the eigenspaces of its generators, is mapped to a state which is still incoherent relative to these eigenspaces.

We show that the set of translationally-covariant operations is a strict subset of the dephasing-covariant operations and thus also a strict subset of the incoherent (incoherence-preserving) operations. This implies that any measure of coherence in the dephasing-covariance proposal is also a measure of coherence in the translational-covariance proposal. We also show that this inclusion relation is strict.

Given these inclusion relations, the question arises of whether the measures of coherence that have been identified recently as valid in the BCP approach were already identified in prior work on the resource theory of asymmetry. We show that this is indeed the case for most such measures of coherence.[7] In addition to noting these relations, we discuss two general techniques for deriving measures of coherence, one that infers them from measures of information and the other that appeals to a certain kind of decomposition of operators into so-called *modes of asymmetry.*

F. The choice of preferred subspaces

The notion of the state of a system being *coherent* is only meaningful relative to a choice of decomposition of the Hilbert space of the system into subspaces. The latter must be dictated by physical considerations, which is to say, *operational criteria.* This is because, from a purely mathematical point of view, any state is coherent in some basis and incoherent in another basis. If $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$, then the state $|+\rangle$ is coherent if one judges relative to the $\{|0\rangle, |1\rangle\}$ basis, but by the same token, $|0\rangle$ is deemed coherent if one judges relative to the $\{|+\rangle, |-\rangle\}$ basis. One consequently has no alternative but to appeal to physical considerations in defining the notion of coherence[8].

Furthermore, physical considerations often dictate that the relevant notion of coherence is relative to a decomposition of the Hilbert space into subspaces that are not 1-dimensional. A few examples serve to illustrate this.

In the context of decoherence theory, environmental decoherence does not always pick out 1-dimensional subspaces of the system Hilbert space. The dimensions of the decohering subspaces depend on which degree of freedom of the system couples to the environment and generally this is a degenerate observable. Indeed, this fact, i.e., the existence of *Decoherence Free Subspaces* with dimension larger than one, has been exploited to protect quantum information against decoherence [71, 72].

Another common example is where the notion of coherence that is of interest is coherence relative to the eigenspaces of some particular physical observable, such as the system's Hamiltonian (as happens when the coherence is a resource for building a quantum clock) or the photon number operator in a particular mode (as happens when the coherence is a resource for phase estimation). Even if the notion of coherence of interest is the speakable one, *some* physical degree of freedom must be used to encode the coherence, and practical considerations might dictate the use of a particular physical observable. And in all such cases, there is a priori no reason that the physical observable should be nondegenerate.

A final example is if there are degrees of freedom over which the experimenter has no control. In this case, coherence in that degree of freedom is neither observable nor usable.

Thus, if the physically relevant observable is L, and l labels its eigenspaces while α is a degeneracy index, the state $|l,\alpha_1\rangle + |l,\alpha_2\rangle$ is an *incoherent* state in the resource theory insofar as it has no coherence *between* the eigenspaces of L[9]. Coherence *within* an eigenspace of L might be made to be a resource as well, but this requires the degeneracy to be broken, for instance, by introducing another physical observable which picks out a basis of that eigenspace.

We conclude that any resource theory of coherence should be able to quantify and characterize coherence, not only with respect to 1-dimensional subspaces, but also with respect to subspaces of arbitrary dimension. As we will see in the following, the resource theory of unspeakable coherence based on translationally-covariant operations has this capability. In the case of speakable coherence, we define dephasing-covariant and incoherence-preserving operations to incorporate this possibility, and we generalize the definition of incoherent operations in BCP [20], which assumed 1-dimensional

[7] Note that the present work expands on some of the comparisons between the BCP approach to coherence and the one based on translational asymmetry made in Ref. [21].

[8] The fact that the direct-sum decomposition of Hilbert space that defines the resource of coherence is determined by the physical problem of interest is completely analogous to how the factorization of the Hilbert space of a composite system that defines the resource of entanglement is so determined.

[9] This is analogous to how entanglement *between* laboratories can be a resource while entanglement *within* a given laboratory is not.

subspaces, to do so as well.

G. Composite systems

How should the resource theory of coherence be defined on composite systems? In particular, how should we define the set of free states and free operations in this case? For instance, suppose we are interested in quantifying coherence with respect to the energy eigenbasis, which is relevant, for instance, in the context of thermodynamics and clock synchronization. Consider two non-interacting systems having identical Hamiltonians, H_A and H_B, with energy eigenbases $\{|E\rangle_A\}$ and $\{|E\rangle_B\}$ respectively. One can then imagine two different ways of defining coherence on the composite system AB. Definition (1): coherence is defined relative to the products of eigenspaces of the single system Hamiltonians. In this case, a joint state of systems A and B is coherent if it contains coherence with respect to either of these two Hamiltonians. In particular, in this approach, the state $(|E_1\rangle_A|E_2\rangle_B + |E_2\rangle_A|E_1\rangle_B)/\sqrt{2}$ is considered a resource. Definition (2): coherence is defined relative to the eigenspaces of the total Hamiltonian, that is, of $H_A \otimes I_B + I_A \otimes H_B$, where I_A and I_B are, respectively, the identity operators on systems A and B. In this case, states which do not contain coherence relative to the total Hamiltonian, such as $(|E_1\rangle_A|E_2\rangle_B+|E_2\rangle_A|E_1\rangle_B)/\sqrt{2}$, are deemed to be incoherent.

As the set of preferred subspaces and incoherent states on a single system should be chosen based on physical considerations, the set of incoherent states on composite systems should also be defined in a similar fashion. It turns out that each of the above definitions can be relevant in some physical scenarios. For instance, in the scenarios where the two subsystems cannot exchange energy (for instance, because they are held by two distant parties) then approach (1) is relevant, and entangled states such as $(|E_1\rangle_A|E_2\rangle_B + |E_2\rangle_A|E_1\rangle_B)/\sqrt{2}$ are resources. On the other hand, in the scenarios where we can easily apply operations that allow energy exchange between the two subsystems, then the relevant observable is the total energy, and not the energy of the individual subsystems. Therefore, in this situation approach (2) is the relevant one, and entangled states such as $(|E_1\rangle_A|E_2\rangle_B+|E_2\rangle_A|E_1\rangle_B)/\sqrt{2}$ are not resources. The fact that the resource of coherence is only defined relative to a choice of basis which depends on the physical scenario is precisely analogous to how the resource of entanglement is only defined relative to a choice of factorization of the Hilbert space which depends on the physical scenario. (For instance, in the distant laboratories paradigm, entanglement between laboratories is a resource, while entanglement between systems in the same laboratory is not.)

H. Outline

The article is organized as follows. Sec. II covers preliminary material, including a discussion of certain features that are common to the various different proposals for a resource theory of coherence, what counts as a physical justification of a proposal for the set of free operations, and the definition of a measure of coherence. Sec. III presents the resource theory of unspeakable coherence that one obtains by taking the free operations to be those that are translationally covariant. In particular, various different characterizations and physical justifications of the free operations are provided. Sec. IV presents the proposal for speakable coherence based on dephasing-covariant operations, together with a discussion of the relation to the translationally-covariant operations and physical justifications. In Sec. V, we review the BCP proposal for speakable coherence, which is defined by the incoherent operations, as well as a related proposal, defined by the set of incoherence-preserving operations. The relation to the dephasing-covariance apporach is considered, as well as possibilities for a physical justification. Finally, in Sec. VI, we consider measures of coherence within the various approaches, and in Sec. VII we provide some concluding remarks.

II. PRELIMINARIES

Any resource theory is specified by a set of *free states* and a set of *free operations*. These are states and operations which are *easy* or *allowed* to prepare and implement under a practical or fundamental constraint.

A. Free states

The notion of coherence is only defined relative to a preferred decomposition of the Hilbert space into subspaces. This preferred decomposition is determined based on practical restrictions or physical considerations, although in some cases a preferred decomposition may be considered as a purely mathematical exercise. For a system with Hilbert space $\mathcal{H}$, we denote the preferred subspaces by $\{\mathcal{H}_l\}_l$, so that $\mathcal{H} = \bigoplus_l \mathcal{H}_l$. Here, the index l may be discrete or continuous. We denote the projectors onto these subspaces by $\{\Pi_l\}_l$.

The free states, which are termed *incoherent states*, are those states which are block-diagonal relative to the preferred subspaces,

$$\rho = \sum_l p_l \Pi_l \ . \tag{2.1}$$

An alternative way of characterizing the set of free states is via map that dephases between the preferred subspaces. This dephasing map has the form

$$\mathcal{D}(\cdot) \equiv \sum_l \Pi_l(\cdot)\Pi_l \ . \tag{2.2}$$

As a superoperator acting on the vector space of operators, $\mathcal{D}$ is a projector, and hence idempotent, $\mathcal{D}^2 = \mathcal{D}$. In fact, it projects onto the subspace of operators that are block-diagonal relative to the decomposition $\{\mathcal{H}_l\}_l$, so that the set of incoherent states can be characterized as those that are invariant under $\mathcal{D}$,

$$\mathcal{D}(\rho) = \rho \ . \tag{2.3}$$

Note that for any choice of preferred subspaces, the set of incoherent states is closed under convex combinations. We will denote this set by $\mathcal{I}$.

B. Free measurements

If a system survives a quantum measurement, then the outcome of the measurement provides the ability to predict the outcomes of future measurements on the system. To do so, one must specify the state update map associated to the measurement. The von Neumann projection postulate is an example. This is a specification of the measurement's *predictive* aspect. Whether the system survives or not, every quantum measurement also allows one to make retrodictions about earlier interventions of the system. We here focus only on the retrodictive aspect of a measurement, which in quantum theory is represented by a postive-operator-valued measure (POVM). An element of a POVM, denoted E, satisfies $I \geq E \geq 0$, and wlll be termed an *effect*.

We will call an effect *incoherent* if it is block-diagonal relative to the preferred subspaces. Although one might have expected all proposals for the resource theory of coherence to allow as free only those POVMs made up of incoherent effects, we will see that this is not the case for the proposal based on incoherence-preserving or incoherent operations.

C. Free operations

We turn now to the set of free operations. We consider not only those operations wherein the input and output spaces are the same (i.e., transformations of a system) but also those where they may be different (in which case the operation involves adding or taking away some or all of the system). In particular, the free operations from a trivial input space to a nontrivial output space specify which *preparations* of the output system can be freely implemented, so that a specification of the free operations implies a specification of the free states.

The minimal property that the set of free operations should have is to be *incoherence-preserving*, which is to say that each free operation takes every incoherent state on the input space to an incoherent state on the output space. Note that the incoherence-preserving property implies that for the special case where the operation is a state preparation, the free set corresponds to the incoherent states. [10]

All proposals we consider here are such that every free operation is incoherence-preserving. Nonetheless, different proposals for how to treat coherence as a resource differ in their choice of the set of free operations, subject to this constraint.

Note that we here use the term *quantum operation* to refer to a trace-nonincreasing completely positive linear map. If the operation is trace-preserving, we will refer to it as a *quantum channel*.

D. Physical justification of the free operations through dilation

It is widely believed that physical systems undergoing closed-system dynamics evolve according to a unitary map. In this view, the only circumstance in which a nonunitary map is used to describe the evolution of a system's state is when the system is known to undergo open-system dynamics, that is, when it interacts with some auxiliary system (perhaps its environment) via a unitary map, but one chooses to not describe the auxiliary system, by marginalizing or post-selecting on it. It is straigthforward to show that in any situation wherein a system interacts unitarily with the auxiliary system and one subsequently marginalizes or post-selects on the latter (through a partial trace or a partial trace with some measurement effect respectively), the effective evolution of the system's state is always described by a completely positive trace-nonincreasing map (what we are here calling a quantum operation). The Stinespring dilation theorem [73] guarantees that the vice-versa is also true: *every* quantum operation on the system can be achieved in this fashion.

For a given triple of state of the auxiliary system, effect measured thereon, and unitary coupling of the system to its auxiliary, we will term the effective quantum operation on the system that it defines the *marginal operation* on the system. For a given quantum operation on the system, we will term any triple of state of the auxiliary, effect on the auxiliary and unitary coupling of system to auxiliary that yields the operation as a marginal, a *dilation* of that operation.

In the context of resource theories, one way to define the free set of operations on a system is by specifying the free states and free effects on the auxiliary system, as well as the free unitaries that couple the system to the

[10] This is because we can think of the input in a state preparation as a 1-dimensional system, which is necessarily in an incoherent state.

auxiliary, and then defining the free set of operations on the system as all and only those that can be obtained as a marginal of these. If a proposal for the free set of operations is *not* defined in this way, then one *can and should* ask whether it admits of such a definition or not. In other words, one should seek to determine whether the free set of operations in a given proposal can be understood as those that admit of a dilation in terms of the free states and effects on, and free unitary couplings with, the auxilary system. We refer to such dilations as *free dilations*.

We shall here ask this question of various proposals for how to choose the free set of operations in a resource theory of coherence. If, for any given proposal, one finds that the free set of operations on the system of interest includes an operation that does *not* admit of a free dilation, then this may imply that some nontrivial resource on the composite of system and auxiliary must be consumed in order to realize the operation.

We will show that in cases where one considers a translation group that acts collectively on all physical systems, the translationally-covariant operations have free dilation. We will also show that the set of incoherence-preserving operations and the set of incoherent operations does not have this property, at least not if we treat all systems even-handedly. For the set of dephasing covariant operations, the question remains open.

E. Measures of coherence

In any resource theory, a measure of the resource is a function from states to real numbers which defines a partial order on the set of states. The essential property that any such function must have is to be a monotone (i.e, to be monotonically nonincreasing) under the free operations[11]. We are therefore going to use the following definition of a measure of coherence:

Definition 1 *A function f from states to real numbers is a measure of coherence according to a given proposal if (i) For any trace-preserving quantum operation $\mathcal{E}$ which is free according to the proposal, it holds that $f(\mathcal{E}(\cdot)) \leq f(\cdot)$. (ii) For any incoherent state $\rho \in \mathcal{I}$, it holds that $f(\rho) = 0$.*

Because any incoherent state can be mapped to any other incoherent state via a free trace-preserving operation, condition (i) implies that the value of the function f must be the same for all incoherent states. Condition (ii) merely expresses a choice of convention for this value: that all incoherent states should be assigned measure zero. Of course, given any function satisfying condition (i), one can define a shift of this function which satisfies condition (ii).

It is worth noting that any measure of a resource f is constant on states that are connected by a free unitary operation. That is, if $\mathcal{U}$ is a free unitary operation, then any resource measure f must satisfy

$$f(\rho) = f(\mathcal{U}(\rho)).$$

The proof is simply that if ρ and σ are connected by a free unitary, then state conversion in both directions are possible under the free operations, $\rho \to \sigma$ and $\sigma \to \rho$, which in turn implies that $f(\rho) \geq f(\sigma)$ and $f(\rho) \leq f(\sigma)$.

We distinguish the three resource-theoretic approaches to coherence that we consider in this article by the set of free operations that define them: translationally-covariant, dephasing-covariant and incoherence-preserving operations. A measure of coherence within a given approach is also defined relative to the set of free operations within that approach. Therefore, we refer to measures of coherence within the different approaches as measures of *TC-coherence*, *DC-coherence*, and *IP-coherence* respectively. In Sec. VI, we provide a list of examples for each type.

III. COHERENCE VIA TRANSLATIONALLY-COVARIANT OPERATIONS

We begin by demonstrating that if one is interested in an *unspeakable* notion of coherence, then coherence can be understood as asymmetry relative to a symmetry group of translations. In this approach, the coherence is defined based on a given observable L, such as the Hamiltonian, the linear momentum, or the angular momentum. Then, to characterize coherence relative to the eigenbasis of L, we consider the asymmetry relative to the group of translations generated by L, i.e., the group of unitaries

$$U_{L,x} \equiv e^{-ixL} : x \in \mathbb{R} \ . \tag{3.1}$$

The superoperator representation of the translation $x \in \mathbb{R}$ is then

$$\begin{aligned} \mathcal{U}_{L,x}(\cdot) \ &\equiv \ U_{L,x}(\cdot)U_{L,x}^{\dagger} \\ &= \ e^{-ixL}(\cdot)e^{ixL}. \end{aligned} \tag{3.2}$$

Note that this group has often a natural physical interpretation. For instance, if L is the Hamiltonian, then it generates the group of time translations, and if L is the component of angular momentum in some direction, then it generates the group of rotations about this direction.[12]

[11] See Ref. [4] for an operational justification of monotonicity, that is, an account of why monotonicity is required if a measure of a resource is to characterize the degree of success achievable in some operational task.

[12] Note that, depending on the spectrum of L, this group could be isomorphic to U(1) or to $\mathbb{R}$ (under addition).

The free states are taken to be those that are *translationally-invariant* or *translationally-symmetric*,

$$\forall x \in \mathbb{R}: \; \mathcal{U}_{L,x}(\rho) = \rho. \tag{3.3}$$

One can easily see that the set of translationally-invariant states coincides with the set of states that are incoherent with respect to the eigenspaces of L, i.e.,

$$\forall x \in \mathbb{R}: \mathcal{U}_{L,x}(\rho) = \rho \iff [L, \rho] = 0. \tag{3.4}$$

Therefore, in the translational-covariance approach to coherence the preferred subspaces relative to which coherence is a resource are the eigenspaces of the generator L.

A. Free operations as translationally-covariant operations

For a given choice of symmetry transformations, the resource theory of asymmetry is defined by taking the set of free operations to be those that are covariant relative to the symmetry transformations. We here particularize this definition to the case of a translational symmetry, and provide several ways of characterizing this set.

1. Definition of translationally-covariant operations

Definition 2 *We say that a quantum operation $\mathcal{E}$ is* translationally-covariant relative to the translational symmetry generated by L *if*

$$\forall x \in \mathbb{R}: \; \mathcal{U}_{L,x} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{U}_{L,x}. \tag{3.5}$$

Note that condition (3.5) is equivalent to

$$\forall x \in \mathbb{R}: \; \mathcal{U}_{L,x} \circ \mathcal{E} \circ \mathcal{U}^{\dagger}_{L,x} = \mathcal{E}. \tag{3.6}$$

If the input and output spaces of the map $\mathcal{E}$ are distinct, then the generator L may be different on the input and output spaces. For instance, in the case where L corresponds to the angular momentum in a certain direction, then this observable may have different representations on the input and output spaces. For simplicity, we do not indicate such differences in our notation. A preparation of the state ρ is an operation with a trivial input space and translational-covariance in this case implies translational-invariance of ρ, confirming that translationally-invariant states are the free states in this approach.

Sec. II C articulated a minimal constraint on the free operations, that they should be incoherence-preserving. Translationally-covariant operations have this property because from Eq. (3.3) one can deduce that for a translationally-covariant operation $\mathcal{E}$, and an incoherent state ρ for input,

$$\forall x \in \mathbb{R}: \; \mathcal{U}_{L,x}(\mathcal{E}(\rho)) = \mathcal{E}(\mathcal{U}_{L,x}(\rho)) = \mathcal{E}(\rho), \tag{3.7}$$

which implies that $\mathcal{E}(\rho)$ is translationally-invariant, hence incoherent. Therefore incoherent states are mapped to incoherent states.

If one thinks of incoherence as translational symmetry, then the incoherence-preserving property formalizes the simple intuition known as *Curie's principle*: If the initial state does not break the translational symmetry and the evolution does not break the translational symmetry either, then the final state cannot break the translational symmetry.

2. Translationally-covariant measurements

If an operation $\mathcal{E}$ has a trivial *output* space, so that it corresponds to tracing with a measurement effect E on the input space, that is, $\mathcal{E}(\cdot) = \mathrm{Tr}(E\cdot)$, then Eq. (3.5) reduces to

$$\begin{aligned}\forall x \in \mathbb{R}: \; \mathrm{Tr}(E(\cdot)) &= \mathrm{Tr}(E\, \mathcal{U}_{L,x}(\cdot)) \\ &= \mathrm{Tr}(\mathcal{U}^{\dagger}_{L,x}(E)\, (\cdot)),\end{aligned} \tag{3.8}$$

which in turn implies

$$\forall x \in \mathbb{R}: \; \mathcal{U}^{\dagger}_{L,x}(E) = E, \tag{3.9}$$

i.e., the effect E is translationally-invariant. This condition is equivalent to $[E, L] = 0$, so that the effect E is incoherent with respect to the eigenspaces of L.

Proposition 3 *A POVM is translationally-covariant if and only if all of its effects are incoherent relative to the eigenspaces of L.*

3. Translationally-covariant unitary operations

Finally, if $\mathcal{V}$ is a unitary translationally-covariant operation, that is, $\mathcal{V}(\cdot) = V(\cdot)V^{\dagger}$ for some unitary operator V (in which case the input and output spaces are necessarily the same), then Eq. (3.5) reduces to

$$\forall x \in \mathbb{R}: U_{L,x} V(\cdot) V^{\dagger} U^{\dagger}_{L,x} = V U_{L,x}(\cdot) U^{\dagger}_{L,x} V^{\dagger}, \tag{3.10}$$

which implies

$$\forall x \in \mathbb{R}: \; U_{L,x} V = e^{i\phi(x)} V U_{L,x}, \tag{3.11}$$

for some phase $\phi(x)$. Taking the traces of both sides, we find that in finite-dimensional Hilbert spaces, this condition can hold if and only if $e^{i\phi(x)} = 1$, that is, if and only if

$$\forall x \in \mathbb{R}: \; [U_{L,x}, V] = 0, \tag{3.12}$$

which is equivalent to $[V, L] = 0$, so that the unitary operator V is also block-diagonal with respect to the eigenspaces of L. If $\{\mathcal{H}_\lambda\}_\lambda$ denotes the set of eigenspaces of L, $\{\Pi_\lambda\}_\lambda$ the projectors onto these, and $\{V_\lambda\}_\lambda$ an arbitrary set of unitaries *within* each such subspace, then any such unitary V can be written as

$$V = \sum_\lambda V_\lambda \Pi_\lambda \ . \tag{3.13}$$

If the Hilbert space is infinite-dimensional, on the other hand, then the characterization above need not apply. Indeed, in this case, there are translationally-covariant unitaries that need not map every eigenspace of L to itself. For instance, if the generator is a charge operator with integer eigenvalues, $Q = \sum_{q=-\infty}^{\infty} q|q\rangle\langle q|$, then the unitary $V_{\Delta q}$ that applies a rigid shift of the charge by an integer Δq, that is,

$$V_{\Delta q} \equiv \sum_{q=-\infty}^{\infty} |q+\Delta q\rangle\langle q|,$$

defines a unitary operation $\mathcal{V}_{\Delta q}(\cdot) = V_{\Delta q}(\cdot)V_{\Delta q}^\dagger$ that is covariant relative to the group of shifts of the phase conjugate to charge, $\{\mathcal{U}_{Q,x} : x \in \mathbb{R}\}$ where $\mathcal{U}_{Q,x}(\cdot) = e^{-ixQ}(\cdot)e^{ixQ}$. As another example, if the system is a particle in one dimension, then the unitary operation that boosts the momentum by Δp is translationally-covariant relative to the group of spatial translations. This is because the unitary operation associated with a boost by Δp, $e^{-i\Delta pX}(\cdot)e^{i\Delta pX}$ where X is the position operator, and the unitary operation associated with a translation by Δx, $e^{-i\Delta xP}(\cdot)e^{i\Delta xP}$ where P is the momentum operator, commute with one another for all $\Delta x, \Delta p \in \mathbb{R}$.

4. Characterization via Stinespring dilation

We show that every translationally-covariant operation on a system can arise by coupling the system to an ancilla in an incoherent (translationally-invariant) state, subjecting the composite to a translationally-covariant unitary, and post-selecting on the outcome of a measurement on the ancilla which is assoicated to an incoherent (translationally-invariant) effect. Such an implementation is termed a translationally-covariant *dilation* of the operation.

To make sense of the notion of a translationally-covariant dilation, however, one needs to specify not only the representation of the translation group on the system and ancilla individually, but on the composite of system and ancilla as well. Recall that we allow the operation $\mathcal{E}$ to have different input and output spaces, so that to make sense of a translationally-covariant dilation, we must also specify the representation of the translation group on the output versions of the system and ancilla.

Some notation is helful here. We denote the Hilbert spaces corresponding to the input and output of the map $\mathcal{E}$ by $\mathcal{H}_{\rm s}$ and $\mathcal{H}_{\rm s'}$ respectively. Denoting the Hilbert space of the ancilla by $\mathcal{H}_{\rm a}$, the composite Hilbert space of system and ancilla is $\mathcal{H}_{\rm s} \otimes \mathcal{H}_{\rm a}$. We denote the subsystem that is complementary to $\mathcal{H}_{\rm s'}$ by $\mathcal{H}_{\rm a'}$ (this is the subsystem over which one traces), so that $\mathcal{H}_{\rm s} \otimes \mathcal{H}_{\rm a} = \mathcal{H}_{\rm s'} \otimes \mathcal{H}_{\rm a'}$.

In the physical situations to which TC-coherence applies—which we will discuss at length in Sec. III B—one can always choose an ancilla system such that translation is represented *collectively* on the composite of system and ancilla. Specifically, if $L_{\rm s}$ is the generator of translations on $\mathcal{H}_s$ and $L_{\rm a}$ is the generator of translations on $\mathcal{H}_a$, then the generator of translations on the composite $\mathcal{H}_{\rm s} \otimes \mathcal{H}_{\rm a}$ is $L = L_{\rm s} \otimes I_{\rm a} + I_{\rm s} \otimes L_{\rm a}$. Similarly, we have $L = L_{\rm s'} \otimes I_{\rm a'} + I_{\rm s'} \otimes L_{\rm a'}$. It follows that the translation operation on the composite is collective on the factorization $\mathcal{H}_{\rm s} \otimes \mathcal{H}_{\rm a}$, that is, $\mathcal{U}_{L,x} = \mathcal{U}_{L_s,x} \otimes \mathcal{U}_{L_a,x}$ and on the factorization $\mathcal{H}_{\rm s'} \otimes \mathcal{H}_{\rm a'}$, that is, $\mathcal{U}_{L,x} = \mathcal{U}_{L_{s'},x} \otimes \mathcal{U}_{L_{a'},x}$. In the discussion below, we use L to denote the generator of translations, regardless of the system it is acting upon.

Proposition 4 *A quantum operation $\mathcal{E}$ is* translationally-covariant *if and only if it can be implemented by coupling the system $\mathcal{H}_{\rm s}$ to an ancilla $\mathcal{H}_{\rm a}$ prepared in an incoherent state σ via a translationally-covariant unitary quantum operation $\mathcal{V}$, and then post-selecting on the outcome of a measurement on the ancilla $\mathcal{H}_{\rm a'}$ which is associated with an incoherent effect E. Formally, the condition is that for all quantum states ρ,*

$$\mathcal{E}(\rho) = \mathrm{tr}_{\rm a'}(E\mathcal{V}(\rho\otimes\sigma)), \tag{3.14}$$

where $[L_{\rm a}, \sigma] = 0$ and $[L_{\rm a'}, E] = 0$ and where $\mathcal{V} \circ \mathcal{U}_{L,x} = \mathcal{U}_{L,x} \circ \mathcal{V}$ for all $x \in \mathbb{R}$.

Proof. The proof that any operation of the form of Eq. (3.14) is translationally-covariant is as follows:

$$\mathcal{E}\big(\mathcal{U}_{L_{\rm s},x}(\rho)\big) \tag{3.15a}$$
$$= \mathrm{tr}_{\rm a'}\Big(E\mathcal{V}\Big(\mathcal{U}_{L_{\rm s},x}(\rho)\otimes\sigma\Big)\Big) \tag{3.15b}$$
$$= \mathrm{tr}_{\rm a'}\Big(E\ \mathcal{V}\Big(\mathcal{U}_{L_{\rm s},x}(\rho)\otimes\mathcal{U}_{L_{\rm a},x}(\sigma)\Big)\Big) \tag{3.15c}$$
$$= \mathrm{tr}_{\rm a'}\Big(E\ \mathcal{U}_{L_{\rm s'},x}\otimes\mathcal{U}_{L_{\rm a'},x}\Big(\mathcal{V}[\rho\otimes\sigma]\Big)\Big) \tag{3.15d}$$
$$= \mathcal{U}_{L_{\rm s'},x}\Big(\mathrm{tr}_{\rm a'}\Big(\mathcal{U}^\dagger_{L_{\rm a'},x}(E)\ \Big(\mathcal{V}[\rho\otimes\sigma]\Big)\Big)\Big) \tag{3.15e}$$
$$= \mathcal{U}_{L_{\rm s'},x}\Big(\mathrm{tr}_{\rm a'}\big(E\ \mathcal{V}[\rho\otimes\sigma]\big)\Big) \tag{3.15f}$$
$$= \mathcal{U}_{L_{\rm s'},x}(\mathcal{E}(\rho)), \tag{3.15g}$$

where in the second equality, we have used that fact that σ is an incoherent state; in the third equality, we have used the fact that $\mathcal{V}$ is a translationally-covariant operation; in the fifth equality, we have used the fact that E is an incoherent effect; and in the last equality, we have used Eq. (3.14).

For the converse implication, we refer the reader to the result on the form of the Stinespring dilation for group-covariant quantum operations by Keyl and Werner [74]. ■

5. Characterization via modes of translational asymmetry

We begin by introducing some technical machinery.

Denote the preferred subspaces of $\mathcal{H}$ relative to which coherence is evaluated, that is, the eigenspaces of L, by $\{\mathcal{H}_{\lambda_n}\}$, where $\{\lambda_n\}_n$ is the set of eigenvalues of L (these may be discrete or continuous). Let the set of *modes* Ω be the set of the gaps between all eigenvalues, i.e. $\{\lambda_n - \lambda_m\}_{n,m}$. In the case where L is the system Hamiltonian, each element of Ω can be interpreted as a *frequency* of the system.

Elements of the set Ω label different modes in the system. For any $\omega \in \Omega$, define the superoperator

$$\mathcal{P}^{(\omega)} = \lim_{x_0\to\infty} \frac{1}{2x_0} \int_{-x_0}^{x_0} \mathrm{d}x \; e^{-i\omega x}\, \mathcal{U}_{L,x} \ , \tag{3.16}$$

where $\mathcal{U}_{L,x}(\cdot) = U_{L,x}(\cdot)U_{L,x}^{\dagger} = e^{-ixL}(\cdot)e^{ixL}$. This superoperator is the projector that erases all the terms in the input operator except those which connect eigenstates of L whose eigenvalues are different by ω, i.e., all except those which are of the form $|\lambda_n,\alpha\rangle\langle\lambda_n+\omega,\beta|$, where $|\lambda_n,\alpha\rangle$ and $|\lambda_n+\omega,\beta\rangle$ are eigenstates of L with eigenvalues λ_n and $\lambda_n+\omega$ respectively. One can easily show that

$$\sum_{\omega\in\Omega} \mathcal{P}^{(\omega)} = \mathcal{I}_{\mathrm{id}} \tag{3.17a}$$

$$\mathcal{P}^{(\omega)} \circ \mathcal{P}^{(\omega')} = \delta_{\omega,\omega'}\mathcal{P}^{(\omega)} \tag{3.17b}$$

$$\mathcal{P}^{(0)} = \mathcal{D} \tag{3.17c}$$

$$\mathcal{U}_{L,x} \circ \mathcal{P}^{(\omega)} = e^{i\omega x}\mathcal{P}^{(\omega)} \ , \tag{3.17d}$$

where $\mathcal{I}_{\mathrm{id}}$ is the identity superoperator, and $\delta_{\omega,\omega'}$ is the Kronecher delta.

The set of superoperators $\{\mathcal{P}^{(\omega)} : \omega \in \Omega\}$ are a complete set of projectors to different subspaces of the operator space $\mathcal{B}$. It can be easily shown that these subspaces are orthogonal according to the Hilbert-Schmidt inner product, defined by $(X,Y) \equiv \mathrm{Tr}(X^{\dagger}Y)$ for arbitrary pair of operators $X, Y \in \mathcal{B}$. Therefore, the operator space $\mathcal{B}$ can be decomposed into a direct sum of operator subspaces, $\mathcal{B} = \bigoplus_{\omega\in\Omega} \mathcal{B}^{(\omega)}$, where each $\mathcal{B}^{(\omega)}$ is the image of $\mathcal{P}^{(\omega)}$.

Note that any operator in the operator subspace $\mathcal{B}^{(\omega)}$ transforms distinctively under translations,

$$A \in \mathcal{B}^{(\omega)} \Longrightarrow \mathcal{U}_{L,x}(A) = e^{i\omega x}A. \tag{3.18}$$

We refer to $\mathcal{B}^{(\omega)}$ as the "mode ω" operator subspace. For any operator A, the component of that operator in the operator subspace $\mathcal{B}^{(\omega)}$, denoted

$$A^{(\omega)} \equiv \mathcal{P}^{(\omega)}(A),$$

is termed the "mode ω component of A".

Clearly, every incoherent (i.e. translationally symmetric) state lies entirely within the mode 0 operator subspace, while a coherent (i.e. translationally asymmetric) state has a component in at least one mode ω operator subspace with $\omega \neq 0$.

Operator subspaces associated with distinct ω values have been called "modes of asymmetry" in Ref. [8], where the decomposition of states, operations and measurements into their different modes was shown to constitute a powerful tool in the resource theory of asymmetry.

Example 5 *Consider the special case where $J_{\hat{z}}$ is the angular operator in the $\hat{z}$ direction. For simplicity, assume that $J_{\hat{z}}$ is non-degenerate and let $\{|m\rangle\}_m$ be its orthonormal eigenbasis, where $|m\rangle$ is the eigenstate of $J_{\hat{z}}$ with eigenvalue m. Since the eigenvalues of the angular momentum operator are all separated by integers, it follows the set of modes Ω is a subset of the integers, $\Omega \subseteq \mathbb{Z}$. Then, for each integer $k \in \Omega$ we have*

$$\mathcal{B}^{(k)} \equiv \mathrm{span}\{|m\rangle\langle m+k|\}_m. \tag{3.19}$$

Furthermore, the mode k component of any operator A is given by

$$A^{(k)} = \sum_m |m\rangle\langle m+k| \ \langle m|A|m+k\rangle \ . \tag{3.20}$$

The mode k of the density operator ρ corresponds to coherence of order k *in the context of magnetic resonance techniques [47].*

With these notions in hand, we can provide the mode-based characterization of the translationally-covariant operations.

Proposition 6 *A quantum operation $\mathcal{E}$ is translationally covariant relative to the generator L if and only if it preserves the modes of asymmetry associated to L, that is, if and only if the mode $\omega \in \Omega$ component of the input state is mapped to the mode $\omega \in \Omega$ component of the output state. Formally, the condition is that whenever $\mathcal{E}(\rho) = \sigma$, we have $\mathcal{E}(\rho^{(\omega)}) = \sigma^{(\omega)}$ where $\rho^{(\omega)} \equiv \mathcal{P}^{(\omega)}(\rho)$ and $\sigma^{(\omega)} \equiv \mathcal{P}^{(\omega)}(\sigma)$. Note that σ is only a normalized state if $\mathcal{E}$ is a channel (i.e. trace-preserving) and is otherwise subnormalized.*

The proof follows immediately from properties listed in Eq.(3.17) (See [8] for further discussion).

6. Characterization via Kraus decomposition

Proposition 7 *A quantum operation $\mathcal{E}$ is translationally covariant if and only if it admits of a Kraus decomposition of the form*

$$\mathcal{E}(\cdot) = \sum_{\omega,\alpha} K_{\omega,\alpha}(\cdot)K_{\omega,\alpha}^{\dagger}, \tag{3.21}$$

where the elements of the set $\{K_{\omega,\alpha}\}_\alpha$ are all mode ω operators.

To see that any quantum operation with such a Kraus decomposition is translationally covariant, we note that

$$\mathcal{U}_{L,x} \circ \mathcal{E} \circ \mathcal{U}_{L,x}^{\dagger} = \sum_{\omega \in \Omega, \alpha} \mathcal{U}_{L,x}(K_{\omega,\alpha})(\cdot)(\mathcal{U}_{L,x}(K_{\omega,\alpha}))^{\dagger} \tag{3.22}$$

and then use the fact that $K_{\omega,\alpha} \in \mathcal{B}^{(\omega)}$ and Eq. (3.18) to infer that $\mathcal{U}_{L,x}(K_{\omega,\alpha}) = e^{i\omega x} K_{\omega,\alpha}$, which in turn implies

$$\forall x \in \mathbb{R} : \mathcal{U}_{L,x} \circ \mathcal{E} \circ \mathcal{U}_{L,x}^{\dagger} = \sum_{\omega,\alpha} K_{\omega,\alpha}(\cdot) K_{\omega,\alpha}^{\dagger} = \mathcal{E}, \tag{3.23}$$

which, from Eq. (3.6), simply asserts the translational covariance of $\mathcal{E}$.

The proof that every translationally-covariant operation has a Kraus decomposition of the form specified can be inferred from a result in Ref. [4] which characterizes the Kraus decomposition of any group-covariant operation, by specializing the result to the case of a translation group. Alternatively, it can be inferred from the Stinespring dilation, Proposition 4, using a slight generalization (from channels to all operations) of the argument provided in Appendix A.1 of Ref. [21].

Proposition 7 also implies:

Corollary 8 *A quantum operation $\mathcal{E}$ is translationally-covariant if and only if it admits of a Kraus decomposition every term of which is translationally-covariant.*

It suffices to note that each term of the Kraus decomposition specified in proposition 7 is translationally-covariant. It follows that if the different terms in this Kraus decomposition correspond to the different outcomes of a measurement, then even one who post-selected on a particular outcome would describe the resulting operation as translationally-covariant.

B. Physical justifications for the restriction to translationally-covariant operations

As noted in the introduction, it is critical that any definition of the restricted set of operations in a resource theory must be justifiable operationally. In this section, we discuss different physical scenarios in which the set of translationally-covariant operations are naturally distinguished as the set of easy or freely-available operations.

1. *Fundamental or effective symmetries of Hamiltonians*

If, for a set of systems, the Hamiltonians one can access are symmetric and the states and measurements that one can implement are also symmetric, then for any given system only symmetric operations are possible[13]. Such symmetry constraints can sometimes be understood to be consequences of fundamental or effective symmetries in the problem.

A constraint of translational symmetry on the Hamiltonian is fundamental if it arises from a fundamental symmetry of nature, such as a symmetry of space-time. It is effective if it arises from practical constraints, for instance, if one is interested in time scales or energy scales for which a symmetry-breaking term in the Hamiltonian becomes negligible. A translational symmetry constraint on the states and measurements can sometimes arise as a consequence of this symmetry of the Hamiltonian. For instance, if the only states that one can freely prepare are those that are thermal, then given that thermal states depend only on the Hamiltonian, any fundamental or effective symmetry of the Hamiltonian is inherited by the thermal states.

2. *Lack of shared reference frames*

The most natural experimental restriction that leads to translationally-covariant operations is when one lacks access to any reference frame relative to which the translations can be defined. Such a lack of access can arise in a few ways.

For a pair of separated parties, each party may have a local reference frame, but no information about the relation between the two reference frames. For instance, a pair of parties may each have access to a Caretesian reference frame (or clock, or phase reference), but not know what rotation (or time-translation or phase-shift) relates one to the other.

Under this kind of restriction, each party essentially lacks access to the reference frame of the other. It has been shown that this lack of a shared reference frame implies that the only operations that one party can implement, relative to the reference frame of the other, are those that are group-covariant (see Refs. [6, 29]). For instance, if two parties lack of a shared phase reference, then the only operations whose descriptions they can agree on are phase-covariant operations.

It is also possible that the reference frame that one requires cannot even be prepared locally, due to technological limitations. For instance, only after the experimental realization of Bose-Einstein condensation in atomic systems [75, 76], was it possible to prepare a system that could serve as a reference frame for the phase conjugate to atom number.

3. *Metrology and phase estimation*

Unspeakable coherence is the main resource for quantum metrology, and in particular phase estimation. In this context, a state is a resource to the extent that it allows one to estimate an unknown translation applied to the state (such as a phase-shift, a rotation, or evolution

[13] This is the easy half of the dilation theorem in Prop. 3.14.

for some time interval). Suppose one prepares a system in the state ρ prior to it being subjected to a unitary translation $\mathcal{U}_{L,x}$, where x is unknown. In this case, one knows that the state after the translation is an element of the ensemble $\{\mathcal{U}_{L,x}(\rho)\}_x$ and the task is to estimate x. Clearly, if ρ is invariant under translations (i.e., incoherent), then it is useless for the estimation task. In this sense, translationally asymmetry, and hence coherence relative to the eigenspaces of the generator of translations, is a necessary resource for metrology.

Furthermore, as we show in the following, in this context, the set of translationally-covariant operations has also a simple and natural interpretation. Suppose one is interested in determining which of two states, ρ and σ, is the better resource for the task of estimating an unknown translation. To do so, one must determine which of the two encodings, $x \to \mathcal{U}_{L,x}(\rho)$ and $x \to \mathcal{U}_{L,x}(\sigma)$, carries more information about x. But suppose there exists a quantum operation $\mathcal{E}$ such that for all x, it transforms $\mathcal{U}_{L,x}(\rho)$ to $\mathcal{U}_{L,x}(\sigma)$, i.e.,

$$\forall x : \quad \mathcal{E}(\mathcal{U}_{L,x}(\rho)) = \mathcal{U}_{L,x}(\sigma) \ . \tag{3.24}$$

Here, the quantum operation $\mathcal{E}$ can be thought as an information processing which we perform on the state before performing the measurement which yields the value of x. If such a quantum operation exists, then we can be sure that the state ρ is more useful than σ for this metrological task. Because any information that we can obtain using the state σ, we can also obtain if we use the state ρ.

It turns out that any such information processing $\mathcal{E}$ can be chosen to be translationally covariant with respect to translation $U_{L,x}$, i.e.

Proposition 9 *For any given pair of states ρ and σ the following statements are equivalent:*
(i) There exists a translationally-covariant quantum operation $\mathcal{E}$ such that $\mathcal{E}(\rho) = \sigma$.
(ii) There exists a quantum operation $\mathcal{E}$ such that $\mathcal{E}\left(\mathcal{U}_{L,x}(\rho)\right) = \mathcal{U}_{L,x}(\sigma)$, for all $x \in \mathbb{R}$.

This is the specialization to the case of a translational symmetry group of a similar proposition for an arbitrary symmetry group the proof of which is presented in Ref. [6], where we have also presented a version of this duality for pure states and unitaries, its interpretation in terms of reference frames, and some of its applications.

Statement (ii) in proposition 9 concerns the relative quality of ρ and σ as resources for metrology, while statement (i) concerns the relative quality of ρ and σ within the resource theory defined by the restriction to translationally-covariant operations. The partial order of quantum states under translationally-covariant operations, therefore, determines their relative worth as resources for metrology. Note that in this context, translationally-covariant operations are *all and only* the operations that are relevant.

It follows from proposition 9 that any function which quantifies the performance of states in this metrological task should be a measure of unspeakable coherence.

4. Thermodynamics

The resource theory of *athermality* seeks to understand states deviating from thermal equilibrium as a resource [14–19, 32, 33] . The free operations defining the theory, termed *thermal operations* are all and only those that can be achieved using thermal states, unitaries that commute with the free Hamiltonian, and the partial trace operation. (The restriction on unitaries is motivated by the fact that were one to allow more general unitaries, one could increase the energy of a system, thereby allowing thermodynamic work to be done for free.)

Noting that: (i) if a unitary commutes with the free Hamiltonian, then it is covariant under time-translations, and (ii) because thermal states are defined in terms of the free Hamiltonian, they are symmetric under time-translations, it follows from the dilation theorem for translationally-covariant operations (Prop. 4) that the restriction to thermal operations implies a restriction to time-translation-covariant operations.

5. Control theory

Suppose we are trying to prepare a quantum system in a desired state by applying a sequence of control pulses to the system. Then, there is an important distinction between the pulses which commute with the system Hamiltonian H, and hence are invariant under time translations, and those which are not. Namely, to apply the pulses which do not commute with the system Hamiltonian, we need be careful about the timing of the pulses, and also the duration that the pulse is acting on the system.

To see this, first assume that the pulses are applied instantaneously, i.e., the width of the pulse is sufficiently small that the intrinsic evolution of the system generated by the Hamiltonian H during the pulse is negligible. Then, if instead of applying the control unitary V at the exact time t, we apply it at time $t+\Delta t$, the effect of applying this pulse would be equivalent to applying the pulse $e^{iH\Delta t}Ve^{-iH\Delta t}$ at time t, instead of the desired pulse V. If V does not commute with the Hamiltonian H, then in general V and $e^{iH\Delta t}Ve^{-iH\Delta t}$ are different unitaries, and so the final state is different from the desired state.

Furthermore, if the control pulse V commutes with the system Hamiltonian H, then dealing with the nonzero width of the pulse is much easier and we do not need to be worried about the intrinsic evolution of the system during the pulse, as we now demonstrate. In general, to apply a control unitary V we need to apply a control field to the system. The effect of this control field can be described by a term $H_{\text{cont}}(t)$ which is added to the system Hamiltonian H. Then, to implement a control unitary V which does not commute with the Hamiltonian H, we need to apply a control field $H_{\text{cont}}(t)$ which does not commute with the Hamiltonian H. In this case, the width of the control pulse, i.e., the duration over which we apply the control field $H_{\rm cont}(t)$, becomes an important parameter. In practice, in many situations we need to choose the control field $H_{\rm cont}(t)$ to be strong enough so that the evolution of the system during the pulse width is negligible. On the other hand, if the control field $H_{\rm cont}(t)$ commutes with the system Hamiltonian, then the effect of finite width can be easily taken into account, and so we do not need to apply strong fields to the system.

It follows that in this context, operations which are covariant under time translations are easy to implement, because for this type of operation, there is no sensistivity to the exact timing and the width of the control pulses. So it is natural to consider the operations that are covariant under time-translations as the set of freely-implementable operations, and this again leads us to treat coherence as translational asymmetry.

C. Covariance with respect to independent translations

It can happen that the set of all systems is partitioned into subsets and that the action of the translation group is only collective for those systems within a given subset, while it is independent for different subsets. Suppose that the subsets are labelled by α and that for the set of systems of type α, the generator of collective translations on these systems is denoted $L^{(\alpha)}$. Consider the group element consisting of a translation by $x_\alpha \in \mathbb{R}$ for all the systems of type α. We label this group element by the independent translation parameters $(x_1, x_2, \ldots, x_A) \in \mathbb{R}^A$ where A denotes the number of different types of system. The unitary representation of this group element is

$$U_{\{L^{(\alpha)},x_\alpha\}} \equiv \bigotimes_\alpha e^{-ix_\alpha L^{(\alpha)}}. \tag{3.25}$$

The superoperator representation of this group element is then

$$\mathcal{U}_{\{L^{(\alpha)},x_\alpha\}}(\cdot) \;\equiv\; U_{\{L^{(\alpha)},x_\alpha\}}(\cdot)U^\dagger_{\{L^{(\alpha)},x_\alpha\}} \tag{3.26}$$

In this case, the set of free states are translationally-invariant relative to translations generated by the set of generators $\{L^{(\alpha)}\}$. These are the states that are block-diagonal relative to the distinct joint eigenspaces of $\{L^{(\alpha)}\}$.

The free operations are those that are translationally-covariant relative to the set of generators $\{L^{(\alpha)}\}$, that is, $\forall (x_1, x_2, \ldots, x_A) \in \mathbb{R}^A$,

$$\mathcal{U}_{\{L^{(\alpha)},x_\alpha\}} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{U}_{\{L^{(\alpha)},x_\alpha\}}. \tag{3.27}$$

Indeed, all of the results expressed in this section can be generalized by substituting the translations $x \in \mathbb{R}$ with $(x_1, x_2, \ldots, x_A) \in \mathbb{R}^A$, the superoperator $\mathcal{U}_{L,x}$ with $\mathcal{U}_{\{L^{(\alpha)},x_\alpha\}}$, and the eigenspaces of L with the joint eigenspaces of $\{L^{(\alpha)}\}$.

IV. COHERENCE VIA DEPHASING-COVARIANT OPERATIONS

Much recent work seeking to quantify coherence as a resource has considered *speakable* coherence. The article of Baumgratz, Cramer and Plenio [20] (BCP) provides one such proposal, which has been taken up by most other authors who have sought to characterize coherence as a resource. Nonetheless, we postpone our discussion of the BCP proposal to Sec. V and instead begin our discussion of speakable coherence with a very different proposal, based on operations that are dephasing-covariant. We here assess the dephasing-covariance approach and compare it to the translational-covariance approach discussed in the last section.

A. Free operations as dephasing-covariant operations

1. Definition of dephasing-covariant operations

As before, suppose that the preferred subspaces relative to which coherence is to be quantified are $\{\mathcal{H}_l\}_l$ and are associated with the projectors $\{\Pi_l\}_l$.

Definition 10 *We say that a quantum operation $\mathcal{E}$ is* dephasing-covariant *relative to the preferred subspaces if it commutes with the associated dephasing operation, $\mathcal{D}$ of Eq.* (2.2)*, i.e., if*

$$\mathcal{E} \circ \mathcal{D} = \mathcal{D} \circ \mathcal{E} \ . \tag{4.1}$$

Note that if the input and output spaces of the map $\mathcal{E}$ are distinct, then the dephasing map is different on the input and output spaces, but we do not indicate this difference in our notation.

Dephasing-covariant quantum operations are easily seen to be incoherence-preserving. It suffices to note that if $\mathcal{E}$ is dephasing-covariant, then for any incoherent state $\rho \in \mathcal{I}$,

$$\mathcal{E}(\rho) = \mathcal{E}(\mathcal{D}(\rho)) = \mathcal{D}\left(\mathcal{E}(\rho)\right), \tag{4.2}$$

and therefore $\mathcal{E}(\rho)$ is invariant under dephasing and hence incoherent.

2. Dephasing-covariant measurements

If the output space is trivial, so that the map corresponds to tracing with a measurement effect E on the input space, that is, $\mathcal{E}(\cdot) = \mathrm{Tr}(E\cdot)$, then Eq. (4.1) reduces to

$$\begin{aligned}\mathrm{Tr}(E(\cdot)) &= \mathrm{Tr}(E\mathcal{D}(\cdot))\\ &= \mathrm{Tr}(\mathcal{D}(E)\ (\cdot)),\end{aligned} \tag{4.3}$$

where we have used the fact that $\mathcal{D}$ is self-adjoint relative to the Hilbert-Schmidt inner product, and this in turn implies

$$\mathcal{D}(E) = E, \tag{4.4}$$

where we have used the fact that the set of all quantum states form a basis of the operator space. Thus E is an incoherent effect, i.e., it is block-diagonal with respect to the preferred subspaces.

Proposition 11 *A POVM is dephasing-covariant if and only if all of its effects are incoherent.*

Comparing to proposition 3, we see that a POVM is dephasing-covariant if and only if it is translationally-covariant.

3. Dephasing-covariant unitary operations

Because the dephasing-covariant operations are all incoherence-preserving, the set of unitary dephasing-covariant operations are included within the set of unitary incoherence-preserving operations. As it turns out, the two sets are in fact equivalent. We postpone the proof until Sec. V B, Proposition 18, where we also present the general form of such unitaries.

4. Considerations regarding the existence of a free dilation

By analogy with the considerations of Sec. III A 4, in order to discuss the possibility of dilating a dephasing-covariant operation with the use of an ancilla in an incoherent state and a dephasing-covariant unitary on the composite of system and ancilla, one needs to specify not only the preferred subspaces (relative to which coherence is defined) on the system and ancilla individually, but on the composite of system and ancilla as well. When the input and output spaces differ this needs to be specified on the outputs as well, as discussed in Sec. III A 4.

Recall that the system input and output spaces are denoted $\mathcal{H}_s$ and $\mathcal{H}_{s'}$, the ancilla input and output spaces are denoted $\mathcal{H}_a$ and $\mathcal{H}_{a'}$, and the composite of system and ancilla is $\mathcal{H}_{sa} = \mathcal{H}_s \otimes \mathcal{H}_a = \mathcal{H}_{s'} \otimes \mathcal{H}_{a'}$. We also denote the associated sets of incoherent states and dephasing maps with the subscripts s, a and sa (or $s'a'$).

We here assume that the preferred subspaces for the composite are just the tensor products of those for the system and for the ancilla, so that

$$\mathcal{D}_{sa} = \mathcal{D}_s \otimes \mathcal{D}_a. \tag{4.5}$$

Proposition 12 *A quantum operation $\mathcal{E}$ is* dephasing-covariant *if it can be implemented by coupling the system to an ancilla in a state σ that is incoherent, via a unitary quantum operation $\mathcal{V}$ that is dephasing-covariant, and then post-selecting on a measurement outcome associated to an incoherent effect E. Suppose $\mathcal{E}$ can be implemented as a dilation of the form*

$$\mathcal{E}(\rho) = \mathrm{tr}_{\mathrm{a}'}(E\, \mathcal{V}(\rho \otimes \sigma)), \tag{4.6}$$

where ρ is a state on $\mathcal{H}_{\mathrm{s}}$, σ is a state on $\mathcal{H}_{\mathrm{a}}$, $\mathcal{V}(\cdot) = V(\cdot)V^\dagger$ for some unitary operator V on $\mathcal{H}_{\mathrm{sa}}$, and E is an effect on $\mathcal{H}_{\mathrm{a}'}$. Then formally, $\mathcal{E}$ is dephasing-covariant if there is such a dilation where $\mathcal{D}_{\mathrm{a}}(\sigma) = \sigma$, $\mathcal{D}_{\mathrm{a}'}(E) = E$ and $\mathcal{V} \circ \mathcal{D}_{\mathrm{sa}} = \mathcal{D}_{\mathrm{sa}} \circ \mathcal{V}$.

Proof. The proof is as follows:

$$\mathcal{E}\big(\mathcal{D}_{\mathrm{s}}(\rho)\big) = \mathrm{tr}_{\mathrm{a}'}\Big(E\, \mathcal{V}[\mathcal{D}_{\mathrm{s}}(\rho) \otimes \sigma]\Big) \tag{4.7}$$
$$= \mathrm{tr}_{\mathrm{a}'}\Big(E\, \mathcal{V}[\mathcal{D}_{\mathrm{sa}}(\rho \otimes \sigma)]\Big) \tag{4.8}$$
$$= \mathrm{tr}_{\mathrm{a}'}\Big(E\, \mathcal{D}_{\mathrm{s}'\mathrm{a}'}\Big(\mathcal{V}[\rho \otimes \sigma]\Big)\Big) \tag{4.9}$$
$$= \mathrm{tr}_{\mathrm{a}'}\Big(\mathcal{D}_{\mathrm{s}'\mathrm{a}'}\Big(E\, \mathcal{V}[\rho \otimes \sigma]\Big)\Big) \tag{4.10}$$
$$= \mathcal{D}_{\mathrm{s}'}\Big(\mathrm{tr}_{\mathrm{a}'}\big(E\, \mathcal{V}[\rho \otimes \sigma]\big)\Big) \tag{4.11}$$
$$= \mathcal{D}_{\mathrm{s}'}(\mathcal{E}(\rho)), \tag{4.12}$$

where in the second line, we have used the fact that $\mathcal{D}_{\mathrm{a}}(\sigma) = \sigma$ together with Eq. (4.5); in the third line, we have used the fact that $\mathcal{V}$ is a dephasing-covariant operation; in the fourth line, we have used the fact that $\mathcal{D}_{\mathrm{a}'}(E) = E$ and Eq. (4.5); in the fifth line, we have used Eq. (4.5) again; and in the sixth line, we have used Eq. (4.16). ■

It is an open question whether *every* dephasing-covariant operation on a system can be implemented in this fashion for a suitable choice of ancilla.

5. Characterization via diagonal and off-diagonal modes

A useful way of distinguishing incoherent states from coherent states is by considering their representation as vectors in $\mathcal{B}$, the space of linear operators on $\mathcal{H}$. The dephasing operations $\mathcal{D}$ is a projector on the operator space, i.e., it satisfies $\mathcal{D} \circ \mathcal{D} = \mathcal{D}$, and it induces a direct sum decomposition on the operator space $\mathcal{B}$ as $\mathcal{B} = \mathcal{B}^{\mathrm{diag}} \oplus \mathcal{B}^{\mathrm{offd}}$, where $\mathcal{B}^{\mathrm{diag}}$ and $\mathcal{B}^{\mathrm{offd}}$ are respectively the image and the kernel of $\mathcal{D}$. For an arbitrary operator A, we define the diagonal component of A to be

$$A^{\mathrm{diag}} \equiv \mathcal{D}(A), \tag{4.13}$$

and the off-diagonal component of A to be

$$A^{\mathrm{offd}} \equiv A - \mathcal{D}(A) = A - A^{\mathrm{diag}}. \tag{4.14}$$

Clearly, all incoherent states lie entirely within $\mathcal{B}^{\mathrm{diag}}$, while every coherent state has some nontrivial component in $\mathcal{B}^{\mathrm{offd}}$.

Then, the fact that dephasing-covariant operations by definition commute with the dephasing operation $\mathcal{D}$, immediately implies that these operations are block-diagonal with respect to this decomposition of the operator space $\mathcal{B}$.

It is useful to consider how a dephasing-covariant operation $\mathcal{E}$ is represented as a matrix on the operator space $\mathcal{B}$. If $\{X_i\}_i$ is an orthonormal basis (with respect to the Hilbert Schmidt inner product) for the space of operators $\mathcal{B}$, then $\mathcal{E}$ can be represented by the matrix elements $\mathcal{E}_{ij} = \mathrm{Tr}(X_i^\dagger \mathcal{E}(X_j))$. $\mathcal{E}$ is dephasing-covariant iff its matrix representation has the following form relative to the decomposition $\mathcal{B} = \mathcal{B}^{\mathrm{diag}} \oplus \mathcal{B}^{\mathrm{offd}}$,

$$\begin{array}{cc} & \begin{array}{cc} \mathcal{B}^{\mathrm{diag}} & \mathcal{B}^{\mathrm{offd}} \end{array} \\ \begin{array}{c} \mathcal{B}^{\mathrm{diag}} \\ \mathcal{B}^{\mathrm{offd}} \end{array} & \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) \end{array}, \tag{4.15}$$

where A and B are matrices.

Alternatively, the mode-based characterization of dephasing-covariant operations can be given as follows.

Proposition 13 *A quantum operation $\mathcal{E}$ is dephasing-covariant relative to a preferred set of subspaces if and only if it preserves the diagonal and off-diagonal modes. Formally, the condition is that whenever $\mathcal{E}(\rho) = \sigma$, we have $\mathcal{E}(\rho^{\mathrm{diag}}) = \sigma^{\mathrm{diag}}$ and $\mathcal{E}(\rho^{\mathrm{offd}}) = \sigma^{\mathrm{offd}}$.*

B. Physical justification for the restriction to dephasing-covariant operations?

We noted that whether *every* dephasing-covariant operation admits of a dilation in terms of an incoherent ancilla state, an incoherent effect on the ancilla, and a dephasing-covariant unitary on the system-ancilla composite is currently an open question.

If its answer is positive, then the problem of finding a physical justification for the dephasing-covariant operations reduces to finding a physical scenario wherein the only free states and effects on the ancilla are incoherent and the only free unitaries on the system-ancilla composite are those that are dephasing-covariant. It is not obvious how to justify the latter constraint in particular. However, even if the answer is negative, there remains the possibility that one can find a physical justification for the free set of operations being the set of dephasing-covariant operations. This is the same possibility that remained for justifying the incoherence-preserving or incoherent operations as the free set, namely, by allowing that different systems may not be treated even-handedly.

Overall, therefore, it is at present unclear whether a restriction to dephasing-covariant operations arises from a natural experimental restriction.

C. Relation of dephasing-covariant operations to translationally-covariant operations

1. Relation between the sets of free operations

Here we study the relation between the dephasing-covariant operations and the translationally-covariant operations *for the same choice of the preferred subspaces.* In the dephasing-covariance approach to coherence, one must begin with a choice of preferred subspaces relative to which dephasing occurs. If one is given a translational symmetry, then one can choose these subspaces to be the eigenspaces of the generator of that symmetry group (or, if the symmetry group incorporates independent commuting translations and therefore multiple commuting generators, then as the joint eigenspaces of the generators). Conversely, if a set of preferred subspaces is given, one can always construct a Hermitian operator that has these subspaces as eigenspaces with distinct eigenvalues and consider this to be a generator of translations. Because a given choice of preferred subspaces might only be physically justified in *one* of the two approaches, the comparison we are making here is best understood as probing the mathematical relation between the two approaches to quantifying coherence as a resource.

To understand this connection it is useful to note that the dephasing operation relative to the eigenspaces of L can be realized by applying a random translation to the system, that is, a translation $\mathcal{U}_{L,x}$ where x is chosen uniformly at random,

$$\mathcal{D}(\cdot) = \lim_{x_0 \to \infty} \frac{1}{2x_0} \int_{-x_0}^{x_0} dx \ \ e^{-ixL}(\cdot)e^{ixL} \ . \tag{4.16}$$

It is also useful to note the connection between the two approaches from the perspective of mode decompositions. The diagonal mode relative to the eigenspaces of L corresponds to the $\omega = 0$ mode of translational asymmetry relative to the generator L,

$$\mathcal{B}^{\mathrm{diag}} = \mathcal{B}^{(0)}, \tag{4.17}$$

while the off-diagonal mode relative to the eigenspaces of L corresponds to the direct sum of the $\omega \neq 0$ modes of translational asymmetry relative to the generator L,

$$\mathcal{B}^{\mathrm{offd}} = \bigoplus_{\omega \neq 0} \mathcal{B}^{(\omega)}. \tag{4.18}$$

Intuitively then, to choose the dephasing-covariant operations as the free set of operations is to disregard the distinction between the different nonzero modes.

Denote the the set of quantum operations that are translationally-covariant with respect to a generator L by TC_L and the set of quantum operations that are dephasing-covariant with respect to the eigenspaces of L (which we will denote $\mathcal{S}(L)$) by $\mathrm{DC}_{\mathcal{S}(L)}$.

Proposition 14 *The operations that are translationally-covariant relative to translations generated by L are a* *proper subset of those that are dephasing-covariant relative to the eigenspaces of L,*

$$TC_L \subset DC_{\mathcal{S}(L)}. \tag{4.19}$$

Proof. The subset relation can be understood easily within any of the characterizations of translationally-covariant and dephasing-covariant operations that we have provided. For instance, starting with the expression for the translational-covariance of an operation $\mathcal{E}$, Eq. (3.5), if one integrates over x, one obtains the expression for the dephasing covariance of $\mathcal{E}$, Eq. (4.1), where we have made use of Eq. (4.16).

To show that the inclusion is strict, it suffices to show that there are dephasing-covariant operations which are not translationally-covariant. Any example of a dephasing-covariant operation wherein one nonzero mode, ω, is mapped to another, distinct, nonzero mode is sufficient.

For example, consider the unitary that swaps a pair of states living in different eigenspaces of L, and leaves the rest of the states unchanged. This operation in general will not be translationally-covariant while it is dephasing-covariant. ■

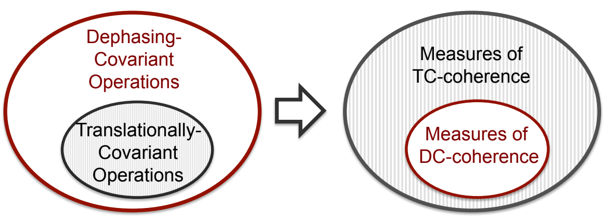

FIG. 1: The relation between the dephasing-covariant and the translationally-covariant operations and the relation between the associated measures of coherence that these sets of operations define. Both inclusions are shown to be strict.

Despite the strict inclusion of translationally-covariant operations in the set of dephasing-covariant operations, if we focus on the POVMs associated to measurements (i.e. the retrodictive aspect of the measurement) the two approaches pick out the same set, as we noted earlier.

2. Relation between measures of coherence

Prop. 14 implies that if a transformation from initial state ρ to final state σ is allowed under $DC_{\mathcal{S}(L)}$ operations, then it is also allowed under TC_L operations. This means that any measure of $DC_{\mathcal{S}(L)}$-coherence is also a measure of TC_L-coherence. In fact, one can show that

Proposition 15 *Any measure of $DC_{\mathcal{S}(L)}$-coherence is also a measure of TC_L-coherence, but not vice-versa.*

The strictness of the inclusion is demonstrated in Sec. VI. This relation is illustrated in Fig. 1.

V. COHERENCE VIA INCOHERENCE-PRESERVING AND INCOHERENT OPERATIONS

In this section, we consider approaches to coherence wherein the free operations are incoherence-preserving or incoherent operations.

A. Free operations as incoherence-preserving operations

Definition 16 *A quantum operation $\mathcal{E}$ is said to be* incoherence-preserving *if it maps incoherent states on the input space to incoherent states on the output space,*

$$\rho \in \mathcal{I}_{\text{in}} \implies \mathcal{E}(\rho) \in \mathcal{I}_{\text{out}}. \tag{5.1}$$

Just as was the case with the dephasing-covariant operations, the incoherence-preserving operations can be characterized in terms of their interaction with the dephasing map:

Proposition 17 *A quantum operation $\mathcal{E}$ is incoherence-preserving if and only if*

$$\mathcal{E} \circ \mathcal{D} = \mathcal{D} \circ \mathcal{E} \circ \mathcal{D} \ . \tag{5.2}$$

We can also characterize incoherence-preserving operations in terms of their representations as matrices on the operator space $\mathcal{B}$, just as we did for dephasing-covariant operations.

We deduce from Eq. (5.2) that an operation $\mathcal{E}$ is incoherence-preserving if and only if its matrix representation has the following form relative to the decomposition $\mathcal{B} = \mathcal{B}^{\text{diag}} \oplus \mathcal{B}^{\text{offd}}$,

$$\begin{array}{cc} & \begin{array}{cc} \mathcal{B}^{\text{diag}} & \mathcal{B}^{\text{offd}} \end{array} \\ \begin{array}{c} \mathcal{B}^{\text{diag}} \\ \mathcal{B}^{\text{offd}} \end{array} & \left(\begin{array}{cc} A & C \\ 0 & B \end{array} \right) \end{array}, \tag{5.3}$$

where A, B and C are matrices. A comparison with the analogous characterization of dephasing-covariant operations, Eq. (4.15), shows how incoherence-preserving operations do not preserve the diagonal and off-diagonal modes.

We postpone our characterization of the incoherence-preserving *measurements* until Sec. V D because it forms the basis of one of our criticisms of this approach.

1. Incoherence-preserving unitary operations

For simplicity, we start with the special case where the preferred subspaces are all 1-dimensional, where the unitary incoherence-preserving operations have a particularly simple form.

Let $\mathcal{V}$ denote a unitary incoherence-preserving operation, and let $\{|l\rangle\langle l|\}_l$ denote the set of projectors onto the elements of the preferred basis. Consider the image of each $|l\rangle\langle l|$ under $\mathcal{V}$. Because a unitary operation preserves the rank of a state, the image must also be a projector onto a 1-dimensional subspace. But given that $|l\rangle\langle l|$ is an incoherent state and $\mathcal{V}$ is incoherence-preserving, it follows that the image must be an incoherent pure state. The only incoherent pure states are those in the set $\{|l\rangle\langle l|\}_l$, therefore $\mathcal{V}$ must map this set to itself, that is, for any l it should hold that

$$\mathcal{V}(|l\rangle\langle l|) = |\pi(l)\rangle\langle\pi(l)| \tag{5.4}$$

where π is a permutation over the set $\{l\}_l$.

If V is the unitary operator that defines the unitary incoherence-preserving operation $\mathcal{V}$ through $\mathcal{V}(\cdot) = V \cdot V^\dagger$, the incoherence-preserving property implies that

$$V = \sum_l e^{i\theta_l} |\pi(l)\rangle\langle l| \tag{5.5}$$

for a set of phases $\{e^{i\theta_l}\}_l$. So, any incoherence-preserving unitary operation can be characterized by a permutation π of the preferred basis, and a set of phases $\{e^{i\theta_l}\}_l$.

The case where the preferred subspaces are not all 1-dimensional is slightly more complicated.

Let $\{|l,\alpha_l\rangle\}_{\alpha_l}$ denote an arbitrary basis for the preferred subspace $\mathcal{H}_l$, so that $\{|l,\alpha_l\rangle\}_{l,\alpha_l}$ is a basis of the entire Hilbert space. Now consider the image of $\{|l,\alpha_l\rangle\}_{\alpha_l}$ under the unitary V. Although each vector $|l,\alpha_l\rangle \in \mathcal{H}_l$ need not be mapped to another vector in $\mathcal{H}_l$, it is still the case that for a given l, there must be some l' such that for *every* vector in $\mathcal{H}_l$, its image is in $\mathcal{H}_{l'}$. The reason is that if this were not the case, it would be possible to identify some vector in $\mathcal{H}_l$ that is mapped to a non-trivial coherent superposition of vectors lying in *different preferred subspaces*, and this would imply a violation of the incoherence-preserving property. Note that the dimension of $\mathcal{H}_{l'}$ must be the same as the dimension of $\mathcal{H}_l$.

Therefore, if π denotes a dimension-preserving permutation of the preferred subspaces, and V_l denotes a unitary that acts arbitrarily *within* the $\mathcal{H}_l$ subspace and as identity on the complementary subspace, then the property of $\mathcal{V}$ being incoherence-preserving implies that

$$V = \sum_{l,\alpha_l} V_{\pi(l)} |\pi(l),\alpha_{\pi(l)}\rangle\langle l,\alpha_l|. \tag{5.6}$$

So, any incoherence-preserving unitary operation can be characterized by a dimension-preserving permutation π among the preferred subspaces, and a set of unitary operators $\{V_l\}_l$.

B. Relation of incoherence-preserving operations to dephasing-covariant operations

We begin by considering unitary operations.

Proposition 18 *The set of unitary dephasing-covariant operations relative to the preferred subspaces $\{\mathcal{H}_l\}_l$ is equivalent to the set of unitary incoherence-preserving operations relative to these same subspaces.*

The proof is as follows. Every dephasing-covariant operation is incoherence-preserving and therefore what must be demonstrated is that for unitary operations, being incoherence-preserving implies being dephasing-covariant. This simply follows from the general form of an incoherence-preserving unitary operation, Eq. (5.6).

In general, however, the dephasing-covariant operations are a strict subset of the incoherence-preserving operations.

To demonstrate this, we provide a simple example of an operation that is incoherence-preserving but not dephasing-covariant. Consider a qubit and denote the preferred basis thereof by $\{|0\rangle,|1\rangle\}$. Let $|\pm\rangle = 2^{-1/2}(|0\rangle \pm |1\rangle)$. Consider the quantum operation defined by

$$\mathcal{E}(\rho) = |0\rangle\langle 0|\mathrm{Tr}(|+\rangle\langle +|\rho) + |1\rangle\langle 1|\mathrm{Tr}(|-\rangle\langle -|\rho), \tag{5.7}$$

$\mathcal{E}$ is clearly incoherence-preserving, because for all input states (and therefore, in particular, incoherent states), the output is always an incoherent state. On the other hand, one can easily show that it is not dephasing-covariant because $\mathcal{D}\circ\mathcal{E} = \mathcal{E}$, while $\mathcal{E}\circ\mathcal{D} \neq \mathcal{E}$. This can be seen, for instance, by noting that while $\mathcal{D}(|+\rangle\langle +|) = \mathcal{D}(|-\rangle\langle -|)$, operation $\mathcal{E}$ maps $|+\rangle$ and $|-\rangle$ to two different states.

This example is related to the fact if one considers operations with trivial output spaces, i.e., destructive measurements, then there are incoherence-preserving operations which are not dephasing-covariant (See Sec. V D).

Denote the operations that are incoherence-preserving (dephasing-covariant) relative to a particular choice $\mathcal{S} \equiv \{\mathcal{H}_l\}_l$ of the preferred subspaces by $IP_\mathcal{S}$ ($DC_\mathcal{S}$). Then we have the following result.

Proposition 19

$$DC_\mathcal{S} \subset \; IP_\mathcal{S}. \tag{5.8}$$

Therefore, any measure of $IP_\mathcal{S}$-coherence is also a measure of $DC_\mathcal{S}$-coherence.

Here the second statement follows from the fact that if $\rho \to \sigma$ is a transformation that is possible under incoherence-preserving operations, then it is also possible under dephasing-covariant operations because the latter set includes the former. This, in turn, implies that any function on states that is nonincreasing under the latter set of operations is nonincreasing under the former set of operations.

At present, it is not clear whether the vice-versa holds, that is, whether or not there exist measures of $\mathrm{DC}_\mathcal{S}$-coherence that are not measures of $\mathrm{IP}_\mathcal{S}$ coherence. This question remains open because although we have examples of operations that are incoherence-preserving but

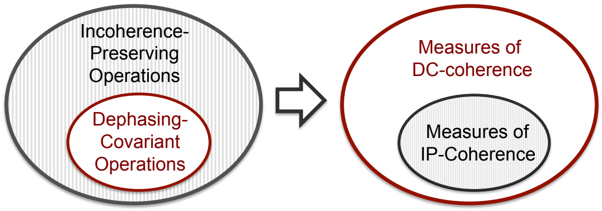

FIG. 2: The relation between the incoherence-preserving and the dephasing-covariant operations and the relation between the associated measures of coherence that these sets of operations define. It is not known whether the inclusion relation among the measures is strict.

not dephasing-covariant, we do not have examples of state transformations $\rho \to \sigma$ which are possible under incoherence-preserving operations but not dephasing-covariant ones. That is, we have no proof that the additional operations in the incoherence-preserving set are in fact helpful in any state-conversion problem. The question is also open if one considers measures of coherence under the incoherent operations rather than the incoherence-preserving operations.

C. Free operations as incoherent operations

The approach to coherence described above is certainly in the same spirit as the approach introduced by BCP. Strictly speaking, however, BCP take the free operations to be the *incoherent operations*, defined as follows:

Definition 20 *A quantum operation $\mathcal{E}$ is said to be* incoherent *if it admits of a Kraus decomposition where each term is incoherence-preserving, that is, if there is a decomposition with Kraus operators $\{K_n\}_n$ such that $K_n \mathcal{I}_{\rm in} K_n^\dagger \subset \mathcal{I}_{\rm out}$ for all n.*

In their article, BCP only considered coherence relative to a decomposition of the Hilbert space into 1-dimensional subspaces, while one may want to consider decompositions into subspaces of arbitrary dimension. Also, BCP did not explicitly consider the possibility that the input and output spaces of $\mathcal{E}$ are different. The definition of incoherent operations we have just provided incorporates these two generalizations of the notion.

Note that because unitary operations have only a single term in their Kraus decomposition, for unitary operations being incoherence-preserving coincides with being incoherent.

D. Criticism of these approaches to defining coherence

As noted in the introduction, a given choice of the set of free operations is only physically justified if it can be understood as arising from some natural restriction on experimental capabilities. The situation where the free operations are the incoherence-preserving or incoherent operations is like that of the dephasing-covariant operations—it is unclear whether there is a natural restriction that picks out these sets.

Nonetheless, we describe two features of the incoherence-preserving or incoherent operations that seem pertinent to the question of whether they can arise from a natural restriction: how they constrain measurements and the nonexistence of a certain kind of Stinespring dilation.

1. *Incoherence-preserving and incoherent measurements*

If the output space of a quantum operation is trivial, so that it corresponds to tracing with a measurement effect E on the input space, that is, $\mathcal{E}(\cdot) = \mathrm{Tr}(E\cdot)$, then the definition of incoherence-preserving in terms of the dephasing map, Eq. (5.2), reduces to a trivial condition that is satisfied by all effects E. Consequently, there is no constraint on the effects in this approach. Similarly, in the case of the incoherent operations proposed by BCP, for any given POVM, the operation associated to a given outcome can always be chosen to prepare the output system in an incoherent state. It follows that the set of incoherent measurements includes *any* POVM.

Proposition 21 *The sets of POVMs associated to the set of incoherence-preserving measurements and the set of incoherent measurements are both the full set of POVMs.*

In these approaches, therefore, there is no limitation on the retrodictive capacity of a free measurement. In particular, measurements in the free set are capable of detecting the presence of coherence in a state. This is a rather counterintuitive feature for free measurements to have. Furthermore, the fact that the free states are incoherent while the free effects are not implies that the proposal has an awkward asymmetry between prediction and retrodiction. Recall that in the approaches based on translationally-covariant or dephasing-covariant operations, by contrast, the free effects are the incoherent effects. These considerations, in our view, suggest that this approach does not have a natural physical justification. The criticism is not conclusive however—an explanation of circumstances in which just this sort of restriction arises may yet be forthcoming.

2. *Considerations regarding the existence of a free dilation*

It turns out that, if one assumes that the preferred subspaces for the system-ancilla composite are the tensor products of the preferred subspaces for the system and for the ancilla, i.e., if all systems are treated even-handedly, then incoherence-preserving and incoherent operations do not have free dilations. For instance, the operation in Eq. (5.7), which distinguishes states $|+\rangle$ and $|-\rangle$, is both incoherent and incoherence-preserving and cannot be implemented in this way.

The lack of a free dilation can be understood as a consequence of the fact that when it comes to unitary operations, there is no distinction between the sets of dephasing-covariant, incoherent, or incoherence-preserving operations (This follows from proposition 18, together with the fact that a unitary operation has a single term in its Kraus decomposition). It follows that one can substitute “incoherence-preserving (or incoherent) unitary operations” for “dephasing-covariant unitary operations” in proposition 12, while preserving its validity. Hence with incoherent states and incoherence-preserving (or incoherent) unitaries, one can only generate dephasing-covariant operations. Because these are, by proposition 14, a proper subset of the incoherent operations, it follows that we cannot implement every incoherence-preserving (incoherent) operation using incoherent states and incoherence-preserving (incoherent) unitaries. Again, we note that a physical justification might still be possible.

3. *Incoherent operations versus incoherence-preserving operations*

Which of various different sets of free operations is the appropriate one for defining a resource theory, for instance, whether to use the incoherence-preserving or the incoherent operatons to define coherence, is a question that can be settled by finding a physical scenario or restriction on experimental capabilities that picks out one or the other. If the set of operations are physically justified by an interaction between the system and an uncontrollable environment (on which one cannot implement any measurements), then incoherence-preserving operations are more natural than incoherent operations. If, on the other hand, the physical justification comes from an interaction between the system and an apparatus with a classical read-out, then incoherent operations are more natural than incoherence-preserving operations.

Furthermore, if one seeks a physical justification in terms of a constraint on the dilation of an operation, then this too bears on the question of whether it is physically more reasonable to take the free operations to be the incoherent operations or the incoherence-perserving operations.

Consider the following (flawed) argument in favour of using the incoherent operations. Take the standard Stinespring dilation of an operation. For any Kraus decomposition of that operation, it is possible to ensure that the effective map on the system is a single term in that Kraus decomposition by implementing the operation through its standard Stinespring dilation and then performing an appropriate measurement on the auxiliary system and post-selecting on a single outcome. Given this possibility of realizing a single term in the Kraus decomposition, so the argument goes, one should require each such term to be incoherence-preserving, rather than just requiring this of their sum.

However, this argument has appealed to the standard Stinespring dilation theorem which only guarantees that for every operation there is *some* unitary on a larger system that realizes it by dilation. In the context of a resource theory, however, one cannot avail oneself of *any* unitary on the larger system because such a unitary might not be free. Similar comments apply to the states and effects on the auxiliary system that appear in the dilation. In a resource theory, if a free operation is implemented by dilation, then it must be implemented by a *free* dilation, which is a strict subset of all possible dilations.

Therefore, to settle this issue by appeal to dilations one must find a physical justification of either the incoherence-preserving or incoherent operations in terms of a restriction on the dilation resources, which is an unsolved problem, as we noted in the previous section.

Finally, we noted in the introduction that our proposal for the set of free operations in a theory of speakable coherence, the dephasing-covariant operations, is closely related to the proposal found in Ref. [22]. In fact, the set of free operations of Ref. [22] stands to the set of dephasing-covariant operations as the set of incoherent operations stands to the set of incoherence-preserving operations. As such, disputes about the relative merits of the fomer two sets are akin to those about the relative merits of the latter two—they will only be resolved when one or the other proposal is given a physical justification as all and only the operations that can be dilated using a restricted set of states, effects and unitaries.

VI. METHODS FOR DERIVING MEASURES OF COHERENCE

In this section, we consider measures of coherence for the various different sets of free operations described in the article, in particular, TC-coherence, DC-coherence and IP-coherence. We have already noted, in Propositions 15 and 19, that if one considers the same choice of preferred subspaces, then every measure of IP-coherence is also a measure of DC-coherence which is also a measure of TC-coherence. Because a measure of TC-coherence is a measure of translational asymmetry, one can immediately obtain many interesting measures of coherence by simply appealing to the known measures of asymmetry. Indeed, we show that most of the recent proposed measures of coherence in the BCP proposal correspond to measures of asymmetry that have been previously studied in Refs. [9] and [10]. Because of the strictness of the inclusions in Proposition 15, however, only a subset of the measures of TC-coherence are measures of DC-coherence or IP-coherence. We will highlight some examples of the strictness of the inclusion.

In addition, we present certain general techniques for deriving measures of coherence, adapted from ideas introduced in the context of asymmetry theory, and we review some of the most important examples of measures of coherence. (Note, however, that the list we provide is not complete; there are many known measures of asymmetry that we do not review here. See e.g. [4, 9, 77, 78].)

A. Measures of coherence based on measures of information

In Sec. III B, we showed that the resource for phase estimation is TC-coherence. We saw that there is a duality between the problem of state transformation in the resource theory of TC-coherence on one hand, and the problem of processing the classical information encoded in the phase shifted versions of the state. This was formalized by proposition 9. This duality can be leveraged to derive measures of unspeakable coherence from measures of information.

We begin by recalling the definition of a measure of the information content of a quantum encoding of a classical message.

Definition 22 *A function I from sets $\{\rho(x)\}_x$ of quantum states to the reals is a measure of the information about x contained in the quantum encoding if*
(i) For any trace-preserving quantum operation $\mathcal{E}$ it holds that $I(\{\mathcal{E}(\rho(x))\}_x) \leq I(\{\rho(x)\}_x)$.
(ii) For any trivial encoding, the elements of which are indistinguishable, I takes the value 0.

If we define a function f_I on states to be such that its value on a state is the measure of information I on the set of states obtained by acting on the state with all elements of the translation group (i.e. the orbit under translations of that state),

$$f_I(\rho) = I(\{\mathcal{U}_{L,x}(\rho)\}_{x\in\mathbb{R}}), \tag{6.1}$$

then by proposition 9 and the definitions of measures of TC-coherence and measures of information, we see that f_I is a measure of TC-coherence if I is a measure of information.

In particular, one can obtain measures of TC-coherence from measures of the distinguishability of any pair of states in the translational orbit of ρ [9, 10].

A similar sort of consideration allows us to infer measures of DC-coherence from measures of information. In particular, for any state ρ, one can encode 1 bit of classical information b as $b \to \rho_b$ where $\rho_0 = \rho$ and $\rho_1 = \mathcal{D}(\rho)$. Then, it can be easily seen that for any measure of information I,

$$f(\rho) = I(\{\rho, \mathcal{D}(\rho)\}), \tag{6.2}$$

i.e., the amount of information about bit b that can be transferred using this encoding is a measure of DC-coherence. This follows from the fact that, by definition, if ρ can be transformed to σ by a dephasing-covariant trace-preserving quantum operation, then the same quantum operation transforms $\mathcal{D}(\rho)$ to $\mathcal{D}(\sigma)$, and hence there is a trace-preserving quantum operation which transforms the ρ-based encoding of b to the σ-based encoding of b. But this in turn implies that the σ-based encoding cannot have more information about b than the ρ-based encoding. (This can be thought of as the analogue of the easy direction of the duality in proposition 9).

We now consider various measures of coherence that can be derived from measures of information.

(i) If one uses the duality described by Eq. (6.1) with the Holevo quantity as the measure of information, then, following the argument of Ref. [10] and assuming a uniform probability density over the translations, one can prove that the function

$$\Gamma(\rho) \equiv S(\mathcal{D}(\rho)) - S(\rho) \ . \tag{6.3}$$

where S is the von Neumann entropy, is a measure of TC-coherence.

Meanwhile, if one uses the duality described by Eq. (6.2) with the quantum relative entropy, $S(\rho||\sigma) \equiv \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma)$, as the measure of information, then one can prove that the function

$$\Gamma'(\rho) = S\left(\rho||\mathcal{D}(\rho)\right) \ , \tag{6.4}$$

is a measure of DC-coherence.

Finally, the function

$$\Gamma''(\rho) = \min_{\sigma\in\mathcal{I}} S\left(\rho||\sigma\right), \tag{6.5}$$

the minimum relative entropy distance of ρ to the set of incoherent states, is clearly nonincreasing under incoherence-preserving operations and is therefore a measure of IP-coherence. It is also nonincreasing under incoherent operations [20].

It turns out that the three measures are all equivalent, that is,

$$\min_{\sigma\in\mathcal{I}} S\left(\rho||\sigma\right) = S\left(\rho||\mathcal{D}(\rho)\right) = S(\mathcal{D}(\rho)) - S(\rho) \ , \tag{6.6}$$

a fact that has been noted by many authors [5, 20, 79]. So this is an example of a function that is a measure of coherence in all of the approaches we have considered.

In the context of asymmetry theory, this function was first introduced by Vaccaro et al., who called it simply *the asymmetry* [80]. It was further studied as a measure of asymmetry in Ref. [5] (see proposition 2) and it was first derived from the Holevo information in Ref. [10], where it was called the *Holevo asymmetry measure*. Ref. [10] also proposed that it and other measures of asymmetry could be used to quantify coherence. BCP noted in Ref. [20] that this function was monotonically nonincreasing under incoherent operations and hence a measure of coherence in their approach, where it has been dubbed the *relative entropy of coherence*. Several years prior both to BCP's work and the work which studied it as a measure of translational asymmetry, this function was proposed as a measure of coherence by Åberg in a paper entitled "Quantifying superposition" [79].

(ii) If we start from Eq. (6.1) using the Holevo quantity, but where the probability distribution over translations is allowed to be arbitrary rather than uniform, it is possible to show that the following function is also a measure of TC-coherence:

$$\Gamma_p(\rho) = S\left(\mathcal{D}_p(\rho)\right) - S(\rho)\ , \tag{6.7}$$

where p is an arbitrary probability density on the real line and $\mathcal{D}_p$ is a "weighted twirling operation" defined by

$$\mathcal{D}_p(\cdot) = \lim_{x_0\to\infty} \frac{1}{2x_0} \int_{-x_0}^{x_0} dx\ p(x) e^{-ixL}(\cdot)e^{ixL}\ . \tag{6.8}$$

This translates into the language of coherence a measure of asymmetry identified in Ref [10].

Note that for any symmetric state ρ and any arbitrary probability density p, $\Gamma_p(\rho) = 0$. In Ref. [10], it is shown that using a nonuniform probability density can be useful in some physical examples.

It turns out that for a general probability density p, the function Γ_p is not a measure of DC-coherence and hence not a measure of IP-coherence either.

(iii) Using Eq. (6.2) while taking as our measure of information the Holevo quantity for a quantum encoding of a classical bit but where the bit values have unequal prior probabilities, we can derive a measure of DC-coherence. Specifically, for any $q \in [0, 1]$, the function

$$\Gamma_q(\rho) = S\left(q\rho + (1-q)\mathcal{D}(\rho)\right) - qS(\rho) - (1-q)S(\mathcal{D}(\rho)), \tag{6.9}$$

is a measure of DC-coherence.

(iv) Starting from the monotonicty of the *relative Renyi entropy* under information processing [81], one can use Eq. (6.1) to show that the function

$$S_{L,s}(\rho) \equiv \mathrm{tr}(\rho L^2) - \mathrm{tr}(\rho^s L \rho^{(1-s)} L) \tag{6.10}$$

$$= -\frac{1}{2}\mathrm{tr}\left([\rho^s, L][\rho^{1-s}, L]\right) \tag{6.11}$$

for $0 < s < 1$ is a measure of TC-coherence. The argument is provided, in the context of asymmetry theory, in Refs. [10] and [9].

Interestingly, this quantity has been introduced before by Wigner and Yanase for $s = 1/2$ [82] (and generalized by Dyson to arbitrary s in $(0, 1)$) and since then some of its interesting properties have been studied. However, its monotonicity under symmetric dynamics, and hence its interpretation as a measure of asymmetry, was not recognized in the past.

It has been claimed by Girolami [83] that this function is a measure of coherence according to the definition of BCP [20], that is, he claimed that it is non-increasing under incoherent operations. However, as is noted in [84] and [21], this claim is incorrect. This can be seen, for instance, by recognizing that in the case of pure states this function is equal to the variance of the observable L, but variance obviously is not invariant under permutations of the eigenspaces of L. In other words, this function is only a measure of unspeakable coherence, not of speakable coherence.

Note that for any incoherent state ρ, it holds that $S_{L,s}(\rho) = 0$. Furthermore, for pure states, the Wigner-Yanase-Dyson skew information reduces to the variance of the observable L, that is,

$$S_{L,s}(|\psi\rangle\langle\psi|) = \langle\psi|L^2|\psi\rangle - \langle\psi|L|\psi\rangle^2. \tag{6.12}$$

For a general mixed state, a nonzero variance over L does not attest to there being coherence between the L eigenspaces because it can also be explained by an incoherent mixture of the latter. The function $S_{L,x}$, on the other hand, seems to succeed in quantifying the amount of variance over L that is coherent, which one might call the "coherent spread" over the eigenspaces of L. It is also worth mentioning that recently, Ref. [83] has proposed a method for measuring this quantity.

Interestingly, it has been noted that the function which is the average over s of the Wigner-Yasane-Dyson skew information for index s, $\int_0^1 ds\ S_{L,s}(\rho)$, has a natural interpretation as the *quantum fluctuations* of the observable L, i.e., as the difference between the total fluctuations $\langle\delta^2 L\rangle$ and the (classical) thermal fluctuations [50]. Furthermore, it has been shown that this quantity can be calculated in several interesting examples of many-body systems [50].

(v) If the relative Renyi entropy is used in Eq. (6.2), we can prove that the function

$$S_s(\rho) \equiv \frac{1}{s-1}\log\left[\mathrm{tr}\left(\rho^s[\mathcal{D}(\rho)]^{1-s}\right)\right] \tag{6.13}$$

for $0 < s < 1$ is a measure of DC-coherence.

(vi) Following an argument presented in Ref. [10], we can use Eq. (6.1) to show that the function

$$F_L(\rho) \equiv \|[\rho, L]\|_1 \tag{6.14}$$

where $\|\cdot\|_1$ is the trace norm (i.e., the sum of the singular values) is a measure of TC-coherence. This measure formalizes the intuition that the coherence of a state with respect to the eigenspaces of L can be quantified by the extent to which the state fails to commute with L. The state ρ has coherence relative to the eigenspaces of L if and only if $[\rho, L] \neq 0$ so in retrospect one would naturally expect that *some* operator norm of the commutator $[\rho, L]$ should be a measure of TC-coherence. This intuition does not, however, tell us *which* operator norm to use. Our result shows that it is the trace norm that does the job [14].

[14] Note that for $s = 1/2$, we have $S_{L,s=1/2} = \|[\rho^{1/2}, L]\|_2/2$ where $\|\cdot\|_2$ is the Frobenius norm, that is, the sum of the squares of the singular values. So, both $\|[\rho^{1/2}, L]\|_2$ and $\|[\rho, L]\|_1$ are measures of asymmetry.

The function F_L reduces to a simple expression for pure states: it is proportional to the square root of the variance of the observable L, that is,

$$F_L(|\psi\rangle\langle\psi|) = 2\left(\langle\psi|L^2|\psi\rangle - \langle\psi|L|\psi\rangle^2\right)^{1/2}. \qquad (6.15)$$

Again, we see that a mixture over the eigenspaces of L has vanishing F_L, while a coherent superposition over these eigenspaces has F_L that depends only on the variance over L. Consequently, this coherence measure, like $S_{L,s}$, in some sense quantifies the coherent spread over the eigenspaces of L.

We see that, when restricted to pure states, the function F_L is a monotonic function of $S_{L,s}$. Given that the latter is neither a measure of DC-coherence nor a measure of IP-coherence, as argued above, it follows that the former is not either.

(vi) Arguments in Refs. [9] and [10] show that the function

$$R_p(\rho) \equiv \|\rho - \mathcal{D}_p(\rho)\|_1 \qquad (6.16)$$

is a measure of TC-coherence for an arbitrary probability distribution p on the reals.

In the special case where p is the uniform distribution, the function becomes

$$R(\rho) \equiv \|\rho - \mathcal{D}(\rho)\|_1. \qquad (6.17)$$

The latter is a measure of DC-coherence, as we show in the next section. However, for a general distribution p, the function R_p can increase under dephasing-covariant operations, and hence it is not a measure of DC-coherence. This can be seen, for instance, using the same example which showed that Γ_p is not in general a measure of DC-coherence. It then follows from proposition 15 that R_p for general distribution p is not a measure of IP-coherence either.

B. Measures of coherence based on mode decompositions

In Sec. III A 5, we introduced the concept of the *mode decomposition* of states and operations, first introduced in Ref. [8] as a useful method in the resource theory of asymmetry. In the language of mode decompositions, the translationally-covariant operations are those such that a given mode component of the input state is mapped to the corresponding mode component of the output state (proposition 6). This implies, in particular, that one can only generate a given output if the input contains all of the necessary modes.

Based on this observation, Ref. [8] noted that one can define a family of asymmetry measures which quantify the amount of asymmetry in each mode. By considering translational symmetry, we obtain measures of TC-coherence. In particular, using the monotonicity of the trace-norm under trace-preserving completely positive maps, we find that for each $\omega \neq 0$, the function

$$\left\|\mathcal{P}^{(\omega)}(\rho)\right\|_1 \qquad (6.18)$$

is a measure of TC-coherence (the $\omega = 0$ case yields a constant function and so is uninteresting). Indeed, we find that any linear function of modes can lead to a different measure of TC-coherence. In other words, for any set of complex numbers $\{c^{(\omega)}\}$, the function

$$\left\|\sum_{\omega\in\Omega} c^{(\omega)}\mathcal{P}^{(\omega)}(\rho)\right\|_1, \qquad (6.19)$$

is a measure of TC-coherence, where Ω is the set of modes corresponding to the generator L.

Proposition 13 established that dephasing-covariant operations preserve the diagonal and off-diagonal modes of a state. It follows that

$$\left\|\rho^{\mathrm{offd}}\right\|_1 = \|[\mathcal{I}_{\mathrm{id}} - \mathcal{D}](\rho)\|_1 \qquad (6.20)$$

is nonincreasing under dephasing-covariant operations and hence is a measure of DC-coherence. Eq. (6.20) is just $R(\rho)$ of Eq. (6.17), and so these considerations have only provided an independent way of seeing that it is a measure of DC-coherence.

Recall that the space of off-diagonal operators is equal to the direct sum of the spaces of mode-ω operators for $\omega \neq 0$, $\mathcal{B}^{\mathrm{offd}} = \bigoplus_{\omega\neq 0}\mathcal{B}_\omega$ (Eq. (4.18)), which implies that the superoperator that projects on the one space also projects onto the other, that is,

$$\mathcal{I}_{\mathrm{id}} - \mathcal{D} = \sum_{\omega\neq 0}\mathcal{P}_\omega. \qquad (6.21)$$

It follows that the function (6.20) is a special case of the function (6.19) where we choose $c^{(0)} = 0$ and $c^{(\omega)} = 1$ for all $\omega \neq 0$, thereby confirming that this particular measure of DC-coherence is also measure of TC-coherence, as Proposition 15 requires.

However, measures of TC-coherence based on Eq. (6.19) will not, in general, be measures of DC-coherence or IP-coherence. In particular, the function (6.18) for some particular ω is an example. It can increase under dephasing-covariant or incoherence-preserving or incoherent operations because these can move weight from other mode components into the mode-ω component. This is yet another proof of the strictness of the inclusion in proposition 15.

Interestingly, measures of TC-coherence of the form of Eq. (6.18) are closely related to a method which is regularly used in NMR for characterizing the coherence of states. Here, the relevant observable L is the magnetic moment in the direction of the quantization axis. The modes corresponding to this observable, i.e., the differences of its eigenvalues, are integers. Then, using NMR techniques, one can experimentally measure functions

$$\left\|\mathcal{P}^{(k)}(\rho)\right\|_2 = \sqrt{\mathrm{Tr}(\mathcal{P}^{(k)}(\rho)\mathcal{P}^{(-k)}(\rho))}\ , \qquad (6.22)$$

for integer k, where $\|\cdot\|_2$ is the Frobenius norm [47–49]. Strictly speaking, these functions are not measures of TC-coherence, i.e. they can increase by translationally-covariant operations such as partial trace. However, these functions provide useful lower and upper bound on $\left\|\mathcal{P}^{(k)}(\rho)\right\|_1$, which *are* measures of TC-coherence, namely,

$$\left\|\mathcal{P}^{(k)}(\rho)\right\|_2 \leq \left\|\mathcal{P}^{(k)}(\rho)\right\|_1 \leq \sqrt{d}\left\|\mathcal{P}^{(k)}(\rho)\right\|_2 , \quad (6.23)$$

where d is the dimension of the Hilbert space.

VII. CONCLUDING REMARKS

We have shown that the translationally-covariant operations define a useful resource theory of unspeakable coherence. The constraint of translationally covariance is seen to arise naturally in many physical scenarios, each motivated by a different application of unspeakable coherence. In the case of speakable coherence, we have explored two sorts of approaches, one based on dephasing-covariant operations and the other based on operations that are incoherence-preserving (the BCP approach is a variant of the latter where a free operation is one that has a Kraus decomposition each term of which is incoherence-preserving). It is currently unclear whether there are physical scenarios that pick out one of these sets of operations as the freely-implementable ones. We have, however, constrained the shape of a putative physical justification.

A possibility worth considering is that speakable coherence, unlike its unspeakable counterpart, *cannot* be usefully understood as a resource. Perhaps the resource that powers tasks involving speakable information is not, in fact, a resource of *coherence*, but rather a different property of quantum states [15]. Even if this different property *implied* having some coherence in the state, it might be that coherence was merely *necessary* but not sufficient for acheiving the task. In this case, it would be incorrect to identify coherence as the resource powering the task.

This is indeed the case for at least one model of quantum computation, namely, the state injection model [85]. Here, the circuit consists entirely of Clifford gates—i.e., those that take the set of Stabilizer states to itself—and one allows injection of nonStabilizer states. The injection of nonStabilizer states is critical for achieving universal quantum computation because, as the Gottesman-Knill theorem shows, a Clifford circuit can be efficiently simulated classically [73]. Note that a Clifford circuit acting only on Stabilizer states is efficiently simulatable *even though* the states throughout the computation have coherence relative to the computational basis. Clearly, then, coherence is not sufficient for achieving universal quantum computation in the state injection model. Furthermore, for the case where the systems have dimension corresponding to an odd prime, it has been shown that a *necessary* condition on the injected states for achieving universal quantum computation is that the circuit should fail to admit of a noncontextual hidden variable model [86]. A Clifford circuit acting only on Stabilizer states admits of such a model via Gross's discrete Wigner representation [87, 88]. As such, the failure of noncontextuality is a much more stringent requirement than the presence of coherence and, unlike coherence, it is a viable candidate for the resource that powers quantum computational advantages in the state injection model.

The prospects for speakable coherence as a resource are better for cryptographic tasks. For instance, the BB84 quantum key distribution protocol [89] requires non-orthogonal states and therefore requires the preparation of states that have coherence relative to the preferred basis (regardless of one's choice of preferred basis). Furthermore, the BB84 protocol uses only stabilizer states and measurements, so that the latter are sufficient for the protocol, unlike the situation for universal quantum computation in the state injection model.

It seems, therefore, that speakable coherence may be a resource for some quantum information-processing tasks and not for others. Greater clarity on the applications of speakable coherence would further the project of finding which sets of free operations that can define speakable coherence and which are physically justified.

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[1] B. Coecke, T. Fritz, and R. W. Spekkens, arXiv:1409.5531 (2014).
[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[3] V. Vedral and M. Plenio, Phys. Rev. A **57**, 1619 (1998), quant-ph/9804075.

[4] G. Gour and R. W. Spekkens, New Journal of Physics **10**, 033023 (2008).

[5] G. Gour, I. Marvian, and R. W. Spekkens, Physical Review A **80**, 012307 (2009).

[6] I. Marvian and R. W. Spekkens, New Journal of Physics **15**, 033001 (2013).

[7] I. Marvian and R. W. Spekkens, Physical Review A **90**, 014102 (2014).

[8] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 062110 (2014), URL `http://link.aps.org/doi/10.1103/PhysRevA.90.062110`.

[9] I. Marvian, Ph.D. thesis, University of Waterloo, https://uwspace.uwaterloo.ca/handle/10012/7088 (2012).

[10] I. Marvian and R. W. Spekkens, Nat Commun **5** (2014), URL `http://dx.doi.org/10.1038/ncomms4821`.

[11] G. Gour, Phys. Rev. A **71**, 012318 (2005).

[12] M. Skotiniotis and G. Gour, New Journal of Physics **14**, 073022 (2012).

[13] V. Narasimhachar and G. Gour, Physical Review A **89**, 033859 (2014).

[14] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, Proceedings of the National Academy of Sciences **112**, 3275 (2015).

[15] F. G. Brandao, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Phys. Rev. Lett. **111**, 250404 (2013).

[16] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern, Physics Reports **583**, 1 (2015).

[17] P. Skrzypczyk, A. J. Short, and S. Popescu, Nature communications **5** (2014).

[18] M. Horodecki and J. Oppenheim, Nature communications **4** (2013).

[19] J. Åberg, Nature communications **4** (2013).

[20] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014), URL `http://link.aps.org/doi/10.1103/PhysRevLett.113.140401`.

[21] I. Marvian, R. W. Spekkens, and P. Zanardi, arXiv:1510.06474 (2015).

[22] B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral, arXiv preprint arXiv:1512.02085 (2015).

[23] B. Yadin and V. Vedral, arXiv preprint arXiv:1505.03792 (2015).

[24] A. Ajoy and P. Cappellaro, Physical Review A **86**, 062104 (2012).

[25] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).

[26] V. Giovannetti, S. Lloyd, and L. Maccone, Nature **412**, 417 (2001).

[27] V. Giovannetti, S. Lloyd, and L. Maccone, Nature Photonics **5**, 222 (2011).

[28] R. Schnabel, N. Mavalvala, D. E. McClelland, and P. K. Lam, Nature communications **1**, 121 (2010).

[29] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Reviews of Modern Physics **79**, 555 (2007).

[30] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Physical Review X **5**, 021001 (2015).

[31] M. Lostaglio, D. Jennings, and T. Rudolph, Nature communications **6** (2015).

[32] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, International Journal of Theoretical Physics **39**, 2717 (2000).

[33] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Phys. Rev. Lett. **115**, 210403 (2015).

[34] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[35] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[36] J. I. Latorre, E. Rico, and G. Vidal, arXiv preprint quant-ph/0304098 (2003).

[37] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).

[38] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).

[39] I. Marvian, arXiv preprint arXiv:1307.6617 (2013).

[40] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006).

[41] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, Journal of High Energy Physics **2013**, 1 (2013).

[42] E. Verlinde and H. Verlinde, arXiv preprint arXiv:1211.6913 (2012).

[43] P. Hayden and J. Preskill, Journal of High Energy Physics **2007**, 120 (2007).

[44] L. Mandelstam and I. Tamm, J. Phys.(USSR) **9**, 1 (1945).

[45] N. Margolus and L.B. Levitin, **120**, 188 (1998).

[46] D. Mondal, C. Datta, and S. Sazim, Physics Letters A **380**, 689 (2016).

[47] P. Cappellaro, in *Quantum State Transfer and Network Engineering* (Springer, 2014), pp. 183–222.

[48] P. Cappellaro, C. Ramanathan, and D. G. Cory, Phys. Rev. Lett. **99**, 250506 (2007).

[49] H. Cho, P. Cappellaro, D. G. Cory, and C. Ramanathan, Physical Review B **74**, 224434 (2006).

[50] I. Frérot and T. Roscilde, p. 16 (2015), 1509.06741, URL `http://arxiv.org/abs/1509.06741`.

[51] W. Ketterle and H.-J. Miesner, Physical Review A **56**, 3291 (1997).

[52] F. Levi, S. Mostarda, F. Rao, and F. Mintert, Reports on Progress in Physics **78**, 082001 (2015).

[53] M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik, The Journal of chemical physics **129**, 174106 (2008).

[54] P. Rebentrost, M. Mohseni, I. Kassal, S. Lloyd, and A. Aspuru-Guzik, New Journal of Physics **11**, 033003 (2009).

[55] G. Karpat, B. Çakmak, and F. F. Fanchini, Physical Review B **90**, 104431 (2014), ISSN 1098-0121, URL `http://journals.aps.org/prb/abstract/10.1103/PhysRevB.90.104431`.

[56] A. Malvezzi, G. Karpat, B. Çakmak, F. Fanchini, T. Debarba, and R. Vianna, arXiv preprint arXiv:1602.03731 (2016).

[57] E. Joos, H. D. Zeh, C. Kiefer, D. J. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the appearance of a classical world in quantum theory* (Springer Science & Business Media, 2013).

[58] A. Peres and P. F. Scudo, arXiv preprint quant-ph/0201017 (2002).

[59] I. Marvian and R. W. Spekkens, arXiv:1104.0018 (2011).

[60] Y. Yang and G. Chiribella, arXiv:1502.00259 (2015).

[61] J. Å berg, Phys. Rev. Lett. **113**, 150402 (2014), ISSN 1079-7114, URL `http://journals.aps.org/prl/abstract/10.1103/PhysRevLett.113.150402`.

[62] A. Winter and D. Yang, arXiv preprint arXiv:1506.07975

(2015).
[63] A. Streltsov, arXiv preprint arXiv:1511.08346 (2015).
[64] D. Girolami and B. Yadin, arXiv preprint arXiv:1509.04131 (2015).
[65] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, arXiv preprint arXiv:1510.06179 (2015).
[66] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[67] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, arXiv preprint arXiv:1601.03781 (2016).
[68] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, arXiv preprint arXiv:1601.03782 (2016).
[69] F. G. Brandao and M. B. Plenio, Nature Physics **4**, 873 (2008).
[70] B. Yadin and V. Vedral, arXiv preprint arXiv:1407.2442 (2014).
[71] D. A. Lidar and K. B. Whaley, *Decoherence-Free Subspaces and Subsystems*, vol. 622 of *Lecture Notes in Physics* (Springer Berlin / Heidelberg, 2003).
[72] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
[73] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge Series on Information and the Natural Sciences (Cambridge University Press, 2000), ISBN 9780521635035.
[74] M. Keyl and R. F. Werner, Journal of Mathematical Physics **40**, 3283 (1999).
[75] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, science **269**, 198 (1995).
[76] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. Van Druten, D. Durfee, D. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
[77] B. Toloui, G. Gour, and B. C. Sanders, Physical Review A **84**, 022322 (2011).
[78] B. Toloui and G. Gour, New Journal of Physics **14**, 123026 (2012).
[79] J. Aberg, arXiv preprint quant-ph/0612146 (2006).
[80] J. A. Vaccaro, F. Anselmi, H. M. Wiseman, and K. Jacobs, Physical Review A **77**, 032114 (2008).
[81] M. Hayashi, *Quantum Information* (Springer, 2006).
[82] E. P. Wigner and M. M. Yanase, Proceedings of the National Academy of Sciences of the United States of America **49**, 910 (1963).
[83] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[84] S. Du and Z. Bai, Annals of Physics **359**, 136 (2015).
[85] S. Bravyi and A. Kitaev, Physical Review A **71**, 022316 (2005).
[86] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Nature **510**, 351 (2014).
[87] D. Gross, Journal of mathematical physics **47**, 122107 (2006).
[88] R. W. Spekkens, Phys. Rev. Lett. **101**, 020401 (2008).
[89] C. H. Bennett and G. Brassard, Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India, 1984 (1984).