Anomalous parity-time–symmetry transition away from an exceptional point

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Anomalous Parity-Time Symmetry Transition away from an Exceptional Point

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Parity-time ($\mathcal{PT}$) symmetric systems have two distinguished phases, e.g., one with real energy eigenvalues and the other with complex conjugate eigenvalues. To enter one phase from the other, it is believed that the system must pass through an exceptional point, which is a non-Hermitian degenerate point with coalesced eigenvalues and eigenvectors. Here we reveal an anomalous $\mathcal{PT}$ transition that takes place away from an exceptional point in a nonlinear system: as the nonlinearity increases, the original linear system evolves along two distinct $\mathcal{PT}$-symmetric trajectories, each of which can have an exceptional point. However, the two trajectories collide and vanish away from these exceptional points, after which the system is left with a $\mathcal{PT}$-broken phase. We first illustrate this phenomenon using a coupled mode theory and then exemplify it using paraxial wave propagation in a transverse periodic potential.

I. INTRODUCTION

Parity-time ($\mathcal{PT}$) symmetry originated in the search for an alternative framework of canonical quantum mechanics and quantum field theory [1–3]. It has since stimulated fast growing interest in optics [4–21], microwaves [24], radio waves [25], acoustics [26], and mechanics [27]. In all these systems, a well-known and intriguing property is the existence of two distinguished phases, e.g., one with real energy eigenvalues ("$\mathcal{PT}$-symmetric phase") and the other with complex conjugate eigenvalues ("$\mathcal{PT}$-broken phase"). The same property is shared with other systems with novel symmetries [28], which is the consequence of having a pseudo-Hermitian Hamiltonian [29].

The two aforementioned phases are separated by exceptional points (EPs) [30–39], which are non-Hermitian degenerate points with coalesced eigenvalues and eigenvectors. While EPs are ubiquitous in non-Hermitian systems, they are singular points in the parameter space and can be reached only by a sweep involving two or more parameters in general. $\mathcal{PT}$-symmetric systems are special in this regard, as they only require sweeping a single parameter to reach an EP. This parameter can be, for example, the gain and loss strength in the system or the effective wavelength of the eigenstates [40]. As such, it is believed that if the system maintains $\mathcal{PT}$ symmetry, then it must pass through an EP in order to enter one phase from the other, regardless of which parameter is varied. To the best of our knowledge, the only exception to this rule occurs when the underlying Hermitian system (i.e., without gain or loss) has genuine degeneracy [16, 17] with identical eigenvalues but distinct eigenstates. This scenario, nevertheless, can be taken as the limiting case of a system with an EP and increasing system size [41].

Here we reveal an anomalous transition from the $\mathcal{PT}$-symmetric phase to the $\mathcal{PT}$-broken phase that takes place *away* from an EP in a nonlinear system: as the nonlinearity increases, the original linear system evolves along two distinct $\mathcal{PT}$-symmetric trajectories, each of which can have an EP. However, the two trajectories collide and vanish away from these EPs, after which the system is left with only a $\mathcal{PT}$-broken phase.

We will refer to this phenomenon as anomalous $\mathcal{PT}$ transition (APT). Below we first illustrate the existence of APT using a coupled mode theory, which cannot be induced by the typical form of nonlinearity considered previously [42, 43]. Instead, APT requires distinct and eigenstate-dependent paths of the effective Hamiltonian as the nonlinearity increases, which we illustrate using nonlinearity-shifted couplings. We then exemplify APT using paraxial wave propagation in a transverse periodic potential, where the $\mathcal{PT}$ transition (and the associated spectral singularity [44]) is no longer due to an exceptional point in the Hilbert space defined by a given transverse momentum. We conclude by discussing how APT can be identified in an experiment and show that it does not occur from the $\mathcal{PT}$-broken phase to the $\mathcal{PT}$-symmetric phase.

II. ROUTE TO $\mathcal{APT}$

We start by considering two identical oscillators with energy E_0. They are subjected to gain and loss at rate $\pm\kappa_0$ and a real-valued coupling g_0 that can be negative. Before we introduce nonlinearity, the effective Hamiltonian of the system can be written as

$$H_0 = \begin{bmatrix} E_0 + i\kappa_0 & g_0 \\ g_0 & E_0 - i\kappa_0 \end{bmatrix}, \tag{1}$$

which is $\mathcal{PT}$-symmetric and well studied. For example, it has been used to describe coupled waveguides [4, 42] and photonic molecule lasers [39, 45], including those made up of coupled InGaAsP microring resonators [18], GaAs/AlGaAs quantum cascade microdisk resonators [46], and erbium-doped silica microtoroid resonators [47]. A generalization of Eq. (1) to include multiple oscillators has been used to describe discrete fiber networks [15, 48] and supersymmetric laser arrays [49]. It is also worth

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noting that Eq. (1) is not a Hamiltonian in the sense of quantum mechanics, since it incorporates gain and loss directly and is hence non-Hermitian, which captures the dynamics of the corresponding systems nevertheless.

The effective Hamiltonian (1) satisfies $\mathcal{PT}H_0\mathcal{PT} = H_0$, where the parity operator $\mathcal{P}$ is represented by a rotation matrix $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ and the time-reversal operator $\mathcal{T}$ by the complex conjugate. The two eigenvalues of H_0 are given by $E^{(1,2)} = E_0 \pm \sqrt{g_0^2 - \kappa_0^2}$, which are real when $|g_0| > \kappa_0$ and the system is in the $\mathcal{PT}$-symmetric phase; they form a complex conjugate pair when $|g_0| < \kappa_0$ and the system is in the $\mathcal{PT}$-broken phase. The EP is located at $|g_0| = \kappa_0$, which the system must pass through to go from one phase to the other.

The two eigenstates of the system can be expressed as $\psi^{(j)} = c_a^{(j)}\varphi_a + c_b^{(j)}\varphi_b\,(j = 1, 2)$, where $\varphi_{a,b}$ are the uncoupled wave functions of the two oscillators. Below we drop the superscript j when ambiguity is unlikely, and we use the normalization $|c_a|^2 + |c_b|^2 \equiv 1$ as usual. We also emphasize that $|c_a|$ and $|c_b|$ are equal in the $\mathcal{PT}$-symmetric phase (given by $2^{-1/2}$), which is not the case in the $\mathcal{PT}$-broken phase.

To illustrate the simplest case where APT arises, we take κ_0 to be independent of the nonlinearity ε. We assume the typical nonlinear energy shift in the effective Hamiltonian, with E_0 replaced by $E_{a,b}(\varepsilon) = E_0 + 2\varepsilon|c_{a,b}|^2$ in the two diagonal elements [42, 43]. Most importantly, we consider nonlinearity-shifted couplings given by

$$g_a(\varepsilon) = g_0 + \varepsilon\beta c_a^* c_b + \varepsilon\gamma|c_a|^2, \tag{2}$$

$$g_b(\varepsilon) = g_0 + \varepsilon\beta c_b^* c_a + \varepsilon\gamma|c_b|^2, \tag{3}$$

which are the key quantities for APT to take place as we will show. Here β, γ are two real constants, and $g_a = g_b^*$ holds by construction when $|c_a| = |c_b|$. The global phase of ψ, which does not bear a physical significance, is eliminated in $g_{a,b}(\varepsilon)$ thanks to the product $c_a^* c_b$ and its complex conjugate. Below we will refer to our nonlinear Hamiltonian as

$$H \equiv \begin{bmatrix} E_a(\varepsilon) + i\kappa_0 & g_a(\varepsilon) \\ g_b(\varepsilon) & E_b(\varepsilon) - i\kappa_0 \end{bmatrix}, \tag{4}$$

and we recover the typical nonlinear Hamiltonian mentioned previously ($\tilde{H}$) when β, γ are taken to be zero (i.e., $g_{a,b}(\varepsilon) = g_0$).

While $\tilde{H}$ displays interesting dynamical effects [42], it does not lead to a transition between the $\mathcal{PT}$-symmetric phase and the $\mathcal{PT}$-broken phase with increasing nonlinearity [23]. In contrast, the nonlinear Hamiltonian H given by Eq. (4) displays a qualitatively different behavior. Let us start in the $\mathcal{PT}$-symmetric phase with $|g_0| > \kappa_0$. As mentioned previously, $|c_a| = |c_b| = 2^{-1/2}$ holds for both linear eigenstates. As $|\varepsilon|$ increases, $c_{a,b}$ evolve continuously from their linear values, and if we assume that they still have the same modulus, then we find $E_a(\varepsilon) = E_b(\varepsilon)$ as well as $g_a^{(j)}(\varepsilon) = [g_b^{(j)}(\varepsilon)]^*$ as mentioned previously. It is important to note that the couplings $g_a^{(1,2)}(\varepsilon)$ [and

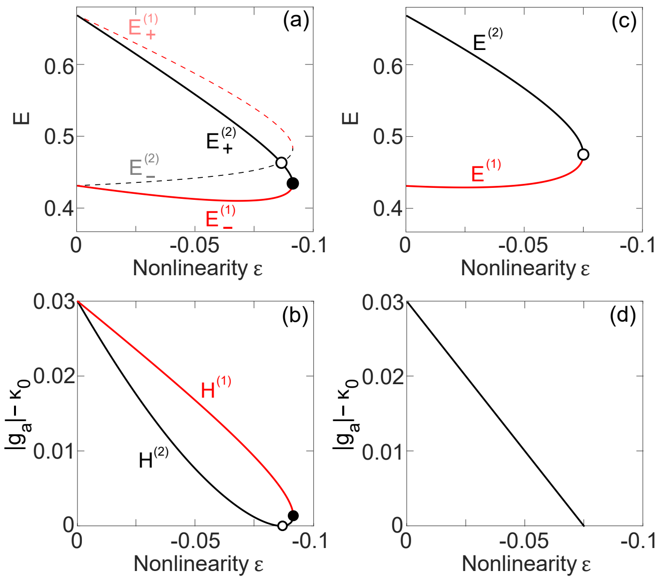

FIG. 1. (Color online) Anomalous $\mathcal{PT}$ transition away from an EP. (a) Two nonlinear energy eigenvalues $E_-^{(1)}$ and $E_+^{(2)}$ in the $\mathcal{PT}$-symmetric phase (solid lines) annihilate each other at the APT point where $\varepsilon = -0.091$ (filled circle). Dashed lines show the two additional eigenvalues $E_+^{(1)}$ and $E_-^{(2)}$ of the linearized Hamiltonians $H^{(1,2)}$, and the open circle shows the EP of $H^{(2)}$. (b) Difference between $|g_a^{(j)}(\varepsilon)|$ and κ_0. Filled and open circles show the APT point and EP, respectively. The parameters used are: $E_0 = 0.55$, $\kappa_0 = 0.22$, $g_0 = -0.25$, $\beta = 0.6$, and $\gamma = -0.8$. (c) Same as (a) but with $\beta = 0$. The APT point is replaced by an EP at $\varepsilon = -2(\kappa_0 + g_0)/\gamma = -0.075$. The difference between the now path-independent $|g_a(\varepsilon)|$ and κ_0 is shown in (d).

$g_b^{(1,2)}(\varepsilon)$] differ, which prompts us to restore the nonlinear mode index j $(j = 1, 2)$. This j-dependence arises from the relative phase between c_a and c_b contained in the product $c_a^* c_b$ and its complex conjugate, which is different for the two eigenstates. As we shall see, this j-dependence, or equivalently a nonzero β, leads to APT.

Along these two j-dependent nonlinear trajectories, the system now has two distinct linearized Hamiltonians $H^{(j)}$, each still being $\mathcal{PT}$-symmetric and satisfying $\mathcal{PT}H^{(j)}\mathcal{PT} = H^{(j)}$. The eigenvalues of $H^{(j)}$ are hence either real or complex conjugates, and they are given by

$$E_\pm^{(j)} = E_0 + \varepsilon \pm \sqrt{|g_a^{(j)}(\varepsilon)|^2 - \kappa_0^2}. \tag{5}$$

The corresponding eigenvectors in the $\mathcal{PT}$-symmetric phase still satisfy $|c_a| = |c_b|$, which is consistent with our assumption. We note that the two linearized Hamiltonians $H^{(1,2)}$ have four eigenvalues in total, but for each $H^{(j)}$, only one of its eigenvalues corresponds to the nonlinear eigenstate $\psi^{(j)}$. These nonlinear eigenstates are stable as can be shown using a standard linear stability analysis suitable for non-Hermitian systems [43, 50], and we denote the corresponding nonlinear eigenvalues by $E_-^{(1)}, E_+^{(2)}$, with the other two spurious ones by $E_+^{(1)}, E_-^{(2)}$

[see Fig. 1(a)].

As is clear from Eq. (5), each of the two $H^{(j)}$ can have an EP at $|g_a^{(j)}(\varepsilon)| = \kappa_0$, which could in principle lead to two $\mathcal{PT}$ transitions to their respective $\mathcal{PT}$-broken phases. However, APT takes place away from these EPs, when $E_-^{(1)}$ and $E_+^{(2)}$ annihilate each other at a different nonlinearity strength [see Fig. 1(a)]. We will refer to this annihilation point as the APT point, beyond which the system is left with only a $\mathcal{PT}$-broken phase, which we will discuss later in Fig. 3.

We have labeled $E_\pm^{(2)}$ by continuity beyond their EP in Fig. 1(a), i.e., with inverted signs before the square root in Eq. (5). It is straightforward to see from Eq. (5) that the annihilation of $E_-^{(1)}$ and $E_+^{(2)}$ is accompanied by $|g_a^{(1)}(\varepsilon)| = |g_a^{(2)}(\varepsilon)|$. In fact not just their moduli, $g_a^{(1,2)}(\varepsilon) \equiv X + iY$ themselves also become the same at the APT point. They are given by the intersections of a circle and a hyperbola in the complex plane, both parametrized by ε:

$$\left(X - g_0 - \varepsilon\frac{\gamma}{2}\right)^2 + Y^2 = \left(\frac{\varepsilon\beta}{2}\right)^2, \tag{6}$$

$$\left(2X - g_0 - \varepsilon\frac{\gamma}{2}\right) Y = -\frac{\varepsilon\beta\kappa_0}{2}. \tag{7}$$

These two conic curves are derived using Eq. (2) and

$$c_b = \frac{\pm\sqrt{|g_a|^2 - \kappa_0^2} - i\kappa_0}{g_a} c_a \tag{8}$$

in the $\mathcal{PT}$-symmetric phase, and they become tangent to each other at a maximum nonlinearity strength $|\varepsilon|_{\max}$, beyond which they no longer intersect. $|\varepsilon|_{\max}$ determines the position of the APT point, and it is 0.091 in the example shown in Figs. 1(a) and 1(b).

To verify that the APT point is not an EP itself, we compute the difference between $|g_a^{(j)}(\varepsilon)|$ and κ_0 along the two nonlinear trajectories. As Fig. 1(b) shows, this difference diminishes as ε reduces, but it does not become zero at the APT point. Instead, it reaches zero at an EP along the trajectory of $H^{(2)}$ before the APT point. It may look surprising at first as to why $E_\pm^{(2)}$ do not enter the $\mathcal{PT}$-broken phase beyond this EP. However, one quickly realizes that since $E_-^{(1)}$ is still in the $\mathcal{PT}$-symmetric phase beyond this EP, $E_+^{(2)}$ has to stay in the $\mathcal{PT}$-symmetric phase also in order to annihilate it at the APT point, where they are both real. In this sense, it is the APT that prevents $E_\pm^{(2)}$ from entering the $\mathcal{PT}$-broken phase beyond its EP. In addition, we note that κ_0 is not just the cut-off of $|g_a^{(2)}(\varepsilon)|$ imposed by the $\mathcal{PT}$-symmetric phase; it is also the true minimum of $|g_a^{(2)}(\varepsilon)|$ which cannot be passed. This is evidenced by the vanishing slope of $|g_a^{(2)}(\varepsilon)|$ at the EP shown in Fig. 1(b), and it can be shown rigorously using a perturbation theory.

Interesting, the EP before the APT point can occur on the trajectory of $H^{(1)}$ instead (see Fig. 2), if $|g_a^{(1)}(\varepsilon)| <$

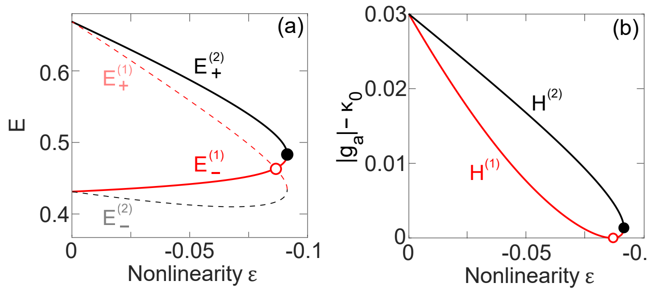

FIG. 2. (Color online) Another example of anomalous $\mathcal{PT}$ transition away from an EP. (a) and (b) are the same as those in Figs. 1(a) and (b) except that the EP before the APT point is now on the nonlinear trajectory of $H^{(1)}$. The parameters are the same as in Fig. 1 except for $\beta = -0.6$.

$|g_a^{(2)}(\varepsilon)|$ in the $\mathcal{PT}$-symmetric phase. In fact, $E_\pm^{(j)}$ only depends on the absolute value of $g_a^{(j)}$ (and $g_b^{(j)}$). Therefore, by noting

$$g_a^{(2)}(-\beta) = \left[g_a^{(1)}(\beta)\right]^* \tag{9}$$

using Eqs. (2) and (8), we find that the values of $E_\pm^{(1)}$ are exchanged with $E_\pm^{(2)}$ when we flip the sign of β. In other words, the four curves in Fig. 2(a) are identical with those in Fig. 1(a) but labeled differently. However, it is the lower lobe that corresponds to the two nonlinear eigenvalues of H in Fig. 1(a), while it is the upper lobe that gives the two nonlinear eigenvalues of H in Fig. 2(a). Consequently, we find that the two sets of nonlinear eigenvalues, one in the system with $+\beta$ and the other in the system with $-\beta$, cross at the EP when plotted together.

The annihilation of two eigenvalues is a generic feature in non-Hermitian and nonlinear systems upon the variation of a parameter. Here this tuning parameter is the nonlinearity itself, and other instances can be, for example, the lengths of the gain and loss regions in a slab laser [36] and a random laser [51]. In fact, this annihilation also happens when $\beta = 0$ [see Fig. 1(c)], with which $g_{a,b}(\varepsilon)$ no longer depend on the nonlinear mode index j in the $\mathcal{PT}$-symmetric phase: they only depend on $|c_a|$ and $|c_b|$, which are the same (i.e., $2^{-1/2}$) for the two nonlinear eigenstates. As a result, these two nonlinear states $\psi^{(1,2)}$ are captured by the same linearized Hamiltonian H, and their eigenvalues are given by $E^{(1,2)} = E_0 + \varepsilon \pm \sqrt{|g_a(\varepsilon)|^2 - \kappa_0^2}$. Therefore, if these two nonlinear eigenstates annihilate, it has to be at an EP where $|g_a(\varepsilon)| = \kappa_0$ [see Fig. 1(d)]. From this comparison we see that a nonzero β, or more generally, a path-dependent evolution of $g_{a,b}$ and H with nonlinearity, leads to the occurrence of APT.

As to the $\mathcal{PT}$-broken phase beyond the APT point, it consists of two additional nonlinear eigenstates $\psi^{(3,4)}$ that spin off from one of the two $\mathcal{PT}$-symmetric eigenstates (see Fig. 3). We note that if $\psi^{(3)} = c_a\varphi_a + c_b\varphi_b$ is a nonlinear eigenstate of H, it is straightforward to show that $c_b^*\varphi_a + c_a^*\varphi_b$ is also a nonlinear eigenstate of H. This is indeed how $\psi^{(3,4)}$ are related, i.e. they satisfy $\mathcal{PT}\psi^{(4)} = \psi^{(3)}(x)$, and their eigenvalues satisfy $E^{(4)} = [E^{(3)}]^*$. These properties are identical to those in a linear $\mathcal{PT}$-broken phase, but we emphasize that $\psi^{(3,4)}$ are eigenstates of two distinct linearized Hamiltonian $H^{(3,4)}$, respectively. Neither of $H^{(3,4)}$ is $\mathcal{PT}$-symmetric, i.e., $\mathcal{PT}H^{(3,4)}\mathcal{PT} \neq H^{(3,4)}$, but they are $\mathcal{PT}$-symmetric partners and satisfy $\mathcal{PT}H^{(3)}\mathcal{PT} = H^{(4)}$.

III. AN EXAMPLE OF $\mathcal{APT}$

To exemplify APT in a model system, we consider paraxial wave propagation with Kerr nonlinearity

$$i\partial_z\psi \equiv H\psi = -\partial_x^2\psi + V_0(x)\psi + \xi|\psi|^2\psi, \qquad (10)$$

where $\psi(x,z)$ is the wave function normalized by $\langle\psi|\psi\rangle \equiv \int_{-D/2}^{D/2} |\psi|^2\, dx = 1$ and z, x are the scaled coordinates of the longitudinal and transverse directions. D is the length of one period of the potential $V_0(x) = V_R(x) + iV_I(x)$, which is $\mathcal{PT}$-symmetric and satisfies $V_R(-x) = V_R(x)$ and $V_I(-x) = -V_I(x)$. For simplicity, we consider $V_R(x) = -\cos(x)^2$ and $V_I(x) = -\tau\sin(2x)$ with $D = \pi$, which have been studied previously in the linear regime [7]. Its first two linear bands (with $\xi = 0$) are in the symmetric phase unless $|\tau|$ is larger than 0.5 [7], with which the modes near the band edge enter the $\mathcal{PT}$-broken phase [see Figs. 4(a) and 4(b)]. This linear model has been realized experimentally using coupled fiber loops with temporal modulated gain and loss, which is equivalent to a synthetic photonic lattice [15]. Strong Kerr nonlinearity exists in fiber optics [52] and hence can be implemented in this system.

In Figs. 4(c) and 4(d) we focus on the two modes $\psi^{(1,2)}(x)$ at $k = 0.77$, which are in the linear $\mathcal{PT}$-symmetric phase with $\tau = 1$. We note that the intensities of these two modes satisfy $|\psi^{(j)}(-x)|^2 = |\psi^{(j)}(x)|^2$, which is equivalent to $|c_a|^2 = |c_b|^2$ in the coupled mode theory discussed previously. As a result, they do not break the $\mathcal{PT}$-symmetry of the system, since now the nonlinearity-modified potential $V^{(j)}(x) = V_0(x) + \xi|\psi^{(j)}(x;k)|^2$ still has a symmetric real part (i.e., $V_R(x)+\xi|\psi^{(j)}(x;k)|^2$) and an antisymmetric imaginary part (i.e., $V_I(x)$). As we have emphasized in the coupled mode theory, APT requires two path-dependent evolutions of the system Hamiltonian with nonlinearity. This property is satisfied here because $|\psi^{(1)}(x)|^2 \neq |\psi^{(2)}(x)|^2$ in the linear case, resulting in different nonlinear potentials $V^{(j)}(x)$ and path-dependent $H^{(j)}$.

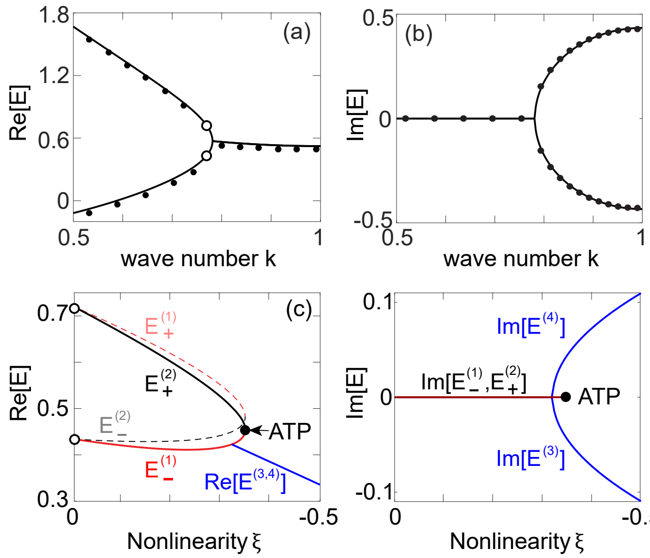

FIG. 4. (Color online) Anomalous $\mathcal{PT}$ transition in a transverse periodic potential $V_0(x) = -\cos(x)^2 - i\sin(2x)$ with Kerr nonlinearity. (a,b) Real and imaginary parts of the first two band near the band edge $k = 1$ (solid lines) in the linear regime. Open circles in (a) and (c) mark the same pair of modes at $k = 0.77$ that we study in the nonlinear regime. The dots show the linear bands calculated using the coupled mode theory (14) with $\xi = 0$. (c,d) Anomalous $\mathcal{PT}$ transition from the $\mathcal{PT}$-symmetric phase to the $\mathcal{PT}$-broken phase when nonlinearity increases, similar to that shown in Figs. 1(a) and (b).

By choosing a focusing nonlinearity ($\xi < 0$) and increasing its strength, we find that $\psi^{(1,2)}(x)$ indeed display APT [see the solid lines in Fig. 4(c) and 4(d)]: they approach each other and annihilate at $\xi = -0.35$, beyond which the system is left with a $\mathcal{PT}$-broken phase. Similar to the situation in the coupled-mode theory, each linearized $H^{(j)}$ has more than one eigenstate, but only one of them corresponds the nonlinear mode $\psi^{(j)}$. The others nevertheless indicate where the EP of $H^{(j)}$ is. As can be seen from Fig. 4(c), the EP of $H^{(2)}$ (where $E_\pm^{(2)}$ crosses) is again located at a smaller nonlinearity strength than the APT point, similar to the scenario shown in Fig. 1(a).

Below we formulate a two-mode coupled mode theory that reproduces the APT in this example. For a given wave number k, the modes of the Hermitian periodic potential $V_R(x)$ are given by the Bloch wave functions $\varphi_i(x;k)\exp(ikx)$, and $\varphi_i(x;k)$ are determined by

$$\left[-\frac{\partial^2}{\partial x^2} - 2ik\frac{\partial}{\partial x} - k^2 + V_R(x)\right]\varphi_i(x;k) = E_i\varphi_i(x;k).$$

We note that the corresponding energy eigenvalue E_i is real. It is straightforward to show that $\langle i|j\rangle \equiv$

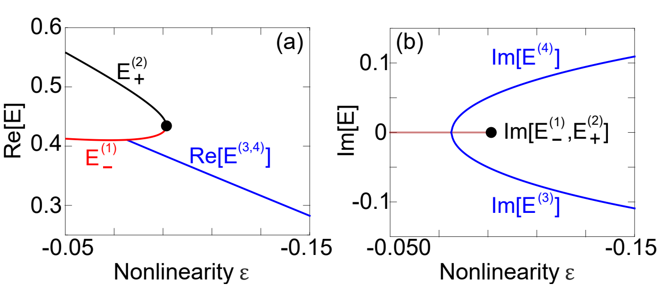

FIG. 3. (Color online) Additional nonlinear eigenvalues $E^{(3,4)}$ not shown in Fig. 1(a). They make up the $\mathcal{PT}$-broken phase beyond the APT point (filled circles).

$\langle\varphi_i(x;k)|\varphi_j(x;k)\rangle = \delta_{ij}$ in the absence of degeneracy. $\langle\cdot|\cdot\rangle$ denotes the Hermitian inner product as usual, with the integration over one period (D) of the $\mathcal{PT}$-symmetric potential. In addition, the equation above is invariant upon the parity operation $x \to -x$ and taking the complex conjugate (note again that E_i is real). Therefore, in principle we can find $\varphi_i(x;k) = \varphi_i^*(-x;k)$. Nevertheless, the global phase of $\varphi_i(x;k)$ is undetermined by its normalization $\langle i|i\rangle = 1$. Thus we find

$$\varphi_i(x;k) = \varphi_i^*(-x;k)\exp(2i\theta_i) \tag{11}$$

instead in general, where θ_i is the phase of $\varphi_i(x=0;k)$.

The coupled-mode theory is formulated using modes $\varphi_{g,e}(x;k)$ with energy $E_{g,e}$ in the first two bands of the Hermitian potential. As we shall see, a convenient choice is to set $\theta_g = 0$ and $\theta_e = \pi/2$, leading to $\varphi_g(x;k) = \varphi_g^*(-x;k)$ and $\varphi_e(x;k) = -\varphi_e^*(-x;k)$. The presence of $V_I(x)$ (and $\xi|\psi|^2$) in principle couples modes of the same wave number k in all bands, but the coupling is the strongest among modes of neighboring bands in general, and we find that the inclusion of $\varphi_{g,e}(x;k)$ is sufficient to demonstrate APT. The basis of our coupled mode theory is chosen as

$$\varphi_{a,b}(x;k) = \frac{1}{\sqrt{2}}[\varphi_g(x;k) \pm \varphi_e(x;k)], \tag{12}$$

which satisfy $\langle a|b\rangle = 0$ and $\langle a|a\rangle = \langle b|b\rangle = 1$. With the phase conventions of $\varphi_{e,g}$ chosen above, the following relation also holds:

$$\varphi_a(-x;k) = \frac{\varphi_g^*(x;k) - \varphi_e^*(x;k)}{\sqrt{2}} = \varphi_b^*(x;k). \tag{13}$$

We then find that $\langle a|V_I|a\rangle = -\langle b|V_I|b\rangle \equiv \kappa_0$ is real using the definition of the Hermitian inner product, which denotes the gain and loss strength.

The effective Hamiltonian of the periodic $\mathcal{PT}$-symmetric system for the first two bands can then be written as

$$H_c = \begin{pmatrix} E_0 + i\kappa_0 & g \\ g & E_0 - i\kappa_0 \end{pmatrix} + \xi \begin{pmatrix} N_a & L_a \\ L_b & N_b \end{pmatrix}. \tag{14}$$

E_0 and the linear coupling g are given by $(E_g \pm E_e)/2$, respectively. Note that $\langle a|V_I|b\rangle = -\langle a|V_I|b\rangle$ and $\langle b|V_I|a\rangle = -\langle b|V_I|a\rangle$ both vanish, which otherwise would have appeared in the off-diagonal elements of the linear part of H_c in Eq. (14). The nonlinear terms N_a, L_a in H_c are given by

$$N_a \equiv \langle aa|aa\rangle|c_a|^2 + \langle ab|aa\rangle c_b^* c_a + 2\langle ab|ab\rangle|c_b|^2, \tag{15}$$

$$L_a \equiv \langle ab|bb\rangle|c_b|^2 + \langle aa|bb\rangle c_a^* c_b + 2\langle aa|ab\rangle|c_a|^2, \tag{16}$$

and N_b, L_b are similarly defined with the subscripts a and b in these expressions exchanged. The quartic inner product here is defined by $\langle ij|i'j'\rangle \equiv \int_{-D/2}^{D/2} \varphi_i^*(x;k)\varphi_j^*(x;k)\varphi_{i'}(x;k)\varphi_{j'}(x;k)\,dx$, from which

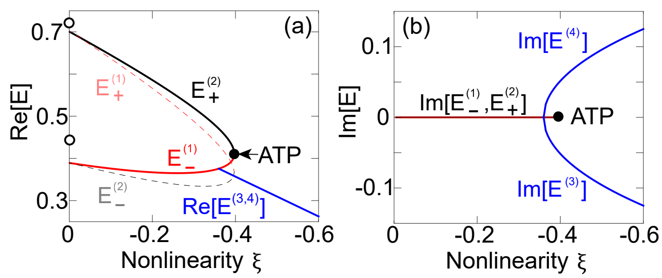

FIG. 5. (a,b) Same as Figs. 4(c) and (d) but reproduced using the coupled-mode theory (14) with $\xi \le 0$. Open circles in (a) show the energies of the two linear bands at $k = 0.77$ by solving Eq. (10) with $\xi = 0$.

we see immediately that $\langle aa|aa\rangle$, $\langle bb|bb\rangle$, $\langle ab|ab\rangle$ are real by definition and $\langle aa|bb\rangle = \langle bb|aa\rangle^*$. In addition, we find $\langle aa|aa\rangle = \langle bb|bb\rangle$ as well as $\langle ab|aa\rangle = \langle ab|bb\rangle^*$, $\langle aa|ab\rangle = \langle bb|ab\rangle^*$ using the relation (13). As a result, we find that $N_a = N_b^*$, $L_a = L_b^*$ and $\mathcal{PT}H_c\mathcal{PT} = H_c$ when $|c_a| = |c_b|$.

Although the nonlinearity represented by $N_{a,b}, L_{a,b}$ does not take the exact form as in Eq. (4), the Hamiltonian given by (14) is $\mathcal{PT}$-symmetric when $|c_a| = |c_b|$ and path-dependent (via $c_a^* c_b$ and its complex conjugate). These two conditions are crucial for APT as we have shown, and they can also be realized, for example, with a gain and loss strength that depends on the nonlinear eigenstates. This coupled mode theory agrees well with the direct numerical solutions of the paraxial equation (10) in the linear case, as we show in Figs. 4(a) and 4(b). For the pair of modes at $k = 0.77$ shown in Fig. 4(c) and (d), we find $E_g = 0.0208$, $E_e = 1.0692$, $\kappa_0 = 0.5006$, $\langle aa|aa\rangle = \langle bb|bb\rangle = 0.4791$, $\langle ab|aa\rangle = \langle bb|ab\rangle = 0.0048 - 0.0088i = \langle ab|bb\rangle^* = \langle aa|ab\rangle^*$, $\langle ab|ab\rangle = 0.1588$, and $\langle aa|bb\rangle = -0.0848 + 0.1343i = \langle bb|aa\rangle^*$. This coupled mode theory reproduces qualitatively the APT shown in Fig. 4 (see Fig. 5), and we note that a deviation occurs due to the omittance of the coupling to higher bands: the EP now appears along the path of $H^{(1)}$ instead of $H^{(2)}$. Similar (small) deviation in the coupled theory can be seen in the linear case as well, as we show in Fig. 5(a) at $\xi = 0$.

IV. CONCLUSION

In summary, we have revealed an anomalous $\mathcal{PT}$ transition from the $\mathcal{PT}$-symmetric phase to the $\mathcal{PT}$-broken phase that takes place away from an EP. We note that the transition in the opposite direction is not an APT: two $\mathcal{PT}$-broken eigenstates belong to two different linearized Hamiltonians as we have mentioned. In order for these complex conjugate eigenvalues to coalesce, they must become real simultaneously at some nonlinearity strength, which is an EP by definition. Hence this transition follows the standard $\mathcal{PT}$ transition mechanism (see Ref. [23], for example). It may look difficult to distinguish APT from a standard $\mathcal{PT}$ transition [cf. Figs. 1(a) and (c)] in an experiment, because the spurious eigenvalues $E_+^{(1)}, E_-^{(2)}$ of the linearized Hamiltonians cannot be accessed to identify the EP. One possibility to overcome this difficulty is to prepare another "conjugate" system, where the sign of β is flipped as we have discussed at the end of Sec. II. The crossing of the two sets of nonlinear eigenvalues in these conjugate systems gives the EP.

[1] C. M. Bender and S. Boettcher, *Real spectra in non-hermitian Hamiltonians having $\mathcal{PT}$ symmetry*, Phys. Rev. Lett. **80**, 5243 (1998).

[2] C. M. Bender, S. Boettcher, and P. N. Meisinger, *$\mathcal{PT}$-symmetric quantum mechanics*, J. Math. Phys. **40**, 2201 (1999).

[3] C. M. Bender, D. C. Brody, and H. F. Jones, *Complex extension of quantum mechanics*, Phys. Rev. Lett. **89**, 270401 (2002).

[4] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, *Theory of coupled optical $\mathcal{PT}$-symmetric structures*, Opt. Lett. **32**, 2632 (2007).

[5] S. Klaiman, U. Gunther, and N. Moiseyev, *Visualization of branch points in $\mathcal{PT}$-symmetric waveguides*, Phys. Rev. Lett. **101**, 080402 (2008).

[6] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, *Optical solitons in $\mathcal{PT}$ periodic potentials*, Phys. Rev. Lett. **100**, 030402 (2008).

[7] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, *Beam dynamics in $\mathcal{PT}$ symmetric optical lattices*, Phys. Rev. Lett. **100**, 103904 (2008).

[8] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, *Observation of PT-symmetry breaking in complex optical potentials*, Phys. Rev. Lett. **103**, 093902 (2009).

[9] S. Longhi, *$\mathcal{PT}$-symmetric laser absorber*, Phys. Rev. A **82**, 031801(R) (2010).

[10] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, *Observation of parity-time symmetry in optics*, Nature Phys. **6**, 192 (2010).

[11] Y. D. Chong, L. Ge, and A. D. Stone, *$\mathcal{PT}$-symmetry breaking and laser-absorber modes in optical scattering systems*, Phys. Rev. Lett. **106**, 093902 (2011).

[12] L. Feng, M. Ayache, J. Huang, Y.-L. Xu, M.-H. Lu, Y.-F. Chen, Y. Fainman, and A. Scherer, *Nonreciprocal light propagation in a silicon photonic circuit*, Science **333**, 729 (2011).

[13] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, *Unidirectional invisibility induced by PT-symmetric periodic structures*, Phys. Rev. Lett. **106**, 213901 (2011).

[14] L. Ge, Y. D. Chong, and A. D. Stone, *Conservation relations and anisotropic transmission resonances in one-dimensional $\mathcal{PT}$-symmetric photonic heterostructures*, Phys. Rev. A **85**, 023802 (2012).

[15] A. Regensburger, C. Bersch, M. A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, *Parity-time synthetic photonic lattices*, Nature (London) **488**, 167 (2012).

[16] L. Ge and A. D. Stone, *Parity-time symmetry breaking beyond one dimension: the role of degeneracy*, Phys. Rev. X **4**, 031011 (2014).

[17] L. Feng, Z. J.Wong, R.-M.Ma, Y.Wang, and X. Zhang, *Singlemode laser by parity-time symmetry breaking*, Science **346**, 972 (2014).

[18] H. Hodaei, M. A. Miri, M. Heinrich, D. N. Christodoulides, and M. Khajavikhan, *Parity-timesymmetric microring lasers*, Science **346**, 975 (2014).

[19] B. Peng, S. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang, *Parity-time-symmetric whispering-gallery microcavities*, Nature Phys, **10**, 394 (2014).

[20] B. Zhen, C. W. Hsu, Y. Igarashi, L. Lu, I. Kaminer, A. Pick, S.-L. Chua, J. D. Joannopoulos, and M. Soljačić, *Spawning rings of exceptional points out of Dirac cones*, Nature **525**, 354 (2015).

[21] W. Sun, Z. Gu, S. Xiao, and Q. Song, *Three-dimensional light confinement in a PT-symmetric nanocavity*, RSC Adv. **6**, 5792 (2016).

[22] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. Musslimani, *PT symmetric optical lattices*, Phys. Rev. A **81**, 063807 (2010).

[23] Y. Lumer, Y. Plotnik, M. Rechtsman, and M. Segev, *Nonlinearly induced PT-transition in photonic systems*, Phys. Rev. Lett. **111**, 263901 (2013).

[24] S. Bittner, B. Dietz, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer, *PT symmetry and spontaneous symmetry breaking in a microwave billiard*, Phys. Rev. Lett. **108**, 024101 (2012).

[25] Z. Lin, J. Schindler, F. M. Ellis, and T. Kottos, *Unidirectional invisibility induced by $\mathcal{PT}$-symmetric periodic structures*, Phys. Rev. A **85**, 050101(R) (2012).

[26] X. Zhu, H. Ramezani, C. Shi, J. Zhu, and X. Zhang, *$\mathcal{PT}$-symmetric acoustics*, Phys. Rev. X **4**, 031042 (2014).

[27] C. M. Bender, B. K. Berntson, D. Parker, and E. Samuel, *Observation of PT phase transition in a simple mechanical system*, Am. J. Phys. **81**, 173 (2013).

[28] L. Ge and H. E. Türeci, *Antisymmetric PT-photonic structures with balanced positive- and negative-index materials*, Phys. Rev. A **88**, 053810 (2013).

[29] A. Mostafazadeh, *Pseudo-hermitian representation of quantum mechanics*, Int. J. Geom. Meth. Mod. Phys. **7**, 1191 (2010).

[30] J. Okolowicz, M. Ploszajczak, and I. Rotter, *Dynamics of quantum systems embedded in a continuum*, Phys. Rep. **374**, 271 (2003).

[31] W. D. Heiss, *Exceptional points of non-Hermitian operators*, J. Phys. A: Math. Gen. **37**, 2455 (2004).

[32] M. V. Berry, *Physics of nonhermitian degeneracies*, Czechoslovak J. Phys. **54**, 1039 (2004).

[33] N. Moiseyev, *Non-Hermitian Quantum Mechanics* (Cambridge, New York, 2011).

[34] C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, *Experimental observation of the topological structure of exceptional points*, Phys. Rev. Lett. **86**, 787 (2001).

[35] J. Wiersig, S.-W. Kim, and M. Hentschel, *Asymmetric scattering and nonorthogonal mode patterns in optical microspirals*, Phys. Rev. A **78**, 053809 (2008).

[36] L. Ge, Y. D. Chong, S. Rotter, H. E. Türeci, and A. D. Stone, *Unconventional modes in lasers with spatially varying gain and loss*, Phys. Rev. A **84**, 023820 (2011).

[37] M. Liertzer, L. Ge, A. Cerjan, A. D. Stone, H. E. Türeci, and S. Rotter, *Pump-induced exceptional points in lasers*, Phys. Rev. Lett. **108**, 173901 (2012).

[38] M. Brandstetter *et al.*, *Reversing the pump dependence of a laser at an exceptional point*, Nature Comm. **5**, 4034 (2014).

[39] R. El-Ganainy, M. Khajavikhan, and L. Ge, *Exceptional points and lasing self-termination in photonic molecules*, Phys. Rev. A **90**, 013802 (2014).

[40] L. Ge, K. G. Makris, D. N. Christodoulides, and L. Feng, *Scattering in $\mathcal{PT}$ and $\mathcal{RT}$ symmetric multimode waveguides: generalized conservation laws and spontaneous symmetry breaking beyond one dimension*, Phys. Rev. A **92**, 062135 (2015).

[41] I. V. Barashenkov, L. Baker, and N. V. Alexeeva, *$\mathcal{PT}$-symmetry breaking in a necklace of coupled optical waveguides*, Phys. Rev. A **87**, 033819 (2013).

[42] H. Ramezani, T. Kottos, R. El-Ganainy, and D. N. Christodoulides, *Unidirectional nonlinear PT-symmetric optical structures*, Phys. Rev. A **82**, 043803 (2010).

[43] E.-M. Graefe, *Stationary states of a PT symmetric two-mode BoseEinstein condensate*, J. Phys. A **48**, 444015 (2012).

[44] S. Longhi, *Spectral singularities and Bragg scattering in complex crystals,* Phys. Rev. A **81**, 022102 (2010).

[45] L. Ge and R. El-Ganainy, *Nonlinear modal interactions in parity-time ($\mathcal{PT}$) symmetric lasers*, Sci. Rep. **6**, 24889 (2016).

[46] M. Brandstetter, M. Liertzer, C. Deutsch, P. Klang, J. Schöberl, H. E. Türeci, G. Strasser, K. Unterrainer, and S. Rotter, *Reversing the pump dependence of a laser at an exceptional point*, Nature Comm. **5**, 4034 (2014).

[47] B. Peng, . K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang, *Parity-time-symmetric whispering-gallery microcavities*, Nature Phys. **10**, 394 (2014).

[48] A. Regensburger, C. Bersch, B. Hinrichs, G. Onishchukov, A. Schreiber, C. Silberhorn, and U. Peschel, *Photon propagation in a discrete fiber network: an interplay of coherence and losses*, Phys. Rev. Lett. **107**, 233902 (2011).

[49] R. El-Ganainy, L. Ge, M. Khajavikhan, and D. N. Christodoulides, *Supersymmetric laser arrays*, Phys. Rev. A **92**, 033818 (2015).

[50] Y. Castin and R. Dum, *Low-temperature Bose-Einstein condensates in time-dependent traps: Beyond the $U(1)$ symmetry-breaking approach*, Phys. Rev. A **57**, 3008 (1998).

[51] J. Andreasen, C. Vanneste, L. Ge, and H. Cao, *Effects of spatially nonuniform gain on lasing modes in weakly scattering random systems*, Phys. Rev. A **81**, 043818 (2010).

[52] G. P. Agrawal, *Nonlinear fiber optics, 3rd ed.* (Academic Press, San Diego, 2011).