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Eigenvalue dynamics in the presence of non-uniform gain and loss

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Loss-induced transmission in waveguides, and reversed pump dependence in lasers, are two prominent examples of counter-intuitive effects in non-Hermitian systems with patterned gain and loss. By analyzing the eigenvalue dynamics of complex symmetric matrices when a system parameter is varied, we introduce a general set of theoretical conditions for these two effects. We show that these effects arise in any irreducible system where the gain or loss is added to a subset of the elements of the system, without the need for parity-time symmetry or for the system to be near an exceptional point. These results are confirmed using full-wave numerical simulations. The conditions presented here vastly expand the design space for observing these effects. We also show that a similarly broad class of systems exhibit a loss-induced narrowing of the density of states.

I. INTRODUCTION

Recently, the study of parity-time (\mathcal{PT}) symmetric optical systems has highlighted the importance of exploring non-Hermitian systems with patterned gain and loss [1–11], and has led to the discovery of a remarkable array of phenomena, such as loss-induced transmission in waveguides [12], unidirectional transport behavior [13–17], reversed pump dependence in lasers [18–20], and band flattening in periodic structures [4, 21–24]. These effects are leading to new possibilities for constructing on-chip integrated photonic circuits for the manipulation of light.

Here, we focus on two of these effects: loss-induced transmission in waveguides, and reversed pump dependence in lasers. Both of these effects are fascinating since they are quite counter-intuitive. They are also of potential practical importance in providing novel mechanisms for optical switching based on gain or loss modulation. In loss-induced transmission, loss is added to one of two otherwise identical, parallel coupled waveguides [12]. After a critical amount of loss is added, further increases in the absorption also increases the total transmission through the waveguide pair. Likewise, reversed pump dependence can be observed in laser systems consisting of two coupled cavities [18–20, 25]. First, one of these cavities is pumped such that the total system begins to lase. However, if the pump in the first cavity is then held constant and the gain in the second cavity is increased, the total output lasing power can be seen to decrease until the total gain distribution becomes relatively uniform, so long as the added gain is sufficiently greater than the losses of the unpumped system. If the laser is close to threshold after gain is added to the first cavity, this mechanism can drive the laser below threshold.

Initially, these types of counter-intuitive effects were found in \mathcal{PT} symmetric systems in their broken phase, and thus the onset of these behaviors was associated with the occurrence of an exceptional point [12]. Subsequent analyses demonstrated that loss-induced transmission in waveguides and reverse pump dependence in lasers could be found in systems which were in the vicinity of an exceptional point, without requiring that the study sys-

tem contain an exceptional point [18–20]. A few recent studies have also interpreted reverse pump dependence in specific laser systems as the result of a loss or gain induced impedance mis-matching, resulting in modes localized to specific regions of the system [25, 26], and have argued that neither an exceptional point nor \mathcal{PT} symmetry breaking are required in order to observe these behaviors. However, given the significance of these counter-intuitive behaviors, it is important to develop a set of general theoretical conditions for a system to exhibit these effects.

In this article, by analyzing the eigenvalue dynamics of the system matrix under modulation, we theoretically prove that both of these phenomena arise in a general class of structures in the presence of non-uniform gain and loss. In particular, loss-induced transmission can be observed in a class of linear systems with dimension no less than two, provided that the following conditions are satisfied:

- 1) The underlying system matrix is irreducible when expressed on the basis set of individual waveguide modes.
- 2) The additional loss modulation is applied only to a subset of the waveguides.

A very similar set of theoretical conditions can be developed for reversed pump dependence in lasers, which are then verified using full-wave simulations. The conditions here are applicable to the known systems which display these effects, but moreover point to a much wider class of systems which exhibit these behaviors without \mathcal{PT} symmetry, or a requirement for the system to be close to an exceptional point. Finally, we show that systems undergoing a spatially non-uniform modulation of loss can also display a loss-induced Purcell effect, where the density of states spectrum of the system narrows as the loss increases. This has potential applications in building narrow-band absorbing and emitting structures.

II. LOSS-INDUCED TRANSMISSION IN WAVEGUIDES

To illuminate the general theoretical principles, we will first consider an example of loss-induced transmission in

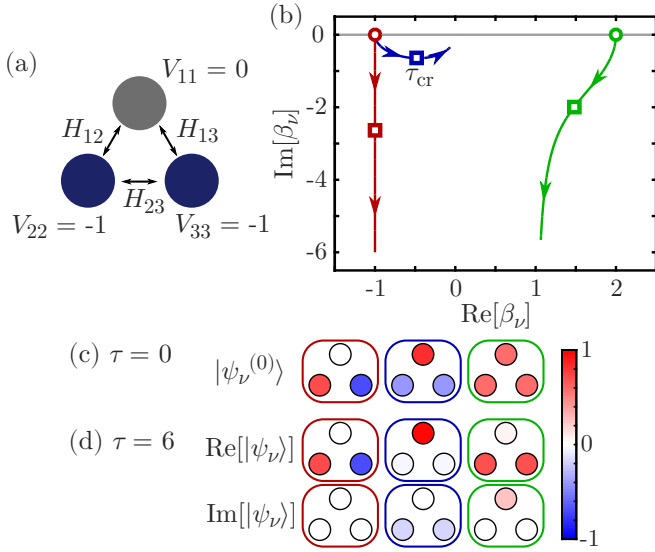


FIG. 1. (a) Schematic of three equally coupled, $H_{nm}^{(0)} = 1$ for $n \neq m$, identical and otherwise lossless, $H_{nn}^{(0)} = 0$, waveguides with $V_{11} = 0$, $V_{22} = V_{33} = -1$. (b) Motion of the eigenvalues, β_ν , through the complex plane as $\tau \in [0, 6]$ is increased, as indicated by the arrows. The circles (squares) indicate the locations of the eigenvalues when $\tau = 0$ ($\tau = \tau_{\text{cr}} \approx 2.64$). (c),(d) Representation of the modes of the three-waveguide systems when $\tau = 0$ (c), and $\tau = 6$ (d). Eigenstate border colors correspond to the eigenvalue trajectories in (b).

a system consisting of three equally coupled, identical, single-mode waveguides in which two of the waveguides can have their loss tuned externally, shown in Fig. 1(a). The system dynamics can then be written in terms of the individual waveguide modes using coupled mode theory as [27]

$$\frac{1}{i} \frac{d}{dz} |c(z)\rangle = H |c(z)\rangle = \left(H^{(0)} + i\tau V \right) |c(z)\rangle. \quad (1)$$

Here, $c_n(z)$ is the modal amplitude on the n th waveguide, and the evolution of the total system is given by H . The underlying system is described by the symmetric matrix $H^{(0)}$, in which the diagonal entries represent the propagation constants of the decoupled waveguides, and the off-diagonal entries represent the coupling strengths between the waveguides. Furthermore, we assume that $H^{(0)}$ is not block-diagonal, i.e. irreducible. The additional loss modulation is described by $i\tau V$, where V is a real diagonal matrix representing the relative strength of the tunable loss in each waveguide, and $\tau \geq 0$ provides an overall scaling to the loss. For the system shown in Fig. 1(a), $V_{11} = 0$, while $V_{22} = V_{33} = -1$. Thus, the additional loss is only applied to a sub-space in the system, spanned by two of the individual waveguide modes. This system therefore satisfies our stated conditions for loss-induced transmission.

The system in Fig. 1(a) indeed exhibits loss-induced transmission. To illustrate, we set $|c(z)\rangle = e^{i\beta z} |\psi\rangle$, and solve Eq. (1) for the propagation eigenvalues, β_ν , and

corresponding eigenstates, $|\psi_\nu\rangle$, of the total system as we vary the loss modulation strength τ . The eigenvalue trajectories for this system are shown in Fig. 1(b) for $\tau \in [0, 6]$. At $\tau = 0$, all of the eigenvalues reside on the real axis. Then, as τ is increased from zero, all of the eigenvalues initially move into the negative half of the complex plane, demonstrating that all of the modes of the system experience increasing loss for increasing τ . However, beyond a critical value, τ_{cr} , one of the eigenvalues begins to return to the real axis. Thus, for values of $\tau > \tau_{\text{cr}}$, this system exhibits loss-induced transmission, as the total transmission through the system becomes dominated by the transmission through the lossless waveguide.

Previously, the onset of loss-induced transmission at $\tau = \tau_{\text{cr}}$ has been associated with the proximity of the system to, though not necessarily at, an exceptional point, where two of the eigenvalues of the system coincide and their corresponding eigenvectors become identical and self-orthogonal [12]. Certainly, if one enlarged the parameter space by varying another component(s) of the system, it is possible to find exceptional points. For example, the system shown in Fig. 1(a) can be tuned to contain an exceptional point by decoupling the two lossy waveguides, $H_{23}^{(0)} = H_{32}^{(0)} = 0$, or detuning the lossless waveguide from the lossy waveguides, $H_{11}^{(0)} = 1$. The eigenvalue trajectories for these two systems as τ is increased are shown in the appendix, and other possible permutations containing exceptional points are likely to exist. It is conceivable that the loss-induced transmission seen in Fig. 1(b) can be explained using an analytic continuation in parameter space from an exceptional point(s), but such a theory has not yet been established. Furthermore, the analysis presented here provides a simpler explanation for the observed loss-induced transmission.

Motivated by the specific example above, we now provide a proof of the general theoretical condition for systems exhibiting loss-induced transmission independent of any symmetry considerations. We consider a general system as described by Eq. (1) with N coupled elements. The underlying system is assumed to either be lossless, or contain an overall uniform loss (or gain), such that $H^{(0)} = \text{Re}[H^{(0)}] + i\delta I$, with $\text{Re}[H^{(0)}]$ symmetric. Thus, the eigenstates of $H^{(0)}$, $|\psi_\nu^{(0)}\rangle$, are entirely determined by $\text{Re}[H^{(0)}]$,

$$\text{Re}[H^{(0)}] |\psi_\nu^{(0)}\rangle = \text{Re}[\beta_\nu^{(0)}] |\psi_\nu^{(0)}\rangle, \quad (2)$$

and its eigenvalues, $\beta_\nu^{(0)}$, all lie on the line $\text{Im}[\beta_\nu^{(0)}] = \delta$ in the complex plane. As loss is only added to a sub-space of the system, V is negative semi-definite in addition to being real diagonal. Moreover, by our condition 2 above, V has a non-trivial kernel. Defining $K = \dim[\ker[V]]$, we then have $K \geq 1$. Here, K represents the number of elements in the system that are individually unaffected by the addition of loss. To both avoid assumptions upon the magnitude of the elements of $H^{(0)}$ and for semantic

convenience, we will use small and large values of τ to refer to the regimes where either the term $H^{(0)}$ or the term $i\tau V$ dominates in H , respectively.

To demonstrate that the system as described above always exhibits loss-induced transmission, we will prove three propositions:

- 1) For small τ , at least a subset, if not the full set, of the eigenvalues of H , which satisfy $\text{Im}[\beta_\nu] = \delta$ at $\tau = 0$, descend from this line as τ is increased. Let L be the number of elements of this subset, $L \leq N$.
- 2) For large τ , there are K eigenvalues of H which remain on, or return to, the line $\text{Im}[\beta_\nu] = \delta$.
- 3) The number of eigenvalues which remain on the line $\text{Im}[\beta_\nu] = \delta$ as τ is initially increased is strictly less than K , i.e. $N - L < K$.

With these three propositions, at least one of the system's eigenvalues which is initially affected by the addition of loss must eventually return to the line $\text{Im}[\beta_\nu] = \delta$, resulting in loss-induced transmission.

When τ is small, the eigenvalues of H can be calculated perturbatively from the eigenvalues and eigenstates of $H^{(0)}$ as

$$\beta_\nu \approx \text{Re}[\beta_\nu^{(0)}] + i\delta + i\tau \langle \psi_\nu^{(0)} | V | \psi_\nu^{(0)} \rangle. \quad (3)$$

As V is negative semi-definite, there will be $L \leq N$ eigenvalues for which $|\psi_\nu^{(0)}\rangle \notin \ker[V]$ such that $\text{Im}[\beta_\nu] < \delta$ as τ is initially increased from zero. Thus we have proved proposition 1 above.

When τ is large, the eigenvalues of the system can be calculated perturbatively from the eigenvalues, v_ν , and eigenstates, $|\varphi_\nu\rangle$ of V , as

$$\beta_\nu \approx i\tau v_\nu + \langle \varphi_\nu | \text{Re}[H^{(0)}] | \varphi_\nu \rangle + i\delta. \quad (4)$$

As V is both real diagonal and negative semi-definite, $v_\nu \leq 0$, $\text{Im}[v_\nu] = 0$, and $|\varphi_\nu\rangle \in \mathbb{R}$. Thus, $\text{Im}[\beta_\nu] \rightarrow -\infty$ as $\tau \rightarrow \infty$ unless $v_\nu = 0$. But, as there are K eigenvalues of V with $v_\nu = 0$, there must be K eigenvalues of H with $\text{Im}[\beta_\nu] = \delta$. Thus we have proved proposition 2 above. The observed splitting of eigenvalues in the large τ limit has been previously referred to as modal dichotomy [28, 29]. Modal dichotomy theory also predicts that transmitting eigenstates exist in $\ker[V]$ for large τ , and avoid the absorption in the system, while decaying eigenstates cannot overlap with the lossless regions, confirmed here in Fig. 1(d). This predicted behavior is similar to the eigenstates observed in \mathcal{PT} symmetric systems in their broken phase [8].

At this point, we are guaranteed that $N - L \leq K$, but for loss-induced transmission to occur at large τ , we must show $N - L < K$. Suppose $N - L = K$. Then there are K eigenstates of $H^{(0)}$ which exist in $\ker[V]$. Moreover, as the eigenstates of $H^{(0)}$ are linearly independent, these K eigenstates must span $\ker[V]$. This means that $H^{(0)}$ can be written as a block diagonal matrix with at least two blocks. However, this contradicts our assumption that $H^{(0)}$ is irreducible. Therefore, there must always be fewer than K eigenstates of $H^{(0)}$ which exist in $\ker[V]$. As such,

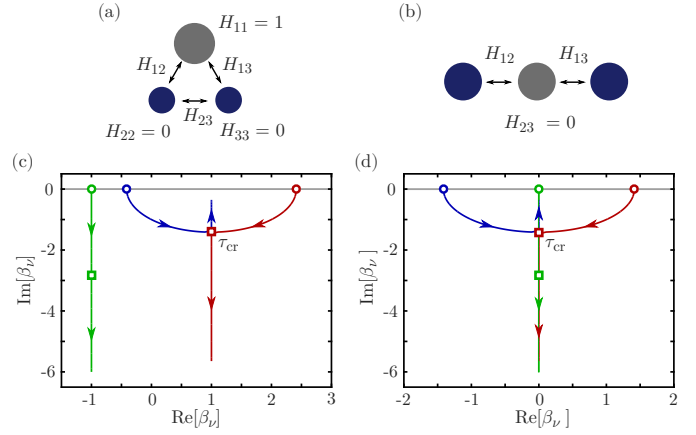


FIG. 2. (a) Schematic of three equally coupled, $H_{nm}^{(0)} = 1$ for $n \neq m$, and otherwise lossless waveguides, where one of the waveguides has a greater decoupled propagation constant, $H_{11}^{(0)} = 1$, $H_{22}^{(0)} = H_{33}^{(0)} = 0$. Loss is then added to the two identical waveguides, $V_{11} = 0$, $V_{22} = V_{33} = 1$. (b) Schematic of three coupled waveguides, $H_{12}^{(0)} = H_{13}^{(0)} = 1$, $H_{23}^{(0)} = 0$, which are otherwise identical and lossless, $H_{nn}^{(0)} = 0$. Loss is then added to the two outer waveguides, $V_{11} = 0$, $V_{22} = V_{33} = 1$. (c),(d) Motion of the eigenvalues, β_ν , through the complex plane as $\tau \in [0, 6]$ is increased, as indicated by the arrows. The circles (squares) indicate the locations of the eigenvalues when $\tau = 0$ ($\tau = \tau_{\text{cr}} \approx 2.83$).

at least one of the eigenstates of H which is initially not in the kernel of V must evolve into an eigenstate which is in the kernel of V as τ increases, and exhibit loss-induced transmission. This proof shows that the onset of loss-induced transmission can be found in a general class of systems, and without searching the parameter space of these systems for exceptional points.

In many previous works the onset of counter-intuitive behavior has been associated with the system being at, or close to, an exceptional point [12, 18–20]. However, the coupled waveguide system studied in Fig. 1(a) is equidistant in parameter space from at least two other systems which display an exceptional point as τ is increased. Figure 2(a) shows a system in which the two lossy waveguides are decoupled, $H_{23}^{(0)} = H_{32}^{(0)} = 0$, and Fig. 2(b) shows a second where the lossless waveguide is not identical to the lossy waveguides, $H_{11}^{(0)} = 1$, but these systems are otherwise identical to the system considered in Fig. 1(a). As can be seen in Figs. 2(c)-(d), both of these systems exhibit an exceptional point as τ is increased.

Thus, to claim that the loss-induced transmission seen in Fig. 1(b) is due to an exceptional point, one then must provide a reason for choosing a specific exceptional point in the broader parameter space of systems as being responsible for this behavior. While this is a seemingly straightforward procedure in systems that have been designed to be close to a particular exceptional point [12, 18–20], in this case there is not an obvious choice. Both of the altered systems of Figs. 2(a)-(b) are unit 1

away from the original system in parameter space. (Note that $H^{(0)}$ is symmetric, so there is only a single degree of freedom changed in the system of Fig. 2(b).) Therefore, without another criteria to select a particular exceptional point, it is unclear how the behavior of the original system can be understood in terms of a single exceptional point.

III. REVERSE PUMP DEPENDENCE IN LASERS

The same mathematical argument given in the previous section can be used to show that reversed pump dependence [18–20] can be found in laser systems with at least two coupled cavities, provided that they satisfy the conditions:

- 1) The underlying system is irreducible when expressed on a basis set of individual cavity modes.
- 2) The gain is added in two parts. First, gain is only increased in a subset of the coupled cavities. Second, gain is added to the remainder of the cavities in system, until the gain in the system is uniform.
- 3) Sufficient gain is added to reach the large τ limit.

To demonstrate the connection with loss induced transmission in waveguides, we decompose a laser system satisfying the above criteria into an underlying system with large uniform gain as described by a matrix $H^{(0)}$ with $\delta > 0$, and an additional loss modulation term $i\tau V$. As reversed pump dependence can be equivalently described as the *increase* of lasing power with the increase of the loss strength τ for some range of τ , it will naturally occur provided that both $H^{(0)}$ and V satisfy the conditions prescribed for loss induced transmission. To see that conditions 2 and 3 above are consistent with the second condition for loss induced transmission, we note that by condition 2 the loss modulation on top of the underlying lasing system is applied only to a subset of the cavities. Furthermore, condition 3 ensures that when only a subset of the system has gain, i.e. after the ‘first’ gain has been added but before the ‘second,’ the system is in the large τ limit. Therefore, any lasing system satisfying conditions 1-3 above must exhibit reversed pump dependence.

In applying the conditions above for real laser systems we have made two simplifications, that the addition of uniform gain effects all of the system’s modes equally, and that the system is linear, which we must now address. The first simplification was made to ensure that all of the resonances of the underlying system, $\omega_\nu^{(0)}$, lie along the line $\text{Im}[\omega_\nu^{(0)}] = \delta$. However, in many cases this condition can be relaxed. To demonstrate this, we perform finite-difference frequency-domain (FDFD) simulations of a one-dimensional system consisting of three cavities of equal lengths, coupled together by air gaps in between them, as shown in Fig. 3. First, only the center cavity is pumped, decreasing the imaginary part of the refractive index from the background in the system, n_{bg} ,

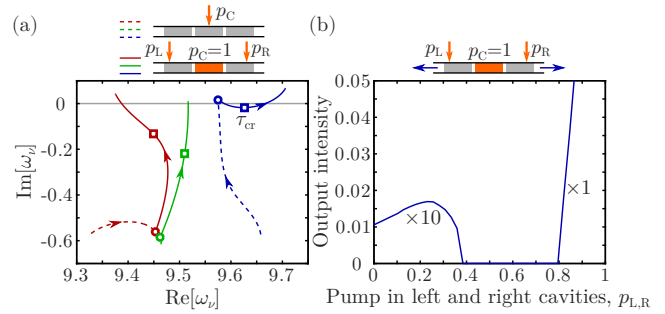


FIG. 3. (a) Motion of the resonances, ω_ν , for three coupled cavities, each with $L = 1\text{mm}$ and unpumped refractive index $n_{\text{bg}} = 3 + 0.163i$, separated by air gaps with $L = 100\mu\text{m}$, as the total gain is increased. First, the gain in the center cavity is increased to $n_{\text{fp}} = 3 - 0.037i$ holding the gain in the left and right cavities constant, $p_C \in [0, 1]$, $p_{L,R} = 0$ (dashed lines). Then, the gain in the center cavity is held constant while the left and right cavities are pumped to n_{fp} , $p_C = 1$, $p_{L,R} \in [0, 1]$ (solid lines). The circles (squares) show the locations of the resonances at $p_C = 1$, $p_{L,R} = 0$ ($p_C = 1$, $p_{L,R} = 0.6$, which corresponds to τ_{cr}). (b) SALT simulations of the same three cavity system showing the total output intensity as the pump in the left and right cavities is increased from $p_{L,R} \in [0, 1]$. The gain medium is assumed to be homogeneously broadened, with a central frequency of $k_a = 9.6\text{mm}^{-1}$, and width $\gamma_\perp = 0.2\text{mm}^{-1}$. The pump level is chosen to realize the refractive indexes from (a), and n_{fp} corresponds to $D_0 = 1.2$ in SALT units [30–32].

to its fully pumped value, n_{fp} . This process is parameterized by $p_C = \text{Im}[n_C - n_{\text{bg}}]/\text{Im}[n_{\text{bg}}] \in [0, 1]$, where n_C is the refractive index of the center cavity. Then, the left and right cavities are pumped, parameterized by $p_{L,R} \in [0, 1]$ which are similarly defined, until the gain in all three cavities is uniform, $p_C = p_{L,R} = 1$. The motion of the resonances of the system through this pumping process is shown in Fig. 3(a). As can be seen, the system begins to lase, $\text{Im}[\omega_\nu] > 0$, when only the center cavity is pumped, $p_C = 1$ and $p_{L,R} = 0$, but falls below threshold as the gain in the left and right cavities is increased. For example, all the eigenvalues have $\text{Im}[\omega_\nu] < 0$ for $p_C = 1$ and $p_{L,R} = 0.6$. When additional gain is added to the left and right cavities, the system starts lasing again.

Second, the underlying proof for reverse pump dependence assumes a linear system. However, lasers are an inherently non-linear system in which gain saturation both prevents any resonance from moving into the positive half of the complex plane, and also reduces the gain available for additional resonances to reach threshold when the system is lasing. We can account for the effects of gain saturation within the FDFD simulations by using the steady-state *ab initio* laser theory (SALT), which solves the set of self-consistent equations for each steady-state lasing mode coupled together through the saturable gain medium [30–32]. SALT simulations of the three coupled cavity system confirm that lasing ceases and then resumes as gain is added to the left and right cavities, as shown in Fig. 3(b). However, the SALT sim-

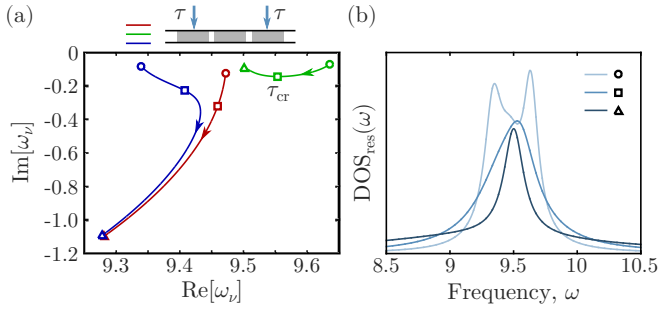


FIG. 4. (a) Motion of the resonances, ω_ν , for three coupled cavities, each with $L = 1\text{mm}$ and refractive index $n = 3 + 0.005i$, separated by air gaps with $L = 125\mu\text{m}$, as the loss in the left and right cavities is increased to $n = 3.02 + 0.336i$. The circles show the locations of the resonances without any additional loss. The squares show the locations of the resonances at τ_{cr} , which corresponds to a refractive index in the left and right cavities of $n = 3 + 0.072i$. The triangles show the locations of the resonances when the refractive index in the left and right cavities is $n = 3.02 + 0.336i$. (b) Plot of the contribution to the density of states, Eq. (5), from the three cavity resonances shown in part (a). The curves show the density of states with the locations of the resonances at the circles (light blue), squares (blue), and triangles (dark blue).

ulations demonstrate that only a single lasing mode is above threshold when the gain in the cavity is uniform, $p_C = p_{L,R} = 1$, rather than all three modes as is predicted in Fig. 3(a). This is due to the gain saturation, which suppresses the motion of the remaining resonances from reaching threshold for the pump values shown.

Together, the theoretical analysis and numerical simulations presented here verify that this system exhibits reverse pump dependence, again without requiring a search of the system parameter space for exceptional points. Thus, two simplifications made in order to arrive at the general theoretical conditions for reversed pump dependence amount to quantitative, not qualitative, corrections, indicating that the simple but intuitive model presented here contains the relevant physics for understanding both loss-induced transmission in waveguides and reverse pump dependence in lasers. While it has been previously shown that exceptional point dynamics are not required to exhibit reverse pump dependence in loss-coupled distributed-feedback lasers [26], the proof presented here demonstrates the breadth of systems which can exhibit this behavior.

IV. LOSS-INDUCED PURCELL EFFECT

The results above indicate that the new physical effects of loss-induced transmission and reversed pump dependence are in general related to spatially non-uniform gain/loss modulation in a photonic structure. We now show that such non-uniform gain/loss modulation may also have important ramifications on the density of states

of a photonic structure. In general, the density of states of a system is proportional to the imaginary part of the Green's function, which is dominated by the resonances of the structure,

$$\text{DOS}_{\text{res}}(\omega) \sim \sum_{\nu} \frac{\gamma_{\nu}}{(\omega - \text{Re}[\omega_{\nu}])^2 + \gamma_{\nu}^2}, \quad (5)$$

where $\gamma_{\nu} = -\text{Im}[\omega_{\nu}]$ is the width of the resonance. When a uniform loss is added to a system, all of the γ_{ν} increase, resulting in the broadening of $\text{DOS}_{\text{res}}(\omega)$, and $\text{DOS}_{\text{res}}(\omega) \rightarrow 0$ as $\gamma_{\nu} \rightarrow \infty$. However, as we have demonstrated above, by adding a non-uniform loss modulation, $i\tau V$, to a loss-less underlying system, $H^{(0)}$, at least one of the resonances which initially broadens for small τ must return to the real axis for large τ , such that $\text{DOS}_{\text{res}}(\omega) > 0$ even as $\tau \rightarrow \infty$, thus exhibiting loss-induced narrowing of the density of states spectrum for some range of τ . Another way to view this is as a loss-induced Purcell effect due to the localization of the eigenstates.

This narrowing of the density of states can be seen in a one-dimensional system consisting of three coupled cavities surrounded by air, depicted in Fig. 4(a). As the loss is increased in the left and right cavities, two of the resonances of the system are seen to descend deep into the lower half of the complex plane, while the remaining resonance initially descends, but then begins to ascend for $\tau > \tau_{cr}$. During this process, the density of states spectrum narrows as the contributions from the two diverging resonances diminish, and as the remaining resonance returns towards the real axis, as shown in Fig. 4(b).

V. CONCLUSION

In conclusion, we have demonstrated that there exists a broad class of systems which exhibit loss-induced transparency in waveguides and reverse pump dependence in lasers. We show that these effects generally arise in a class of systems with non-uniformly modulated gain/loss, and that these systems need not have parity-time symmetry or be close to exceptional points. Moreover, the same class of systems can also exhibit a narrowing of the density of state spectrum with increasing loss. In addition, the criteria for observing bending eigenvalue trajectories in the complex plane for complex symmetric matrices may also have applications in other branches of physics, for example, in the resonance trapping effect observed in open billiard systems [33–36]. Finally, our results demonstrate the breadth of potential device designs with non-uniform gain and loss which can be used to create counter-intuitive optical effects with potentially important applications.

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