Optical-reciprocity-induced symmetry in photonic heterostructures and its manifestation in scattering PT-symmetry breaking

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Optical reciprocity induced symmetry in photonic heterostructures and its manifestation in scattering $\mathcal{PT}$ symmetry breaking

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The scattering matrix S obeys the symmetry property $\mathcal{PT}S\mathcal{PT} = S^{-1}$ in a Parity-Time ($\mathcal{PT}$) symmetric system and the unitary relation $S^\dagger S = 1$ in the absence of gain and loss. Here we report a different symmetry relation of S in a one-dimensional heterostructure, which is given by the amplitude ratio of the incident waves in the scattering eigenstates. It originates from the optical reciprocity and holds independent of the presence of gain and loss in the system. Guided by this symmetry relation, we probe the reminiscence of the spontaneous symmetry breaking of a $\mathcal{PT}$-symmetric S matrix, when the system does not have exact $\mathcal{PT}$ symmetry due to unbalanced gain and loss and even in the absence of gain. We show that the additional symmetry relation provides a clear evidence of a quasi-transition, even when all previously found signatures of the $\mathcal{PT}$ symmetry breaking of S are completely erased.

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Parity-Time ($\mathcal{PT}$) symmetric optical systems have attracted growing interest in the past few years. These systems are non-Hermitian due to the presence of gain and loss, which are delicately balanced such that the refractive index satisfies $n(x) = n^*(-x)$ with respect to a symmetry plane at $x = 0$. The plethora of findings in these systems are tied to the spontaneous symmetry breaking of the effective Hamiltonian at an exceptional point (EP) [1–12]. This spontaneous symmetry breaking was first suggested in non-Hermitian quantum mechanism [14–16] and later realized in wave propagation in the paraxial regime [17–22], which takes the system from a regime of real energy eigenvalues to complex conjugate pairs of eigenvalues. It has been shown that qualitatively similar behaviors exist even when such systems do not have exact $\mathcal{PT}$ symmetry, which leads to, for example, enhanced transmission with increased loss [8], reduced lasing emission with increased gain [10–12] and other interesting phenomena [13].

Recently another type of spontaneous $\mathcal{PT}$ symmetry breaking was found for the scattering matrix of a $\mathcal{PT}$-symmetric system [23], independent of its shape and dimension: the eigenvalues of the scattering (S) matrix can remain on the unit circle in the complex plane, conserving optical flux despite the non-Hermiticity; the symmetry breaking results in pairs of scattering eigenvalues with inverse moduli [23–25]. However, unlike in previous studies of the effective Hamiltonian, it was believed that all signatures of this symmetry breaking are erased if $\mathcal{PT}$ symmetry is non-exact, e.g., in the presence of unbalanced gain and loss [26]. Given the difficulty of maintaining an exact $\mathcal{PT}$ symmetry and dealing with a strong optical gain, the second kind of $\mathcal{PT}$ symmetry breaking has not been demonstrated to date among other reasons.

In this report we tackle this outstanding problem from a new perspective, i.e., we search for another symmetry property of the scattering system that can be utilized to reveal the reminiscence of $\mathcal{PT}$ symmetry breaking in the presence of unbalanced gain and loss. The key relation, we find, relies on optical reciprocity [27–29], and more specifically, the identical transmission coefficient through a one-dimensional (1D) photonic heterostructure that is independent of the propagation direction [30]. Optical reciprocity has been known since the early days of electromagnetism, and recently its study has been revitalized in the quest of on-chip optical isolators for optical computing [31–34]. We find that optical reciprocity leads to a novel symmetry property of the S matrix eigenstates in a 1D heterostructure, which is given in terms of the amplitude ratio of the incident waves (referred to as ν below) and independent of $\mathcal{PT}$ symmetry of the system. We note that this new symmetry applies to the scattering eigenstates instead of the system itself. Therefore, it plays a different role than $\mathcal{PT}$ symmetry and cannot become spontaneously broken.

Guided by this relation, we probe the reminiscence of the aforementioned scattering $\mathcal{PT}$ symmetry breaking using the behavior exhibited by ν when the system does not have exact $\mathcal{PT}$ symmetry. When there is $\mathcal{PT}$ symmetry, $|\nu|$ undergoes a bifurcation at the EP where the spontaneous $\mathcal{PT}$ symmetry breaking of the scattering matrix takes place. The EP persists with unbalanced gain and loss and even in the absence of gain, and we show that the additional symmetry relation enables a clear visualization of $|\nu|$ when it undergoes a quasi-bifurcation near the EP, even when all previously found signatures of $\mathcal{PT}$ symmetry breaking are completely erased, including the bifurcation of the moduli of the scattering eigenvalues.

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Finally, we show the existence of a "final" exceptional point in a multi-layer heterostructure, which is attributed to asymmetric reflections from the two sides of the heterostructure.

Before we introduce this optical reciprocity induced symmetry property, it is worth reviewing the spontaneous symmetry breaking of the S matrix in a 1D $\mathcal{PT}$-symmetric heterostructure. The S matrix connects the incident waves to the scattered waves (see the inset in Fig. 1), e.g.,

$$\begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} r_L & t \\ t & r_R \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix} \equiv S \begin{pmatrix} B \\ C \end{pmatrix}, \tag{1}$$

where $t \equiv t_L = t_R$ and $r_{L,R}$ are the transmission and reflection coefficients from the left and right sides [35]. Using the parametrization introduced in Ref. [24], i.e.,

$$S = \frac{1}{a} \begin{pmatrix} ib & 1 \\ 1 & ic \end{pmatrix}, \tag{2}$$

the eigenvalues of the S matrix are given by

$$\sigma_\pm = \frac{i}{2a} \left[(c+b) \pm \sqrt{(c-b)^2 - 4} \right]. \tag{3}$$

When the system is $\mathcal{PT}$-symmetric, b, c are two real parameters and a is complex parameter. They satisfy $|a|^2 - 1 = bc$, which is another way of writing the generalized conservation law $|T - 1| = \sqrt{R_L R_R}$ [24], where $T \equiv |t|^2, R_{L,R} = |r_{L,R}|^2$ are the transmittance and reflectances. One finds $|\sigma_\pm| = 1$ when $|c - b| < 2$, which is the $\mathcal{PT}$-symmetric phase of the S matrix; when $|c-b| > 2$, one finds that $\sigma_\pm$ have the same phase angle but their moduli are no longer 1, which is the $\mathcal{PT}$-broken phase of the S matrix.

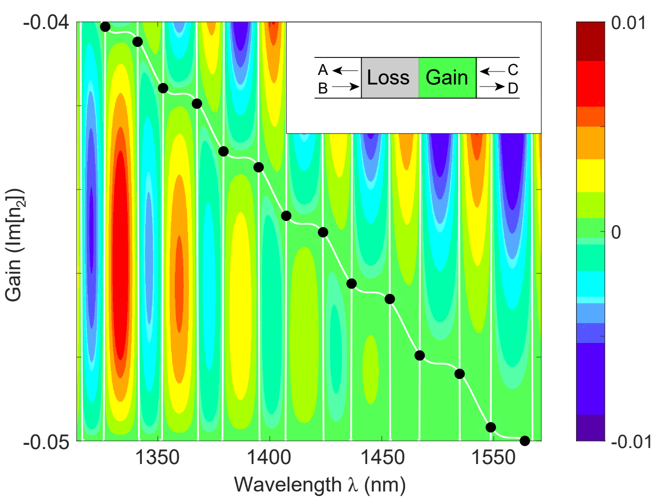

FIG. 1. (Color online) Exceptional points in a two-layer heterostructure (black dots) with weak unbalanced gain and loss. Loss is fixed at $\mathrm{Im}[n_1] = 0.05$ in the left half and the gain is reduced from $\mathrm{Im}[n_2] = -0.05$ to -0.04 in the gain half. $\mathrm{Re}[n] = 3$ in the system and its length is $L = 23\,\mu m$. False color plot of the product $\mathrm{Re}[\mathcal{G} - i]\mathrm{Im}[\mathcal{G} - i]$ is also shown. Nearly vertical and wavy diagonal lines show the zeros of $\mathrm{Re}[\mathcal{G} - i]$ and $\mathrm{Im}[\mathcal{G} - i]$, respectively. Their intersections show the locations of the exceptional points. $\mathcal{G} = -i$ does not hold in this region. Inset: Schematic of scattering from a two-layer heterostructure.

Another manifestation of the spontaneous $\mathcal{PT}$ symmetry breaking, which is more relevant for the additional symmetry of the S matrix we will introduce shortly, is exhibited in the amplitude ratios of the incident waves from the left and right sides in the two scattering eigenstates [i.e., $\nu \equiv B/C = A/D$ in Eq. (1)]. They are given by

$$\nu_\pm = \frac{i}{2} \left[(b-c) \pm \sqrt{(c-b)^2 - 4} \right], \tag{4}$$

which display the same qualitative change as $\sigma_\pm$ when the value of $|c - b|$ crosses 2. The latter is an exceptional point, at which $\sigma_\pm$ coalesce and so do $\nu_\pm$. We note that this condition for an exceptional point, as well as both Eq. (3) and (4), holds even when the system is *not $\mathcal{PT}$-symmetric*, in which case a, b, c are three complex parameters in general.

Now let us return to the $\mathcal{PT}$-symmetric case. It can be easily checked that $|\sigma_+\sigma_-| = 1$ and

$$|\nu_+\nu_-| = 1. \tag{5}$$

These two relations, however, have very different origins. On the one hand, $|\sigma_+\sigma_-| = 1$ holds only when $|a^2| - 1 = bc$, i.e., it is due to $\mathcal{PT}$ symmetry and breaks down when the gain and loss become unbalanced. $|\nu_+\nu_-| = 1$, on the other hand, only requires that the S matrix is a symmetric matrix with two identical off-diagonal elements (i.e., the transmission coefficient t), which is a result of optical reciprocity as we have mentioned in the introduction [27–30]. Since the optical reciprocity holds in general and does not rely on $\mathcal{PT}$ symmetry, $|\nu_+\nu_-| = 1$ holds also with unbalanced gain and loss and even in the absence of gain.

$|\nu_+\nu_-| = 1$ is the symmetry relation that will guide us to probe the reminiscence of the spontaneous $\mathcal{PT}$ symmetry breaking of the S matrix when the system no longer has $\mathcal{PT}$ symmetry. We start by considering the simplest case, a heterostructure with two layers of equal width, the refractive index in which is n_1 and n_2, respectively. The analytical expression of S is given by

$$\boldsymbol{S} = \frac{1}{\mathcal{D}} \begin{pmatrix} \mathcal{G} + i\mathcal{F} & 1 \\ 1 & -\mathcal{G} + i\mathcal{F} \end{pmatrix}, \tag{6}$$

where $\mathcal{D} \equiv c_1c_2 - gs_1s_2 - i(h_1s_1c_2 + h_2s_2c_1)$, $\mathcal{G} \equiv qs_1s_2$, $\mathcal{F} \equiv u_1s_1c_2 + u_2s_2c_1$, and $g = (n_1/n_2 + n_2/n_1)/2$, $q = (n_1/n_2 - n_2/n_1)/2$, $h_j = (n_j + 1/n_j)/2$, $u_j = (n_j - 1/n_j)/2$, $s_j = \sin(n_j\omega L_j/2c)$, $c_j = \cos(n_j\omega L_j/2c)$ $(j = 1, 2)$. ω is the frequency of the incident light, and we note that s_j, c_j are complex if n_j is complex, i.e., when there is gain or loss. The eigenvalues of this S matrix are given by

$$\sigma_\pm = \frac{i\mathcal{F} \pm \sqrt{1 + \mathcal{G}^2}}{\mathcal{D}}, \tag{7}$$

which indicates that if there is an exceptional point, then it occurs at

$$\mathcal{G} = \pm i, \tag{8}$$

where the radicand in Eq. (7) vanishes.

In the $\mathcal{PT}$-symmetric case we have $n_1 = n_2^* \equiv n + i\tau$, and it is straightforward to show that the S matrix given by Eq. (6) satisfies $\mathcal{PT}S\mathcal{PT} = S^{-1}$ [23], or simply $PS^*P = S^{-1}$, using $s_1 = s_2^*$, $c_1 = c_2^*$, $h_1 = h_2^*$, $u_1 = u_2^*$, $\mathrm{Re}[q] = 0$, and $\mathrm{Im}[g] = 0$. The superscript "*" denotes the complex conjugate as usual, and $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the matrix representation of the parity operator $\mathcal{P}$; it exchanges the incoming/outgoing waves on the left side of the heterostructure with those on the right side. We also note that $\mathcal{G} = in\tau|s_1|^2/(n^2+\tau^2)$ is purely imaginary in this case, and hence the above condition (8) for an exceptional point is reachable even if only one system parameter is varied, in contrast to the general requirement of sweeping at least a two-dimensional parameter space in non-$\mathcal{PT}$ systems [2]. This property guarantees the two distinct phases of the S matrix.

In the absence of $\mathcal{PT}$ symmetry, the S matrix still has exceptional points for complex values n_1 and n_2, since it is a non-Hermitian matrix [2]. For example, in Fig. 1 we show a two-layer heterostructure with fixed loss ($\mathrm{Im}[n_1] = 0.05$) in one half and weakly unbalanced gain ($\mathrm{Im}[n_2]$) in the other. We found that its exceptional points are given by $\mathcal{G} = i$ in this regime, which exist at discrete pairs of $\{\mathrm{Im}[n_2], \lambda\}$. $\lambda = 2\pi c/\omega$ is the wavelength in vacuum, and its value at the exceptional points reduces with $\mathrm{Im}[n_2]$. Since $\mathcal{G}$ now is complex in general and has an arbitrary phase angle, it no longer leads to two distinct phases of the S matrix.

To be more specific, we note that $|\sigma_\pm|$ display a bifurcation at an exceptional point when the system is $\mathcal{PT}$-symmetric [see Fig. 2(a)], which delineates the two phases of S mentioned previously. We also note that the relation $|\sigma_+\sigma_-| = 1$ mentioned previously is satisfied in both phases of S. This bifurcation no longer exists when there is a weak imbalance between gain and loss [see Fig. 2(b)], letting alone the case in which there is only loss in the heterostructure. Another indication of the spontaneous $\mathcal{PT}$ symmetry breaking is the transition of the difference $(R_L+R_R)/2-T$ from sub-unitary to super-unitary at an exceptional point [see Fig. 2(c)], which was derived using $|c-b| = 2$ at an exceptional point in Eq. (3) and the $\mathcal{PT}$ symmetry relation $|a|^2 - 1 = bc$ mentioned previously [24]. This signature is also erased completely even when the gain and loss are weakly unbalanced [see Fig. 2(d)].

Now using the symmetry relation (5) of the scattering eigenstates, the spontaneous $\mathcal{PT}$ symmetry breaking of the S matrix can also be visualized as a bifurcation of $|\nu_\pm|$ where the system is $\mathcal{PT}$-symmetric [see Fig. 2(e)]: they are equal in the $\mathcal{PT}$-symmetric phase and reciprocal of each other in the broken-$\mathcal{PT}$ phase. This behavior survives qualitatively when there is a weak imbalance between gain and loss, as we show in Fig. 2(f). We note that

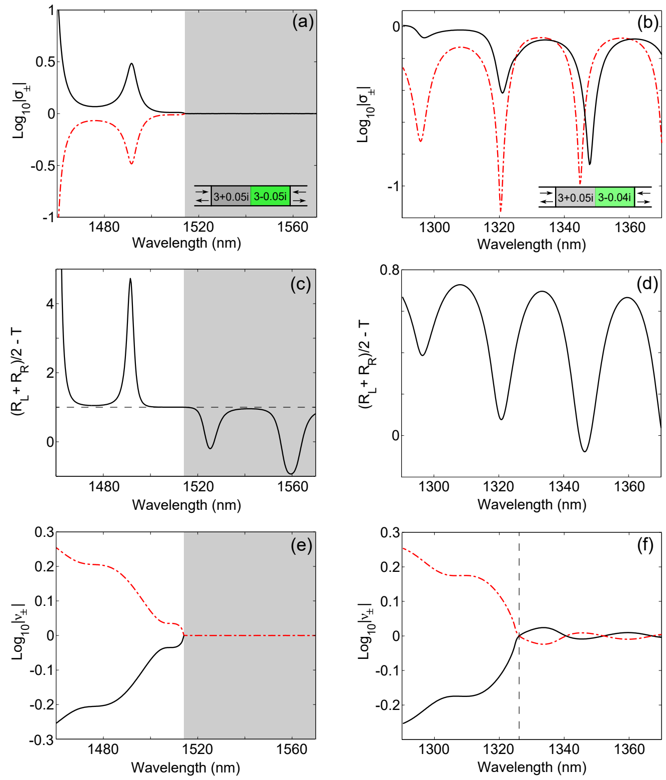

FIG. 2. (Color online) Contrast of the scattering behaviors when gain and loss are balanced (a,c,e; $\mathrm{Im}[n_2] = -\mathrm{Im}[n_1] = -0.05$) or weakly unbalanced (b,d,f; $\mathrm{Im}[n_1] = 0.05$ and $\mathrm{Im}[n_2] = -0.04$). $\mathrm{Re}[n] = 3$ and $L = 23\,\mu m$ as in Fig. 1. Shadowed areas in (a,c,e) indicate the broken symmetry phase. Dashed line in (f) marks the wavelength of the closest exceptional point at $\{\mathrm{Im}[n_2] = -0.0401, \lambda{=}1326 \text{ nm}\}$ shown in Fig. 1. Note that we have chosen $\mathrm{Im}[n_2]$ to be slightly different from the value of this EP intentionally, since the exact value of the latter is unknown in an actual setup before we conduct the experiment.

the quasi-transition point shown in Fig. 2(f) moves to a shorter wavelength with $\mathrm{Im}[n_2] = -0.04$ when compared with the $\mathcal{PT}$-symmetric case (where $\mathrm{Im}[n_2] = -0.05$). This is due to the blueshift of the exceptional point with reduced gain mentioned above (see Fig. 1). We can also check explicitly that the symmetry relation (5) holds here: it is easy to convince oneself that the eigenstates of the S matrix given by Eq. (6) are the same as those of $\left(\begin{smallmatrix} \mathcal{G} & 1 \\ 1 & -\mathcal{G} \end{smallmatrix}\right)$, and we find

$$\nu_\pm = \mathcal{G} \pm \sqrt{\mathcal{G}^2 + 1}; \tag{9}$$

their product is indeed -1.

As the imbalance between gain and loss increases, so does the amplitude of the oscillations of $|\nu_\pm|$ shown Fig. 2(f). They weaken the distinctiveness of the quasi-transition but do not smear out the latter completely (see Fig. 3(a) at $\mathrm{Im}[n_2] = -0.17$, for example). Interestingly, this observation holds even if the system only has loss, i.e., with both $\mathrm{Im}[n_1], \mathrm{Im}[n_2] > 0$. In Fig. 3(c) we

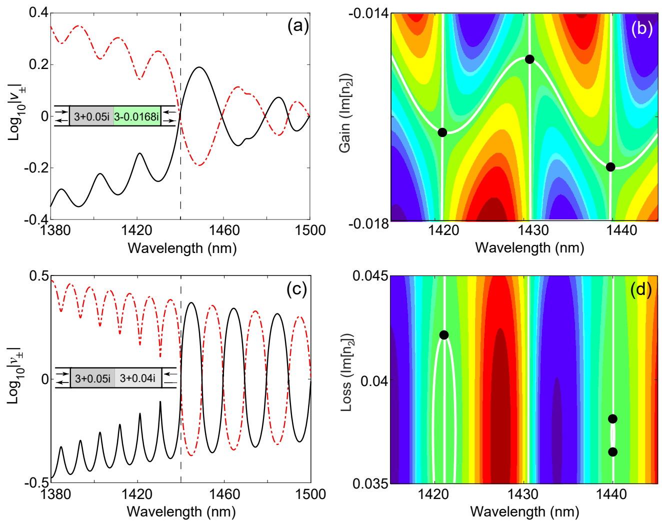

FIG. 3. (Color online) Amplitude ratios ν in the scattering eigenstates of the S matrix with strongly unbalanced gain and loss (a) and loss only (c). $\mathrm{Im}[n_2] = -0.0168$ in (a) and 0.04 in (c), and $\mathrm{Im}[n_1] = 0.05$ is fixed. Dashed lines in (a) and (c) mark the wavelength of the closest exceptional point at $\{\mathrm{Im}[n_2] = -0.0170, \lambda=1440$ nm$\}$ and $\{\mathrm{Im}[n_2] = 0.038, \lambda=1440$ nm$\}$, respectively (see (b) and (d)). System length is chosen to be $L = 36\,\mu$m and the other parameters are the same as in Fig. 1. (b,d) Similar to Fig. 1 for (a) and (c), showing the EPs near the quasi-transition of $|\nu_\pm|$. In (d) $\mathcal{G} - i$ is replaced by $\mathcal{G} + i$.

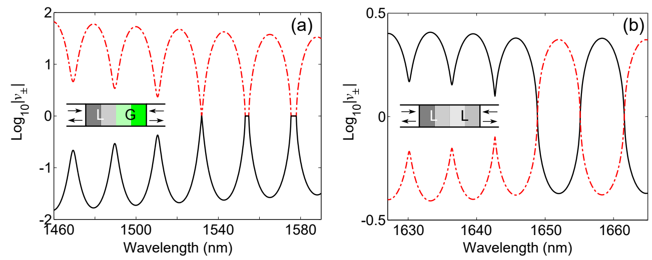

FIG. 4. (Color online) Amplitude ratios ν in the scattering eigenstates of the S matrix in a 4-layer heterostructure. $\mathrm{Re}[n] = 3$ and $L = 72\,\mu$m, and the four layers have equal length. (a) $\mathrm{Im}[n] = 0.05, 0.01, -0.01, -0.05$ for a $\mathcal{PT}$-symmetric heterostructure [37]. (b) $\mathrm{Im}[n] = 0.05, 0.01, 0.005, 0.025$ for a loss-only waveguide.

show the case when $\mathrm{Im}[n_2] = 0.04$, where $|\nu_\pm|$ approach each other and become interwoven beyond an exceptional point. We note that this exceptional point is now given by $\mathcal{G} = -i$, instead of $\mathcal{G} = i$ in the quasi-$\mathcal{PT}$ symmetric case shown in Figs. 1 and 3(b). There is one special point at which the quasi-transition of $|\nu_\pm|$ vanishes completely, that is when $\mathrm{Im}[n_1] = \mathrm{Im}[n_2]$. The system at this point is parity symmetric about the center of the heterostructure ($x = 0$), and the two scattering eigenstates are even and odd functions of x, i.e., $|\nu_\pm|$ is always 1. We note that exceptional points in a loss-only system was previously studied in transmission [8] and reflection [22] experiments.

Next we discuss heterostructures with more than two layers. If the additional layers are identical and attached symmetrically to the two sides of the central region, we find that $\nu_\pm$ do not change their values and hence the quasi-transition of $|\nu_\pm|$ persists, no matter whether the additional layers have gain or loss. This observation can be shown analytically by generalizing the "mirror theorem" in $\mathcal{PT}$-symmetric heterostructures [25], with the central layers now having unbalanced gain and loss. For this purpose we utilize the transfer matrix M, which is defined by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{t} \begin{pmatrix} t^2 - r_L r_R & r_L \\ -r_R & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \equiv M \begin{pmatrix} C \\ D \end{pmatrix} \tag{10}$$

using the same notations as in Eq. (1) for the central region. Likewise, a transfer matrix M_L and M_R can be defined for the added left and right layers, and here they satisfy $PM_LP = M_R^{-1}$, where P is the same matrix representing the parity operator introduced before. The total transfer matrix of the system with the mirrors is then given by $M' = M_L M M_R$. As Eq. (4) shows, $\nu_\pm$ of the central $\mathcal{PT}$-symmetric region only depend on $(c - b)$, or equivalently $\Delta \equiv (r_L - r_R)/t$, which is the sum of the two off-diagonal elements of M in Eq. (10). Therefore, to prove that $\nu_\pm$ do not change with the added mirrors, we only need to show that the sum of the two off-diagonal elements of M', denoted by $\Delta' \equiv (r'_L - r'_R)/t$, equals Δ. It is straightforward to show that $\Delta' = \det(M_R)\Delta$. Since the determinant of a 1D transfer matrix is 1 in general [36], this result concludes our proof.

When the two layers added are different, the S matrix of the $\mathcal{PT}$-symmetric system has multiple regions of symmetric and broken symmetry phases in general [23], each bounded by two exceptional points. The separations of these exceptional points in terms of wavelength are comparable to the oscillation periods of $|\nu_\pm|$ and can be fairly close. Hence these oscillations become more detrimental and obscure the bifurcations of $|\nu_\pm|$. However, in the strong gain/loss limit of a $\mathcal{PT}$-symmetric heterostructure, achieved with either a large τ, a short wavelength, or a long system size, there seems to be a "final" exceptional point, beyond which the system stays in the broken symmetry phase [see Fig. 4(a)]. The existence of this final bifurcation point persists with unbalanced gain and loss and even in the absence of gain [see Fig. 4(b)], similar to the simplest two-layer waveguide discussed above.

This final exceptional point provides a good opportunity to gain a deeper understanding of the correspondence between the scattering behaviors in $\mathcal{PT}$-symmetric and non-$\mathcal{PT}$ heterostructure. As we have discussed, the exceptional points of the S matrix is given by $c - b = \pm 2$ in Eqs. (3) and (4), or equivalently, $r_L - r_R = \pm 2it$. In a $\mathcal{PT}$-symmetric heterostructure, r_L and r_R are in phase if $|t| < 1$ and π out-of-phase otherwise [24]. When combined with a different form of the generalized conservation law, i.e., $|t|^2 - 1 = -r_L^* r_R = -r_L r_R^*$, the above condition for the exceptional points becomes

$$||r_L| - |r_R|| = 2|t|,\ |r_L| + |r_R| = 2 \qquad (\text{if } |t| < 1), \quad (11)$$
$$||r_L| - |r_R|| = 2, \quad |r_L| + |r_R| = 2|t| \quad (\text{if } |t| > 1). \quad (12)$$

For all the final exceptional point in $\mathcal{PT}$-symmetric heterostructures, including those in Fig. 2(e) and 4(a), we always find the first scenario above [i.e., Eq. (11)] to be true, which indicates a significant difference of $|r_L|, |r_R|$ when compared with $|t|$. In other words, it is this asymmetric reflection that leads to the final broken phase of the S matrix in terms of the wavelength. Such asymmetric reflection does occur when the system is not $\mathcal{PT}$-symmetric, for example, when one half of the system has loss and the other half has unbalanced gain, or when the two halves have different average losses. This is especially the case in the short wavelength or large system limit, where the reflection from one side does not "see" the other side of the system and $|t| \to 0$.

In conclusion, we have shown that the optical reciprocity leads to the symmetry relation $|\nu_+\nu_-| = 1$, which holds in all 1D heterostructure. It is accompanied by a bifurcation of $|\nu_\pm|$ in $\mathcal{PT}$-symmetric systems when the spontaneous symmetry breaking of the S matrix takes place, and this bifurcation persists qualitatively for the final exceptional point with unbalanced gain and loss and even in the absence of gain. Since tuning into the scattering eigenstates requires comparing the amplitudes *and* phases of the scattered waves to those of the incident waves, measuring $\nu_\pm$ directly in the scattering eigenstates is rather inconvenient. One alternative is to measure $\nu_\pm$ indirectly using Eq. (4), with b, c replaced by $-ir_L/t, -ir_R/t$. Experimental designs on a silicon platform with Cr/Ge structures on top are currently underway, and the results will be reported elsewhere once properly characterized and measured.

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