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Self-tallying Quantum Anonymous Voting

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Anonymous voting is a voting method of hiding the link between a vote and a voter, the context of which ranges from governmental elections to decision making in small groups like councils or companies. In this paper, we propose a quantum anonymous voting protocol assisted by two kinds of entangled quantum states. Particularly, we provide a mechanism of opening and permuting the ordered votes of all the voters in an anonymous manner; any party, who is interested in the voting results, can acquire a permutation copy, and then obtains the voting result through simple calculation. Unlike all previous quantum works on anonymous voting, our quantum anonymous protocol firstly possesses the properties of privacy, self-tallying, non-reusability, verifiability and fairness at the same time. Besides, we demonstrate that the entanglement of the novel quantum states used in our protocol makes the attack from outside eavesdropper and inside dishonest voters impossible. We also generalize our protocol to execute the task of anonymous multi-party computation, such as anonymous broadcast and anonymous ranking.

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I. INTRODUCTION

Science of cryptography studies how to prevent valuable information from being leaked to unauthorized parties. In practice, most cryptographic protocols are designed to protect message from being eavesdropped by an adversary when they are sent from one party to another. However, in some situations, to keep the identity of message senders private is just as important as to keep the message secret. One example is anonymous voting, in which each voter votes for one of candidates anonymously. Therefore, no one but himself or herself could know which candidate he or she votes. The context of voting ranges from governmental elections to decision making in rather small groups like councils or companies. To be reliable and useful in practice, voting protocols should have some desirable properties (see [1] for more details) like privacy, non-reusability, verifiability, fairness and eligibility as follows.

(1) *Privacy*. Only the individual voter knows how he or she votes.

(2) *Non-reusability*. Each voter can vote only once and cannot change the vote of someone else.

(3) *Verifiability*. Each voter can verify whether his or her vote has been counted properly, but cannot prove to anyone else how he or she is voting.

(4) *Fairness*. Nobody can obtain a partial vote tally before the end of the protocol.

(5) *Eligibility*. Only eligible voters can vote.

In the past decades, a number of voting protocols pursuing the above properties have been proposed. The first voting protocol to guarantee voting privacy was proposed by Chaum in 1981 [2]. Since then various voting protocols based on some cryptographic primitives, such as homomorphic encryption and blind signature, were proposed. Most of these voting protocols adopt public-key cryptographic primitives like large integer factorization and discrete logarithm. However, with the advent of quantum algorithm, they are no longer security anymore [4, 5]. To battle with the power of quantum computer, quantum cryptography was born to encrypt information based upon principle of quantum mechanics. Surprisedly, some of these fundamental principles like no-cloning theorem and the observer effect could guarantee unconditional security. Since the first quantum key distribution protocol was proposed in 1984 by Bennett and Brassard [6], a variety of quantum cryptographic protocols have been proposed, including those for key distribution [7], secret sharing [8, 9], coin flipping [10, 11], private query [12–15], and so on.

In recent years, researchers have investigated how to use quantum mechanics to preserve the anonymity of senders and receivers in communication tasks. The first quantum protocol to anonymously broadcast classical bits and qubits was proposed by Christandl and Wehner [16]. Subsequently, much attention has been paid to perform anonymous voting by using quantum principle. In 2007, Vaccaro et al presented a quantum anonymous voting protocol [3]. Subsequently, several quantum anonymous voting protocols [17–19] based on entangled states were put forward. Afterwards, Horoshko and Kilin [20] gave a quantum anonymous voting protocol which simply utilized single-particle qubit states to vote and Bell states to check the anonymity. More recently, a series of quantum anonymous voting protocols based on continuous variables have been proposed[21]. However, these protocols are function-limited from two aspects: (1) most of them only consider two candidates; (2) most of them

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are designed to achieve only the property of privacy and the other properties are rarely pursed. In special, the property of self-tallying proposed in classical voting protocol by Kiayias and Yung [22] makes anyone who is interested in the voting result can tally votes by himself or herself. The functionality of self-tallying avoids the introducing a third party thus reducing the potential risk of security. As far as we know all previous quantum anonymous voting protocols do not have this property, which needs at least one third party to tally votes, and most of them neglect the cheating of third party, e.g., the third party tampers with the voting results.

Is there a quantum voting protocol which not only overcome the above limitations but also satisfy all these favorable properties. We address this question in this paper. We propose the first quantum anonymous voting protocol for any number of candidates meeting privacy, non-reusability, verifiability, fairness and self-tallying at the same time. With slightly generalization, we show that our protocol can be used for any anonymous multiparty computation (AMC) task. This paper is structured as follows. In Sect. II, we introduce two kinds of entangled quantum states which will be the key resources in our protocol. We present our self-tallying quantum anonymous voting (SQAV) protocol in Sect.III. Then we analyze the security of our protocol in Sect.IV. In Sect.V, we generalize our protocol to AMC and briefly discuss two possible applications. Finally we discuss the properties of self-tallying, non-reusability, verifiability and fairness our protocol satisfied in the Discussion and draw a conclusion in the last section.

II. QUANTUM RESOURCES OF THE PROTOCOL

The security of our SQAV protocol relies on the fact that we use two classes of quantum multiparticle entangled states to distribute the ballot boxes and index numbers to each voter. In this section we introduce these states and some properties of them, which are quite useful in our protocol.

Consider a system in m levels with computational basis $\{|j\rangle_C, j = 0, 1, \cdots, m-1\}$. The fourier basis $\{|j'\rangle_F, j = 0, 1, \cdots, m-1\}$, which can be obtained by applying fourier operation on computational basis, is defined as

$$|j'\rangle_F = \mathcal{F}|j\rangle_C = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} \exp(\frac{2\pi ijk}{m})|k\rangle_C \ . \quad (1)$$

Now we give the first quantum entangled state in our protocol, which has been dexterously applied to implement the tasks of anonymous voting [18] and anonymous ranking [23].

The m level n-particle state $|\mathcal{X}_n\rangle$ is defined as

$$|\mathcal{X}_n\rangle \equiv \frac{1}{m^{\frac{n-1}{2}}} \sum_{\sum_{k=0}^{n-1} j_k \bmod m=0} |j_0\rangle_C |j_1\rangle_C \cdots |j_{n-1}\rangle_C, \quad (2)$$

where $|j_k\rangle$ is the state of jth particle in the computational state and $j_k \in \mathbb{Z}_m := \{0, 1, \cdots, m-1\}$.

$|\mathcal{X}_n\rangle$ has an interesting property that it has the form of GHZ state in the fourier basis,

$$|\mathcal{X}_n\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j'\rangle_F |j'\rangle_F \cdots |j'\rangle_F \ . \quad (3)$$

Therefore $|\mathcal{X}_n\rangle$ has two nice properties. (1) When the state is measured in the computational basis, the summation of the outcomes of all particles modulo m is equal to zero. (2) When the state is measured in the fourier basis, the outcomes of all particles are always the same. To take advantage of the above two properties to protect the voting process being eavesdropped or attacked, we need to use the following result [23].

Theorem 1 *A n-particle m-level quantum state is in the form of $|\mathcal{X}_n\rangle$ if and only if both of the following two conditions are true: (1) when each particle is measured in the computational basis, the sum over all the n measurement outcomes modulo m is equal to zero; (2) when each particle is measured in the fourier basis, the measurement outcomes are all the same.*

The other quantum entangled states we will use in the voting protocol is defined as follows.

A n-level n-particle singlet state $|\mathcal{S}_n\rangle$ is defined as

$$|\mathcal{S}_n\rangle \equiv \frac{1}{\sqrt{n!}} \sum_{S \in \mathcal{P}_n^n} (-1)^{\tau(S)} |s_0\rangle |s_1\rangle \cdots |s_{n-1}\rangle. \quad (4)$$

Here $\mathcal{P}_n^n$ is the set of all permutations of $\mathbb{Z}_n := \{0, 1, \cdots, n-1\}$, S is a permutation (or sequence) in the form $S = s_0 s_1 \cdots s_{n-1}$. $\tau(S)$, named inverse number, is defined as the number of transpositions of pairs of elements of S that must be composed to place the elements in canonical order, $012 \cdots n-1$.

$|\mathcal{S}_n\rangle$ is n-lateral rotationally invariant, which means the measurements of all particles are all different in any basis [24]. In the Appendix. A, we give a proof of this property. Specifically,

$$|\mathcal{S}_n\rangle_C = e^{i\phi} |\mathcal{S}_n\rangle_F \ , \quad (5)$$

where ϕ is a phase factor. This property will be exploited to ensure the security of our voting protocol based on theorem 2.

Theorem 2 *A n-particle n-level quantum state is in the form of $|S_n\rangle$, if and only if the following condition is satisfied: whenever the state is measured in the computational basis or the fourier basis, the permutation of the outcomes of n particles $\{s_0, s_1, \cdots, s_{n-1}\}$ is a random element of the set $\mathcal{P}_n^n$.*

We give a proof of theorem 2 in Appendix. B.

III. QUANTUM ANONYMOUS VOTING PROTOCOL

We first briefly outline our quantum anonymous voting protocol before delving into details. Assume there are n voters labeled as $V_0, V_1, \cdots, V_{n-1}$. Each voter can vote for m candidates labeled by integer $0, 1, \cdots, m-1$. Our protocol consists of three steps. First, a number of n-particle entangled states $|\mathcal{X}_n\rangle$ are distributed to n voters, with each voter holding one particle for each state. After security test for checking eavesdropping, each voter obtains n random numbers, called *ballot numbers*, from n secret "ballot boxes" by measuring left n states $|\mathcal{X}_n\rangle$. Second, a number of n-particle entangled states $|\mathcal{S}_n\rangle$ are distributed to n voters, and each voter also holds one particle for each state. After security test, each voter gets a random number, called *index number*, through measuring left one state $|\mathcal{S}_n\rangle$, which decides which ballot box each voter will use for voting. Finally, each voter casts a vote to his or her indicated ballot box anonymously and all voters open all ballot boxes at the same time. By this method, a random permutation of all votes appears and any party, who is interested in the voting result, can obtain a copy of permutation thus disclosing the voting result. The details of our protocol are presented as follows and the communications in our protocol are shown in Fig. 1.

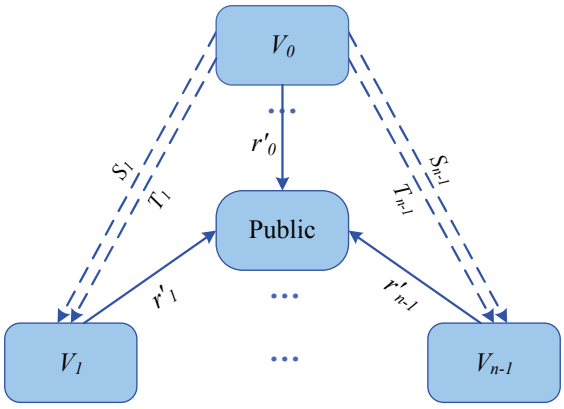

FIG. 1. Communications in our protocol. For simplicity, communications in the eavesdropping checks are not considered. The dashed lines represent quantum channels and the solid lines represent classical simultaneous broadcast channels.

A. Procedure of the protocol

Step 1. Distributing secret ballot boxes.

(1.1) *Prepare quantum states.*
One of n voters is chosen randomly to prepare $n + n\delta_0$ copies of quantum state $|\mathcal{X}_n\rangle$, where δ_0 is the security strength. Without loss of the generality, we assume V_0 is appointed as the distributor. The jth copy of state $|\mathcal{X}_n\rangle$ lives in the Hilbert space of n particles, $p_{j,0}, p_{j,1}, \cdots, p_{j,n-1}$. Therefore we have a *particle matrix*, $p_{j,k}$ with $0 \le j \le n + n\delta_0 - 1, 0 \le k \le n-1$.

(1.2) *Distribute to each voter*
The distributor V_0 sends each column of the particle matrix, $S_k = \{p_{0,k}, p_{1,k}, \cdots, p_{n+n\delta_0-1,k}\}$, to each voter V_k (V_0 keeps S_0).

(1.3) *Security test*
After each voter has received his or her particle sequence, each voter as the checker performs the security check processes to ensure the state distributed is intact. Start from voter V_0 (the order does not matter), then he or she randomly picks out δ_0 particles as the test particles,

$$\vec{p}^{\,0}_{\text{test}} = p_{i_0,0} p_{i_1,0} \cdots p_{i_{\delta_0-1},0} \ . \tag{6}$$

V_0 also needs to choose randomly from computational basis or fourier basis with uniform distribution for each test state, in which he or she will measure his or her test particles with chosen basis. Then he or she publishes the row index of his or her test particles and the measurement basis he or she chosen to do the measurement. After receiving this information, all other voters are required to measure their particles with the same row index,

$$\vec{p}^{\,k}_{\text{test}} = p_{i_0,k} p_{i_1,k} \cdots p_{i_{\delta_0-1},k}, \quad k = 1, 2, \cdots, n-1 \ , \tag{7}$$

in the basis picked by the checker V_0. In other words, the i_0th, i_1th, $\cdots$, i_{δ_0-1}th copies of $|\mathcal{X}_n\rangle$ are samples and measured in either the computational basis or fourier basis. Then all voters send their measurement outcomes to the checker V_0 in the order designed by V_0. Let's label the result of measuring each test particle as $r_{i_j,k}$. If V_0 chooses the computational basis, he or she then needs to check if $\sum_{j=0}^{n-1} r_{i_j,k} \mod m = 0$. If V_0 chooses the fourier basis, he or she needs to verify whether $r_{i_j,0}, r_{i_j,1}, \cdots, r_{i_j,n-1}$ are all same. If the test is failed, V_0 informs the other voters to abort the protocol. If the test is passed, the same test procedure is performed by the next checker. Repeat the same procedure until the test performed by each voter is passed or abort the protocol in some intermediate step.

(1.4) *Generate ballot numbers*
If the security test passes, each voter now has n particles left after discarding all test particles. Each voter then measures his or her left n particles in the computational basis. This will generate n ballot numbers for each voter. Ballot numbers of all voters form a *ballot matrix*, $r_{j,k} \in \{0, 1, \cdots, m-1\}$. The kth column contains n private ballot numbers for V_k. Since the security is passed, each left copy of $|\mathcal{X}_n\rangle$ is intact, according to theorem 1, ballot numbers must satisfy the condition

$$\sum_{k=0}^{n-1} r_{j,k} \mod m = 0 \ . \tag{8}$$

for $j = 0, 1, \cdots, n-1$.

Step 2. Distributing secret indexes.

(2.1) *Prepare quantum states.*
Similarly to step (1.1), one of n voters is chosen randomly to prepare $1 + n\delta_1$ copies of quantum state $|\mathcal{S}_n\rangle$, where δ_1 is the security strength. The jth copy of state $|\mathcal{S}_n\rangle$ lives in the Hilbert space of n particles, $t_{j,0}, t_{j,1}, \cdots, t_{j,n-1}$. Therefore we have a *particle matrix*, $t_{j,k}$ with $0 \leq j \leq n\delta_1, 0 \leq k \leq n-1$.

(2.2) *Distribute to each voter*
The distributor sends each column of the particle matrix, $T_k = \{t_{0,k}, t_{1,k}, \cdots, t_{n\delta_1,k}\}$, to the voter V_k .

(2.3) *Security test*
After each voter has received his or her particle sequence, each voter performs the security check processes to ensure the state distributed is intact. Start from voter V_0 (the order does not matter), then he or she randomly picks out δ_1 particles as the test particles,

$$\vec{t}^{\,0}_{\text{test}} = t_{i_0,0}, t_{i_1,0}, \cdots, t_{i_{\delta_1-1},0} \ . \tag{9}$$

V_0 also needs to choose randomly from computational basis or fourier basis with uniform distribution for each test particle, in which he or she will measure his or her test particle with chosen basis. Then he or she publishes the row index of his or her test particles and the corresponding measurement basis he or she chosen to do the measurement. After receiving this information, all other voters are required to measure their particles with the same row index,

$$\vec{t}^{\,k}_{\text{test}} = t_{i_0,k}, t_{i_1,k}, \cdots, t_{i_{\delta_1-1},k} \ , \tag{10}$$

for $k = 0, 1, 2, \cdots, n-1$ in the basis picked by the checker V_0 and send their measurement outcomes to the checker V_0 in the order appointed by V_0. That is, the i_0th, i_1th, $\cdots$, i_{δ_1-1}th copies of $|\mathcal{S}_n\rangle$ are measured in either the computational basis or the fourier basis. Label the result of measuring each test particle as $d_{i_j,k}$. No matter V_0 chooses the computational basis or the fourier basis, he or she then needs to check if $\{d_{i_j,0}, d_{i_j,1}, \cdots, d_{i_j,n-1}\} \in \mathcal{P}_n^n$ according to theorem 2. If the test passes, the same test procedure is performed by the next checker. If the test fails, V_0 informs the other voters to abort the protocol. The same procedure is repeated until the test performed by each voter is passed or the protocol is aborted in some certain intermediate step.

(2.4) *Generate index numbers*
If the security test passes and then discards all test particles, each voter now has only one particle left. Each voter then measures his or her particle in the computational basis. This will generate an index number for each voter. Index numbers of all voters form an *index array*, $d_k \in \{0, 1, \cdots, m-1\}$. d_k indicates anonymously that d_kth ballot box is the box for V_k to cast vote. Since the security has tested, the only left copy of $|\mathcal{S}_n\rangle$ is intact according to theorem 2. Here $d_0, d_1, \cdots, d_{n-1} \in \mathcal{P}_n^n$.

Step 3. Vote casting.

(3.1) *Vote casting*
After steps 1 and 2, each voter V_k has n ballot numbers, $r_{0,k}, r_{1,k}, \cdots, r_{n-1,k}$, and one index number, d_k. Now voter V_k votes to the candidate $v_k \in \{0, 1, \cdots, m-1\}$, by adding v_k to $r_{d_k,k}$. He or she then renews ballot numbers $r'_{jk} = (r'_{0,k}, r'_{1,k}, \cdots, r'_{n-1,k})$, in which

$$r'_{j,k} = \begin{cases} r_{j,k} + v_k \bmod \text{m} & \text{if } j = d_k \ , \\ r_{j,k} & \text{if } j \neq d_k \ . \end{cases} \tag{11}$$

All voters publish all the updated ballot numbers through simulation broadcast channels [25, 27]. At last we have a *vote matrix*, $r'_{j,k}$, which is available for every party at the same time.

(3.2) *Self-tallying*
With the vote matrix, each party, who is interested in the voting result, can count the votes for each candidate. They take the summation of each row,

$$R_j = \sum_{k=0}^{n-1} r'_{j,k} \bmod m \ , \tag{12}$$

$$= \sum_{k=0}^{n-1} r_{j,k} + v_{k_0} \bmod m \ . \tag{13}$$

Here $d_{k_0} = j$. Therefore $\{R_0, R_1, \cdots, R_{n-1}\}$ is a permutation of the votes $\{v_0, v_1, \cdots, v_{n-1}\}$. The number of votes candidate V_i got is given by

$$N_i = \sum_{R_j = i} 1 \ , \tag{14}$$

for $i = 0, 1, \cdots, m-1$.

(3.3) *Security check*
Each voter V_k needs to verify that $R_{d_k} = v_k$. If the answer is yes, it indicates that his or her vote is counted correctly; otherwise the protocol is aborted since the voting step is compromised.

B. Example

To illustrate the protocol, we give a simple example (see Table. I) with $n = 4$ voters and $m = 3$ candidates. For simplicity, we assume there is no eavesdrop or attack happened. Thus we ignore the security tests (steps (1.3), (2.3) and (3.3)). After executing step 1, suppose ballot matrix held by 4 voters are

$$r_{j,k} = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \ . \tag{15}$$

After step 2, assume the index numbers are

$$(d_0, d_1, d_2, d_3) = (1, 0, 3, 2) \ . \tag{16}$$

Then in step 3, assume the four voters V_0, V_1, V_2 and V_3 cast votes

$$(v_0, v_1, v_2, v_3) = (1, 2, 1, 0). \tag{17}$$

The voting and self-tallying processes are described in Table. I. The final published results are

$$(R_0, R_1, R_2, R_3) = (2, 1, 0, 1) \tag{18}$$

which is indeed a permutation of the votes v_k as we expected.

Example of SQAV

	V_0	V_1	V_2	V_3	R_j
$r'_{0,k}$	0	1+2	2	0	2
$r'_{1,k}$	2+1	2	1	1	1
$r'_{2,k}$	1	0	2	0+0	0
$r'_{3,k}$	0	1	1+1	1	1

TABLE I. A simple example of SQAV with $n = 4$ and $m = 3$. Each voter adds his or her votes to the ballot assigned by his or her index number. The tallying results are calculated according to Eq. (12).

IV. PRIVACY ANALYSIS

Privacy is the primary property of a SQAV protocol. In this section, we focus on discussing the privacy of our SQAV and other properties will be given in section VI. Generally, the top priority is to protect the privacy of each voter. That is, no outsider or voters should know which vote is cast by whom, except the one by himself or herself. In our SQAV, the attacker could be an outside eavesdropper, one dishonest voter [28, 29] or an adversary which includes some dishonest voters. If an attacker successfully eavesdrops the ballot random numbers or index number of the voter V_k without being detected, he or she can easily know which candidate V_k votes for. Therefore, preserving privacy in our SQAV requires to prevent ballot numbers and index numbers from being eavesdropped. The security tests in steps (1.3) and (2.3) are designed to protect the ballot matrix, index array and the voting process from being compromised.

A. Outside eavesdropper

For outside eavesdropper, Eve could intercept S_k or T_k during step (1.2) or (2.2). Let's consider that Eve intercepts arbitrary x particles she would like to in S_k. If $x < n$, then there is a chance that all x particles are happen to be among the n particles which are not included in the tests. Actually the probability of this happening is

$$\begin{aligned} P_e &= \binom{n}{x} / \binom{n + n\delta_0}{x} \\ &= \frac{n!}{(n-x)!} \frac{(n + n\delta_0 - x)!}{(n + n\delta_0)!} \\ &= \prod_{k=n}^{n-x+1} \frac{k}{k + n\delta_0} \qquad (19) \\ &\sim \mathcal{O}((\frac{1}{\delta_0})^x) \ , \qquad (20) \end{aligned}$$

which is approaching to zero if we make the security strength δ_0 large enough. Actually the more particles Eve intercepts, the faster the probability that she could pass the security check goes to zero. Similarly we could argue that the probability of Eve intercepting and modifying T_k in Step 2 without being found is negligible. Therefore, for large enough δ_0, δ_1, the disturbed particles cannot escape from the security tests in steps (1.3) and (2.3).

Let's consider another scenario. Assume Eve intercepts and modifies $p_{j_0,k}$ in S_k thus changing the j_0th copy of $|\mathcal{X}_n\rangle$. Suppose that the new state due to Eve's disturbance is $|\phi_e\rangle$. The probability of all security tests in Step (2.3) are passed is

$$P_e = (\frac{1}{2}P_C + \frac{1}{2}P_F)^{n\delta_0} \ , \tag{21}$$

where

$$P_C = \sum_{\sum_k j_k \bmod m = 0} |\langle \phi_e | j_0, j_1, \cdots, j_{n-1} \rangle_C|^2 \ , \tag{22}$$

$$P_F = \sum_{j=0}^{m-1} |\langle \phi_e | j, j, \cdots, j \rangle_F|^2 \ . \tag{23}$$

Since $\langle \phi_e | \mathcal{X}_n \rangle \neq 1$ according to theorem 1, $P_C + P_F < 1$. Therefore, for large enough δ_0,

$$P_e \to 0 \ . \tag{24}$$

The argument for Eve modifying the index number is similar. Eve cannot pass the security tests if δ_1 is large enough based on theorem 2. In summary, as long as the security strength δ_0, δ_1 are large enough, the attack from outside eavesdropper can be prevented.

B. The dishonest voters cannot eavesdrop the ballot numbers without being detected

In the step 1, to gain the information of ballot numbers of honest voters, the dishonest voters could cooperate to attack the particles during their transmission in step (1.2) and announce the wrong results to avoid being detected by the honest voters in step (1.3). Since V_0 is the only voter who prepares and distributes the quantum states, it seems that V_0 plays a different role from the other voters.

To analyze the possible attacks from dishonest voters in more detail, two cases: (1) V_0 is honest and (2) V_0 is dishonest, are considered.

For the case (1), without loss of generality, we suppose there are l dishonest voters, $V_{i_0}, V_{i_1}, \cdots, V_{i_{l-1}}$. The most general attack by the dishonest voters is that they intercept some particles during the transmission from V_0 to honest voters and then they perform an unitary operation (attack operation) on the intercepted particles and an auxiliary system to yield a new state, denoted by $|\Phi\rangle$, of the composite system. To avoid being detected by the honest voters in step (1.3) when they measure their particles in their hands with the fourier basis and the measurement outcomes are required to be the same, $|\Phi\rangle$ should be in the form

$$|\Phi\rangle = \frac{\sum_{j=0}^{m-1} |j'\rangle_0 |j'\rangle_{j_0} \cdots |j'\rangle_{j_{n-l-2}} |\phi_j\rangle}{\sqrt{m}}, \qquad (25)$$

where $|\phi_j\rangle$ are the states of the composite system of l particles sent from V_0 to the dishonest voters and the auxiliary system (denoted by system E_0), and the subscripts $0, j_0, j_1, \cdots, j_{n-l-2}$ represent the particles held by honest voters $V_0, V_{j_0}, V_{j_1}, \cdots, V_{j_{n-l-2}}$. It can be rewritten in the computational basis as

$$|\Phi\rangle = \sum_{k_0,k_{j_0},\cdots,k_{j_{n-l-2}}=0}^{m-1} \frac{|k_0\rangle|k_{j_0}\rangle\cdots|k_{j_{n-l-2}}\rangle}{m^{\frac{n-l+1}{2}}} \otimes |\varphi_{k_0 k_{j_0}\cdots k_{j_{n-l-2}}}\rangle, \qquad (26)$$

where $|\varphi_{k_0 k_{j_0}\cdots k_{j_{n-l-2}}}\rangle = \sum_{j=0}^{m-1} \exp\left(\frac{2\pi i j(k_0+k_{j_0}+\cdots+k_{j_{n-l-2}})}{m}\right)|\phi_j\rangle$ is the unnormalized state vector of system E_0. The dishonest voters could measure the system E_0 and obtain some $|\varphi_{k_0 k_{j_0}\cdots k_{j_{n-l-2}}}\rangle$ to infer the measurement outcomes $k_0 k_{j_0}\cdots k_{j_{n-l-2}}$ of honest voters in step (1.4). From the form of $|\varphi_{k_0 k_{j_0}\cdots k_{j_{n-l-2}}}\rangle$, it is easy to see that, for any two different outcomes $k_0 k_{j_0}\cdots k_{j_{n-l-2}}$ and $k'_0 k'_{j_0}\cdots k'_{j_{n-l-2}}$ such that $k_0 k_{j_0}\cdots k_{j_{n-l-2}} = k'_0 k'_{j_0}\cdots k'_{j_{n-l-2}} \bmod m$, $|\varphi_{k_0 k_{j_0}\cdots k_{j_{n-l-2}}}\rangle = |\varphi_{k'_0 k'_{j_0}\cdots k'_{j_{n-l-2}}}\rangle$. This means that the dishonest voters can only at most know the information about the sum $k_0 k_{j_0}\cdots k_{j_{n-l-2}} \bmod m$ by measuring the system E_0. However, this attack is trivial in the sense that without any eavesdropping attack the dishonest voter can cooperate to directly infer the sum of measurement outcomes (ballot numbers) of honest voters after executing the step (1.4).

For the case (2) that V_0 is dishonest, we assume there are other l dishonest voters $V_{i_0}, V_{i_1}, \cdots, V_{i_{l-1}}$. The most general attack for them are similar to the case (1). The only difference could be that the dishonest voters can directly prepare and distribute fake states to the honest voters instead of intercepting the particles. To avoid being detected by honest voters, these states should be in the form similar to Eq. (25) or (26). From the above analysis for case (1), it is not hard to draw the same conclusion as case (1) that, in order to avoid being detected, the dishonest voters can only perform a trivial attack to obtain the sum of ballot numbers of the honest voters.

C. The dishonest voters cannot eavesdrop the index numbers without being detected

In step 2, to eavesdrop the information of index numbers of honest voters, the dishonest voters could also attack the particles during their transmissions in step (2.2) and announce the wrong results to avoid being detected by the honest voters in step (2.3). Just as analyzing eavesdropping the ballot numbers in the last subsection, we also consider two cases: (1) V_0 is honest and (2) V_0 is dishonest.

For the case (1), we also assume there are l dishonest voters, $V_{i_0}, V_{i_1}, \cdots, V_{i_{l-1}}$. The most general attack for them is that, they first intercept some transmitted particles in step (2.2), entangle them with an auxiliary system prepared in advance and then return the operated particles to honest voters. The state of the whole composite system is denoted by $|\Psi\rangle$. To elude detection in step (2.3), it is required that all the measurement outcomes should be distinct when measuring each particle held by honest voter in the fourier basis, and thus $|\Psi\rangle$ should be of the form

$$|\Psi\rangle = \sum_{S\in\mathcal{P}_n^{n-l}} \frac{(-1)^{\tau(S)}\mathcal{F}^{\otimes(n-l)}|S\rangle}{\sqrt{|\mathcal{P}_n^{n-l}|}} \otimes |u_S\rangle, \qquad (27)$$

where $S = s_0 s_{j_0}\cdots s_{j_{n-l-2}}$. $|u_S\rangle$ is the state of composite system (denoted by E_1) of l particles sent to the dishonest voters and auxiliary system. $\mathcal{P}_n^{n-l} = \{x_0 x_1\cdots x_{n-l-1} | x_0, x_1, \cdots, x_{n-l-1} \in \mathbb{Z}_n, \forall j \neq k, x_j \neq x_k\}$ is the set of all the $(n-l)$-permutations of $\mathbb{Z}_n$ and $|\mathcal{P}_n^{n-l}| = \frac{n!}{l!}$ is its size. $\mathcal{P}_n^{n-l}$ can be divided into $\binom{n}{n-l} = \frac{n!}{(n-l)!l!}$ subsets, each of which corresponds to the set of all the $(n-l)!$ permutations of a $(n-l)$-combination of $\mathbb{Z}_n$. In addition, any two states $|u_{S_0}\rangle$ and $|u_{S_1}\rangle$ such that $S_0 \in \mathcal{P}_n^{n-l,w_0}$, $S_1 \in \mathcal{P}_n^{n-l,w_1}$ and $w_0 \neq w_1$ should be orthogonal to each other, i.e., $\langle u_{S_0}|u_{S_1}\rangle = 0$. If not, the dishonest voters cannot deterministically know subset $\mathcal{P}_n^{n-l,w}$ in which the honest voters' measurement outcomes are, and thus they cannot announce the correct measurement outcomes to avoid being detected. Rewrite $|\Psi\rangle$ in the computational basis, we have

$$|\Psi\rangle = \frac{n^{-\frac{n-l}{2}}}{\sqrt{|\mathcal{P}_n^{n-l}|}} \sum_{T\in\mathcal{R}_n^{n-l}} |T\rangle \otimes |v_T\rangle, \qquad (28)$$

where $T = t_0 t_{j_0}\cdots t_{j_{n-l-2}}$. $\mathcal{R}_n^{n-l} =$

$\{x_0x_1\cdots x_{n-l-1}|x_0,x_1,\cdots,x_{n-l-1}\in\mathbb{Z}_n\}$ and

$$|v_T\rangle=\sum_{S\in\mathcal{P}_n^{n-l}}(-1)^{\tau(S)}\exp(\frac{2\pi i(s_0t_0+\sum_{k=0}^{n-l-2}s_{j_k}t_{j_k})}{n})|u_S\rangle$$
$$=\sum_w\sum_{S\in\mathcal{P}_n^{n-l,w}}(-1)^{\tau(S)}\exp(\frac{2\pi i(s_0t_0+\sum_{k=0}^{n-l-2}s_{j_k}t_{j_k})}{n})|u_S\rangle.$$

$\{|v_T\rangle\}$ are the unnormalized state vectors of system E_1. To avoid being detected by the honest voters who measure their particles in the computational basis in the step (2.3) and the measurement outcomes are required to be distinct, two conditions should be satisfied: (a) in Eq. (28) there is no terms $|v_T\rangle$ for $T\notin\mathcal{P}_n^{n-l}$, or equivalently, $T\in\mathcal{Q}_n^{n-l}=\{x_0x_1\cdots x_{n-l-1}|x_0,x_1,\cdots,x_{n-l-1}\in\mathbb{Z}_n,\exists j\neq k,x_j=x_k\}$; (b) any two states $|v_{T_0}\rangle$ and $|v_{T_1}\rangle$ for $T_0\in\mathcal{P}_n^{n-l,w_0}$, $T_1\in\mathcal{P}_n^{n-l,w_1}$ and $w_0\neq w_1$ should be orthogonal to each other, i.e., $\langle v_{T_0}|v_{T_1}\rangle=0$. Here we focus on analyzing what $|\Psi\rangle$ (in Eq. (28)) should be to satisfy the condition (a). Since $\langle u_{S_0}|u_{S_1}\rangle=0$ for $S_0\in\mathcal{P}_n^{n-l,w_0}$, $S_1\in\mathcal{P}_n^{n-l,w_1}$ and $w_0\neq w_1$, the condition (a) is equivalent to the one that $\sum_{S\in\mathcal{P}_n^{n-l,w}}(-1)^{\tau(S)}\exp(\frac{2\pi i(s_0t_0+\sum_{k=0}^{n-l-2}s_{j_k}t_{j_k})}{n})|u_S\rangle=0$ for arbitrary w and arbitrary $T\in\mathcal{Q}_n^{n-l}$. To satisfy this condition, for arbitrary w, all the $|u_S\rangle$ such that $S\in\mathcal{P}_n^{n-l,w}$ should be equal (denoted by $|u_w\rangle$), which is implied by the the corollary 1 of Appendix. Thus $|v_T\rangle$ can be rewritten as

$$|v_T\rangle=\sum_w\sum_{S\in\mathcal{P}_n^{n-l,w}}(-1)^{\tau(S)}\exp(\frac{2\pi i(s_0t_0+\sum_{k=0}^{n-l-2}s_{j_k}t_{j_k})}{n})|u_w\rangle. \quad (29)$$

Once the dishonest voters successfully elude the eavesdropping check process in step (2.3), they could measure the system E_1 and get some $|v_T\rangle$ to infer the index numbers $T=t_0t_{j_0},\cdots,t_{j_{n-l-2}}$ of honest voters in step (2.4). However, from the form of $|v_T\rangle$ in Eq. (29), it is easy to verify that for any two sequences T_0, T_1 which are in the same subset $\mathcal{P}_n^{n-l,w}$, $|v_{T_0}\rangle=|v_{T_1}\rangle$. Therefore, the dishonest voters can at most know the information about which subset (i.e., w) the honest voters' index numbers are in. However, this general entangle-measure attack is trivial in the sense that the dishonest can cooperate to obtain this information without any attack.

For the case (2) that V_0 is dishonest, the general attack performed by them would be the same as the case (1) except that the dishonest voters would prepare and distribute the fake states in the form similar to the Eq. (27) to the honest voters instead of intercepting the particles in step (2.2). According to the analysis in case (1), we can conclude that the dishonest voters cannot obtain the index numbers of honest voters without being detected.

V. GENERALIZE TO ANONYMOUS MULTI-PARTY COMPUTATION

One important feature of SQAV is to make each vote open without any relation with any voter. Actually it provides a mechanism to implement a class of multi-party tasks. That is, our protocol can be as useful as for voting as long as a multi-party activity which requires to broadcast the data of each party anonymously. Therefore we define a more general class of problem, anonymous multi-party computation (AMC) as follows.

Definition *Anonymous multi-party computation is a task to compute a function of the form* $f(y_0^0,\cdots,y_0^{i_0-1},y_1^0,\cdots,y_1^{i_1-1},y_{n-1}^0,\cdots,y_{n-1}^{i_{n-1}-1})$ *by* n *parties. The function* f *is invariant under the permutation of integer inputs* $\{y_k^i\}$*. Each party,* P_k*, feeds* $y_k^0,\cdots,y_k^{i_k-1}$ *in the function anonymously and obtains the result without the other person assisted. All the inputs are bounded by* $0\leq y_k<m$*.*

The protocol for AMC is very similar to the SQAV.

Step 1. P_0 prepares $\bar{n}+n\delta_2$ copies of m level n-particle state $|\mathcal{X}_n\rangle$, where $\bar{n}=\sum_{k=0}^{n-1}i_k$. Then P_0 keeps the column S_0 to himself and then distribute S_k to P_k. Here the particle columns S_k are defined as in the step(1.2) of our previous quantum anonymous voting protocol. After distribution, each party P_k executes the security test procedure in step (1.3). If all n tests are passed, each party P_k measures his or her $\bar{n}$ particles so again there is a ballot column

$$r_{j,k}=\begin{pmatrix}r_{0,k}\\r_{1,k}\\\vdots\\r_{\bar{n}-1,k}\end{pmatrix}. \quad (30)$$

Step 2. P_0 prepares $1+n\delta_3$ copies of $|\mathcal{S}_{\bar{n}}\rangle$ and distributes particle columns $T_{\sum_{t=0}^{k-1}i_t},\cdots,T_{\sum_{t=0}^{k}i_t-1}$ to P_k ($k\geq1$), while keeping the particle columns $T_0,\cdots,T_{i_0-1}$. Here the particle columns T_k are defined as in the step(2.2) of our previous quantum anonymous voting protocol. In order to protect from attack, each party is required to choose δ_3 copies of $|\mathcal{S}_{\bar{n}}\rangle$ to exam if $|\mathcal{S}_{\bar{n}}\rangle$ is intact. If all tests are passed, each party P_k measures the remaining particles with computation basis and then there are index arrays $d_{\sum_{t=0}^{k-1}i_t,k},\cdots,d_{\sum_{t=0}^{k}i_t-1,k}$, where $d_{i,k}\in\{0,1,\cdots,\bar{n}-1\}$.

Step 3. Finally each party addes each of his or her data to the ballot number decided by the corresponding index number. And we have a *data matrix* $r'_{j,k}$. Finally every party could calculate

$$R_j=\sum_{k=0}^{n}r'_{j,k}\mod m, \quad (31)$$

$\{R_j\}$ is a permutation of all the data $\bigcup y_j^i$. Therefore all the data are broadcasted anonymously.

Step 4. With holding all data, each party can obtain the result of

$$f(y_0^0,...,y_0^{i_0-1},y_1^0,...,y_1^{i_1-1},y_{n-1}^0,\cdots,y_{n-1}^{i_{n-1}-1})$$

through simple calculation by himself or herself.

Actually, AMC is a subclass of secure multi-party computation (SMC) problem, in which a number of parties also jointly compute a function over their inputs while the inputs are kept private. SMC focuses on function result without publication of all inputs. To illustrate it, we give a simple example in which three parties want to jointly compute the function $f(y_0,y_1,y_2)=y_0+y_1+y_2$ over their inputs y_0, y_1 and y_2. Suppose $y_0=2,y_1=3,y_2=6$, by SMC, they have the result $f(y_0,y_1,y_2)=11$. However, each party can only know the sum of the inputs of the other two parties. By AMC, in addition to obtain the result $f(y_0,y_1,y_2)=11$, every party also gets a permutation of the original inputs of others. For example, $(3,6,2)$ and the index of his or her own input is only known to himself or herself. As a result, P_0 knows $(y_1,y_2)=(3,6)$ or $(6,3)$, P_1 knows $(y_0,y_2)=(2,6)$ or $(6,3)$ and P_2 knows $(y_0,y_1)=(2,3)$ or $(3,2)$. In fact, for some particular tasks, the function result leads to open all inputs. In this sense, there is no difference between AMC and SMC. In the following, we give two examples to explain this.

A. Anonymous broadcast

The simplest application of AMC is to implement anonymous broadcast (AB). AB channels are primitives of many anonymous communication protocols.

An anonymous n-party broadcast task [16] is to publish the datum $y_k\in\{0,1,\cdots,m-2\}$ held by sender P_k anonymously and all parties obtain y_k at the same time. In this scenario, the protocol is basic same as SQAV with m candidate and n voters. If a sender would like to broadcast message y, he or she just needs to 'vote' for the 'candidate' y following the protocol in Sec. III. However, if a party does not want to send any message, he or she just needs to 'vote' for the 'candidate' $m-1$. Finally each $R_k\in\{0,1,\cdots,m-2\}$ will be the message sent by one of the senders. Therefore, each sender broadcasts the intended message anonymously.

B. Anonymous ranking

Anonymous ranking (AR) [23] is an important problem in AMC and has significant practical applications [23]. An AR task generally involves two steps. 1) each party needs to broadcast his or her data $y_k=\{y_k^0,y_k^1,...,y_k^{i_k-1}\}$ to the community anonymously. 2) Each one could rank the published data by himself or herself and obtain the rank of his (her) data anonymously. Obviously the first step could be done safely by using our AMC protocol. Finally, similar to the self-tallying in SQAV, self-ranking is obtained.

VI. DISCUSSION

We discuss in detail how our SQVA ensures privacy in Sec. IV. However, except being able to keep privacy for each voter, our protocol has several other nice properties which are not fulfilled by other existing protocols [3, 17–21] at the same time.

1) *Self-tallying.* In our protocol, any voter or other third party, who is interested in voting results, can tally the votes by himself or herself by counting the votes in $\{R_j\}$ in step (3.2). Through simple calculation, they can obtain the voting result.

2) *Non-reusability.* In our voting protocol, each voter cannot cast more than one vote. More specifically, a voter cannot vote one candidate more than once or vote more than one candidate. Suppose voter V_k wants to vote twice v_k and v_e in step (3.1). To do so, he or she first casts v_k to the ballot box decided by his or her index number, d_k as usual. Then he or she casts v_e to another ballot box labeled by d_e. However since the index array $\{d_k\}$ is a permutation of $\mathbb{Z}_n$, d_e must be the index number of another voter V_j. Therefore V_j will find that $R_{d_j}=v_j+v_e\neq v_j \mod m$ and knows that someone cheats thus aborting the voting protocol. Our protocol ensures that each voter only has one vote and he or she can only use it once.

3) *Verifiability.* In the step (3.3) of our protocol, each voter can verify if his or her vote has been modified by attackers. As long as V_k finds out $R_{d_k}\neq v_k$, he or she knows that his or her vote has not been counted correctly.

4) *Fairness.* If a voter could know some useful information about other votes beforehand, he or she might change his or her mind thus voting for another candidate to his or her benefit. In our protocol, the voters vote only in the step (3) and the vote tally is obtained by doing statistics on R_k which is the sum over the numbers $r'_{j,k}$. However, the numbers $r'_{j,k}$ are announced via simultaneous broad channels in the step (3.1), which means that each voter cannot acquire the other voters' information on $r'_{j,k}$ and thus cannot obtain a partial vote tally beforehand. Therefore, fairness can be maintained.

VII. CONCLUSION

We have presented a quantum protocol for implementing the task of anonymous voting with the help of two entangled quantum states, $|\mathcal{X}_n\rangle$ and $|\mathcal{S}_n\rangle$. Through our protocol, any individual party can acquire a permutation of all the votes, which makes anyone can tally the votes by himself or herself without resorting to a third-party tally man. The protocol has been demonstrated to possess the properties of privacy, self-tallying, non-reusability, verifiability and fairness. We also generalize our SQAV to the

more general AMC task. Our generalized protocol could let each party broadcast his or her data anonymously and safely to be further fed into AMC function.

An interesting open question is whether our protocol can be used to implement more tasks on AMC or SMC. This deserves further investigations in the future.

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Appendix A: Proof $|\mathcal{S}_n\rangle$ is n-lateral rotationally invariant

Property 1 A n dimensional quantum state on Hilbert space $\mathcal{H}_n$ is the superposition of computational basis $\{|i\rangle_C|\ i=0,1,\cdots,n-1\}$. Consider state $|\mathcal{S}_n\rangle$ of n such particles on $\mathcal{H}_n^{\otimes n}$ in the following form

$$|\mathcal{S}_n\rangle = \sum_{S\in\mathcal{P}_n^n}(-)^{\tau(S)}|S\rangle \tag{A1}$$

$$\equiv \sum_{S\in\mathcal{P}_n^n}(-)^{\tau(S)}|s_0 s_1,\cdots,s_{n-1}\rangle. \tag{A2}$$

Consider another basis $\{|i'\rangle\}$ connected with the computational basis by a unitary transformation U, where

$$|i\rangle = \sum_j U_{ji}|j'\rangle\ . \tag{A3}$$

Then in this new basis the state $|\mathcal{S}_n\rangle$ takes the same form up to a global phase factor ϕ. That is,

$$|\mathcal{S}_n\rangle = e^{i\phi}\sum_{M\in\mathcal{P}_n^n}(-)^{\tau(M)}|M'\rangle \tag{A4}$$

$$\equiv e^{i\phi}\sum_{M\in\mathcal{P}_n^n}(-)^{\tau(M)}|m_0'm_1'\cdots m_{n-1}'\rangle\ . \tag{A5}$$

Here $\mathcal{P}_n^n = \{x_0x_1\cdots x_{n-1}|x_0,x_1,\cdots,x_{n-1}\in\mathbb{Z}_n,\forall j\neq k, x_j\neq x_k\}$ and the phase factor is given by

$$e^{i\phi} = \det(U)\ . \tag{A6}$$

Proof: Expand Eq.(A2) in the new basis by using the unitary transformation Eq.(A3), we have

$$|\mathcal{S}_n\rangle = \sum_{S\in\mathcal{P}_n^n}(-)^{\tau(S)}\sum_{m_0=0}^{n-1}U_{m_0,s_0}|m_0'\rangle\otimes\cdots\otimes \sum_{m_{n-1}=0}^{n-1}U_{m_{n-1},s_{n-1}}|m_{n-1}'\rangle \tag{A7}$$

$$= (\sum_{M\in\mathcal{P}_n^n}+\sum_{M\notin\mathcal{P}_n^n})\big[\sum_{S\in\mathcal{P}_n^n}(-)^{\tau(S)}U_{m_0,s_0}U_{m_1,s_1}\cdots U_{m_{n-1},s_{n-1}}\big]|M\rangle \tag{A8}$$

$$= (\sum_{M\in\mathcal{P}_n^n}+\sum_{M\notin\mathcal{P}_n^n})\det(U_{m_j,s_i})|M\rangle \tag{A9}$$

if $M\notin\mathcal{P}_n^n$, $\exists s\neq t$ such that $m_s=m_t$, then there are two same columns for matrix U_{m_j,s_i}. It means $U_{m_s,s_i}=U_{m_t,s_i}$. Therefore $\det U_{m_j,s_i}=0$ and we have

$$\begin{aligned}|\mathcal{S}_n\rangle &= \sum_{M\in\mathcal{P}_n^n}\det(U_{m_j,s_i})|M\rangle\\ &= \sum_{M\in\mathcal{P}_n^n}(-)^{\tau(M)}\det(U_{j,s_i})|M\rangle\\ &= \sum_{M\in\mathcal{P}_n^n}(-)^{\tau(M)}\det(U)|M\rangle\\ &= e^{i\phi}\sum_{M\in\mathcal{P}_n^n}(-)^{\tau(M)}|M\rangle\end{aligned} \tag{A10}$$

■

Appendix B: Proof of Theorem 2

To prove the theorem 2, we first give two lemmas and one corollary.

Lemma 1 Let q be an arbitrary element of $\{1,2,\cdots,n-1\}$, and $s_0,s_1,\cdots,s_{q-1}\in\mathbb{Z}_n$ be distinct. If $\sum_{j=0}^{q-1}\exp(\frac{2\pi i s_j t}{n})\alpha_j=0$ always holds for any $t\in\mathbb{Z}_n$, we have $\alpha_0=\alpha_1=\cdots=\alpha_{q-1}=0$.

Proof: If $\sum_{j=0}^{q}\exp(\frac{2\pi i s_j t}{n})\alpha_j=0$ always holds for any $t\in\mathbb{Z}_n$, we have linear equations

$$A\begin{pmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{q-1}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix}, \tag{B1}$$

where A is a $n\times q$ matrix with elements $A_{jk}=\exp(\frac{2\pi i(j-1)s_k}{n}) = (\exp(\frac{2\pi i s_k}{n}))^{j-1}$. Taking the first q rows of A as a new square matrix $\overline{A}$ with size $q\times q$, it is easy to see that $\overline{A}$ is a Vandermonde matrix [30]. Since $s_0,s_1,\cdots,s_{q-1}$ are distinct, the determinant of $\overline{A}$ is non-zero and thus the rank of A is q.

Consequently, the above Eq. (B1) has the only solution $\alpha_0 = \alpha_1 = \cdots = \alpha_{q-1} = 0$.

Lemma 2 Let m be an arbitrary element of $\{2, 3, \cdots, n\}$, $\mathcal{R}_n^m = \{x_0x_1\cdots x_{m-1} | x_0, x_1, \cdots, x_{m-1} \in \mathbb{Z}_n\}$, $\mathcal{P}_n^m = \{x_0x_1\cdots x_{m-1} | x_0, x_1, \cdots, x_{m-1} \in \mathbb{Z}_n, \forall j \neq k, x_j \neq x_k\}$ and $\mathcal{Q}_n^m = \{x_0x_1\cdots x_{m-1} | x_0, x_1, \cdots, x_{m-1} \in \mathbb{Z}_n, \exists j \neq k, x_j = x_k\}$. Apparently, $\mathcal{P}_n^m \cap \mathcal{Q}_n^m = \emptyset$ and $\mathcal{R}_n^m = \mathcal{P}_n^m \cup \mathcal{Q}_n^m$. Divide $\mathcal{P}_n^m$ into $\binom{n}{m} = \frac{n!}{(n-m)!m!}$ subsets, each of which corresponds to the set of all the $m!$ permutations of a m-combination of $\mathbb{Z}_n$, denoted by $\mathcal{P}_n^{n,w}$ ($w = 0, 1, \cdots, \binom{n}{m} - 1$). For an arbitrary subset $\mathcal{P}_n^{m,w}$, if the equation

$$\sum_{S\in\mathcal{P}_n^{m,w}} (-1)^{\tau(S)} \prod_{j=0}^{m-1} \exp(\frac{2\pi i s_j t_j}{n})\beta_S = 0 \qquad \text{(B2)}$$

holds for any $t_0t_1\cdots t_{m-1} \in \mathcal{Q}_n^m$, we have that all the β_S for $S \in \mathcal{P}_n^{m,w}$ are equal.

Proof: We use the method of induction to prove this lemma.

For $m = 2$, suppose $\mathcal{P}_n^{2,w} = \{\hat{s}_0\hat{s}_1, \hat{s}_1\hat{s}_0\}$ with $\hat{s}_0 < \hat{s}_1$, $\mathcal{Q}_n^2 = \{t_0t_1 | t_0 = t_1 = t \in \mathbb{Z}_n\}$ and the equation $\sum_{s_0s_1\in\mathcal{P}_n^{2,w}} (-1)^{\tau(s_0s_1)} \exp(\frac{2\pi i(s_0t_0+s_1t_1)}{n})\beta_{s_0s_1} = 0$ holds for any $t_0t_1 \in \mathcal{Q}_n^2$. Since $t_0 = t_1 = t$, the equation can also be written as $\exp(\frac{2\pi i(\hat{s}_0+\hat{s}_1)t}{n})\beta_{\hat{s}_0\hat{s}_1} - \exp(\frac{2\pi i(\hat{s}_0+\hat{s}_1)t}{n})\beta_{\hat{s}_1\hat{s}_0} = 0$. Obviously, $\beta_{\hat{s}_0\hat{s}_1} = \beta_{\hat{s}_1\hat{s}_0}$ is obtained.

We assume that, for $m = k$ and an arbitrary subset $\mathcal{P}_n^{k,w}$, if the Eq. (B2) always holds for any $t_0t_1\cdots t_{k-1} \in \mathcal{Q}_n^k$, all the $\beta_{s_0s_1\cdots s_{k-1}}$ for $s_0s_1\cdots s_{k-1} \in \mathcal{P}_n^{k,w}$ are equal. Now we analyze the case for $m = k+1$. We suppose the $(k+1)$-combination is $\mathcal{P}_n^{k+1,w}$ corresponding to the set $\hat{S} = \{\hat{s}_0, \hat{s}_1, \cdots, \hat{s}_k\}$ with $\hat{s}_0 < \hat{s}_1 < \cdots < \hat{s}_k$. Namely, $\mathcal{P}_n^{k+1,w}$ is the set of all the $(k+1)!$ permutations of the $\hat{S}$. In this case, observing that s_p ($p \in \{0, 1, \cdots, k\}$) can take each value from $\hat{S}$ in the Eq. (B2), the equation can be written as

$$\sum_{l=0}^{k} \left(\sum_{S\in\mathcal{P}_n^{k+1,w}, s_p=\hat{s}_l} (-1)^{\tau(S)} \exp(\frac{2\pi i\hat{s}_l t_p}{n}) \prod_{j=0, j\neq p}^{k} \exp(\frac{2\pi i s_j t_j}{n})\beta_S \right) = 0. \qquad \text{(B3)}$$

Noting that $(-1)^{\tau(S)} = (-1)^{l-p}(-1)^{\tau(s_0\cdots s_{p-1}s_{p+1}\cdots s_k)}$, the Eq. (B3) can also be written as

$$\sum_{l=0}^{k}(-1)^{l-p} \exp(\frac{2\pi i\hat{s}_l t_p}{n}) \left(\sum_{S\in\mathcal{P}_n^{k+1,w}, s_p=\hat{s}_l} (-1)^{\tau(s_0\cdots s_{p-1}s_{p+1}\cdots s_k)} \prod_{j=0, j\neq p}^{k} \exp(\frac{2\pi i s_j t_j}{n})\beta_S \right) = 0. \qquad \text{(B4)}$$

We now prove that, if the Eq. (B4) holds for any $t_0t_1\cdots t_k \in \mathcal{Q}_n^{k+1}$, all the β_S for $S \in \mathcal{P}_n^{k+1,w}$ are equal. Specially, when $t_0\cdots t_{p-1}t_{p+1}\cdots t_k \in \mathcal{Q}_n^k$ is fixed and t_p takes every value from $\mathbb{Z}_n$, the Eq. (B4) always holds. Hence, according to the lemma 1, we can derive that for arbitrary $l \in \{0, 1, 2, \cdots, k\}$,

$$\sum_{S\in\mathcal{P}_n^{k+1,w}, s_p=\hat{s}_l} (-1)^{\tau(s_0\cdots s_{p-1}s_{p+1}\cdots s_k)} \prod_{j=0, j\neq p}^{k} \exp(\frac{2\pi i s_j t_j}{n})\beta_{s_0s_1\cdots s_k} = 0. \qquad \text{(B5)}$$

Here the Eq. (B5) holds for arbitrary $t_0\cdots t_{p-1}t_{p+1}\cdots t_k \in \mathcal{P}_n^{k,w}$. Based on the previous assumption for the case $m = k$, all the β_S for $S \in \mathcal{P}_n^{k+1,w}$ and $s_p = \hat{s}_l$ are equal. If the equation (B2) holds for any $t_0t_1\cdots t_k \in \mathcal{Q}_n^{k+1}$ when $m = k+1$, l and p can take arbitrary values from $\{0, 1, 2, \cdots, k\}$, we can draw the conclusion that all the β_S for $S \in \mathcal{P}_n^{k+1,w}$ are equal.

By mathematical induction above, we can derive that for an arbitrary $m \in \{2, \cdots, n\}$, if the Eq. (B2) holds for any $t_0t_1\cdots t_{m-1} \in \mathcal{Q}_n^m$, all the $\beta_{s_0s_1\cdots s_{m-1}}$ for $s_0s_1\cdots s_{m-1} \in \mathcal{P}_n^{m,w}$ are equal.

Now we give a corollary of lemma 2 below.

Corollary 1 Let m, $\mathcal{R}_n^m$, $\mathcal{P}_n^m$, and $\mathcal{Q}_n^m$ be defined in lemma 2. For an arbitrary subset $\mathcal{P}_n^{m,w}$, if the equation

$$\sum_{s_0s_1\cdots s_{m-1}\in\mathcal{P}_n^{m,w}} (-1)^{\tau(s_0s_1\cdots s_{m-1})} \prod_{j=0}^{m-1} \exp(\frac{2\pi i s_j t_j}{n})\vec{\beta}_{s_0s_1\cdots s_{m-1}} = \vec{0} \qquad \text{(B6)}$$

holds for any $t_0t_1\cdots t_{m-1} \in \mathcal{Q}_n^m$, where $\vec{\beta}_{s_0s_1\cdots s_{m-1}}$ are vectors and $\vec{0}$ is zero vector, all the $\vec{\beta}_{s_0s_1\cdots s_{m-1}}$ for $s_0s_1\cdots s_{m-1} \in \mathcal{P}_n^{m,w}$ are equal.

The only difference between this corollary and lemma 2 is that $\beta_{s_0s_1\cdots s_{m-1}}$ is generalized to the vector $\vec{\beta}_{s_0s_1\cdots s_{m-1}}$. Hence, the corollary can be directly proved.

Now we use lemma 2 to prove theorem 2.

Proof: Restricting the measurement basis to computation basis or fourier basis, the necessity of our theorem can be directly obtained from the property 1.

Now we prove the sufficiency. On one hand, to satisfy the condition that all the measurement outcomes are distinct when measuring each particle of $|\Theta\rangle$ in fourier basis, $|\Theta\rangle$ must be in the form

$$\begin{aligned}|\Theta\rangle &= \sum_{S\in\mathcal{P}_n^n}(-1)^{\tau(S)}\beta_S(\mathcal{F}|s_0\rangle)\otimes\cdots\otimes(\mathcal{F}|s_{n-1}\rangle)\\ &= \sum_{S\in\mathcal{P}_n^n}(-1)^{\tau(S)}\beta_S(\sum_{t_0}\frac{\exp(\frac{2\pi i s_0 t_0}{n})}{\sqrt{n}}|t_0\rangle)\otimes\cdots\otimes(\sum_{t_{n-1}}\frac{\exp(\frac{2\pi i s_{n-1} t_{n-1}}{n})}{\sqrt{n}}|t_{n-1}\rangle)\\ &= \sum_{t_0,t_1,\cdots,t_{n-1}}\sum_{S\in\mathcal{P}_n^n}(\frac{(-1)^{\tau(S)}}{n^{\frac{n}{2}}}\prod_{j=0}^{n-1}\exp(\frac{2\pi i s_j t_j}{n})\beta_S)|t_0t_1\cdots t_{n-1}\rangle, \end{aligned} \tag{B7}$$

where $S = s_0s_1\cdots s_{n-1}$. On the other hand, to meet the condition that all the measurement outcomes are distinct when measuring each particle of $|\Theta\rangle$ in computation basis, the terms $\sum_{S\in\mathcal{P}_n^n}(\beta_S\prod_{j=0}^{n-1}\exp(\frac{2\pi i s_j t_j}{n}))$ for $t_0t_1\cdots t_{n-1}\in\mathcal{Q}_n^n$ are required to be equal to zero. From lemma 2 (when $m=n$), to satisfy this requirement, we can see that all the β_S for $S\in\mathcal{P}_n^n$ are equal. Moreover, to keep normalization of $|\Theta\rangle$, we have

$$\beta_S = \frac{1}{\sqrt{n!}}. \tag{B8}$$

For any $t_0t_1\cdots t_{n-1}\in\mathcal{Q}_n^n$, according to the definition of square matrix determinant, $\sum_{S\in\mathcal{P}_n^n}\frac{(-1)^{\tau(S)}}{n^{\frac{n}{2}}}\prod_{j=0}^{n-1}\exp(\frac{2\pi i s_j t_j}{n})$ is in fact the determinant of the $n\times n$ matrix $\overline{V}$ with elements $\overline{V}_{jk}=\frac{\exp(\frac{2\pi i t_j k}{n})}{\sqrt{n}}$. Namely,

$$\sum_{S\in\mathcal{P}_n^n}\frac{(-1)^{\tau(S)}}{n^{\frac{n}{2}}}\prod_{j=0}^{n-1}\exp(\frac{2\pi i s_j t_j}{n}) = \det(\overline{V}). \tag{B9}$$

Transposing pairs of rows of $\overline{V}$ to generate a new $n\times n$ matrix $\widetilde{V}$ with elements $\widetilde{V}_{jk}=\frac{\exp(\frac{2\pi i j k}{n})}{\sqrt{n}}$, we have

$$\det(\overline{V}) = (-1)^{\tau(t_0t_1\cdots t_{n-1})}\det(\widetilde{V}). \tag{B10}$$

Taking the Eqs. (B8), (B9) and (B10) to the Eq. (B7) and discarding the terms for $t_0t_1\cdots t_{n-1}\in\mathcal{Q}_n^n$ in the Eq. (B7), we have

$$|\Theta\rangle = \sum_{T\in\mathcal{P}_n^n}\frac{(-1)^{\tau(T)}}{\sqrt{n!}}|T\rangle, \tag{B11}$$

up to the global factor $\det(\widetilde{V})$, where $T=t_0t_1\cdots t_{n-1}$. Therefore, $|\Theta\rangle$ has the same form as $|\mathcal{S}_n\rangle$ and the theorem 2 is proved.

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