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# Detecting Entanglement of Continuous Variables with Three Mutually Unbiased Bases

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An uncertainty relation is introduced for a symmetric arrangement of three mutually unbiased bases in continuous variable phase space, and then used to derive a bipartite entanglement criterion based on the variance of global operators composed of these three phase space variables. We test this criterion using spatial variables of photon pairs, and show that the entangled photons are correlated in three pairs of bases.

## I. INTRODUCTION

Entanglement detection in bipartite continuous-variable (CV) systems has typically been achieved using one out of several entanglement criteria involving measurement of two canonically conjugated variables, such as position and momentum. These criteria are usually cast in terms of the variance [1, 2], covariance matrix [3], moments matrix [4, 5] or entropy [6, 7] of global variables, defined as linear combinations of the local variables.

In principle, bipartite separable quantum states can be perfectly correlated in one basis. However, this perfect correlation implies that there is no correlation in the conjugate basis. Entangled quantum states, on the other hand, can be well correlated in both global position and global momentum variables. The seminal example is the Einstein-Podolsky-Rosen (EPR) state, which as originally proposed is a simultaneous eigenstate of the relative position ( $x_1 - x_2$ ) and total momentum ( $p_1 + p_2$ ) [8].

Of course it is possible to investigate quantum correlations in variables other than position and momentum. For example, using dimensionless  $x$  and  $p$ , one can consider rotated (dimensionless) operators of the form

$$\mathbf{q}_\theta = \cos \theta \mathbf{x} + \sin \theta \mathbf{p}. \quad (1)$$

Using superscripts 1 and 2 to refer to the two parts of the bipartite system, it is well known that the EPR state is an eigenstate of the relative coordinate  $\mathbf{q}_{\theta_1}^1 - \mathbf{q}_{\theta_2}^2$  when  $\theta_1 + \theta_2$  is an integer multiple of  $2\pi$ , and an eigenstate of the sum coordinate  $\mathbf{q}_{\theta_1}^1 + \mathbf{q}_{\theta_2}^2$  when  $\theta_1 + \theta_2$  is an odd multiple of  $\pi$ . In the context of transverse spatial correlations of photon pairs, this type of correlation was demonstrated experimentally in Refs. [9, 10], where the variances of canonically conjugate  $\mathbf{q}_{\theta_j}^j$  and  $\mathbf{q}_{\theta'_j}^j$  were chosen such that  $\theta_j - \theta'_j \equiv \pm\pi/2 \pmod{2\pi}$ . However, as will be discussed in the next section, for entanglement detection it is not necessary to perform measurements satisfying this restriction.

The sets of eigenstates corresponding to canonically conjugate variables are examples of *mutually unbiased*

*bases* (MUBs) [11]. This means that, if a quantum system is an eigenstate of the observable—say— $x$ , then the probability distribution of the state with respect to the conjugate observable  $p$  is uniform. In other words, the precise knowledge of the state in one basis corresponds to complete ignorance in the conjugate basis. Indeed, as we discuss in more detail below, any two CV operators  $\mathbf{q}_\theta$  and  $\mathbf{q}_{\theta'}$  of the form given in Eq. (1), with  $\theta' \not\equiv \theta \pmod{\pi}$  define a pair of MUBs. This fact has been implicitly used in CV measurement-based quantum computation [12–14]. Though it may have been known for some time that any pair of bases composed of the eigenstates of non-parallel (or non-antiparallel) phase space operators are MUBs, to our knowledge Weigert and Wilkinson [15] were the first to show that one can define up to three MUBs in the phase space of one CV bosonic mode.

In this work we study the use of a symmetric MUB triple for the investigation of quantum entanglement. We derive the relevant uncertainty relations for these variables and simple entanglement criteria based on the positive partial transpose (PPT) criterion [3, 16, 17], and use it to identify entanglement experimentally between two spatially entangled photons produced by Spontaneous Parametric Down Conversion (SPDC).

## II. MUTUALLY UNBIASED PHASE SPACE TRIPLE

Two bases  $\{|e\rangle\}$  and  $\{|f\rangle\}$  are mutually unbiased if all of their eigenstates have the same overlap  $|\langle e|f\rangle|$ . Position and momentum eigenstates are related via a Fourier transform, and consequently they satisfy  $|\langle x|p\rangle| = 1/\sqrt{2\pi}$  (we set  $\hbar = 1$ ). The position and momentum operators  $x$  and  $p$  satisfy the well-known Heisenberg-Robertson uncertainty relations [18]

$$\frac{1}{2}[(\Delta x)^2 + (\Delta p)^2] \geq \Delta x \Delta p \geq \frac{1}{2}|\langle [x, p] \rangle| = \frac{1}{2}. \quad (2)$$

Consider now two operators  $\mathbf{q}$  and  $\mathbf{q}'$  given by Eq. (1). These operators are related via a rotation in phase

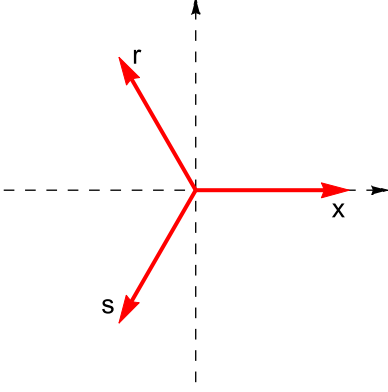


FIG. 1: Three phase space variables  $x$ ,  $r$  and  $s$  defining a mutually unbiased triple. Each variable is rotated  $120^\circ$  from the other two.

space, which is equivalent to a fractional Fourier transform (FRFT) [19, 20]. Let us define  $\theta_d = \theta' - \theta$ . Then we have

$$\mathbf{q}_{\theta'} = \mathbf{F}_{\theta_d}^\dagger \mathbf{q}_\theta \mathbf{F}_{\theta_d}, \quad (3)$$

where  $\mathbf{F}_{\theta_d}$  is the FRFT operator. Moreover, it is straightforward to show that

$$\Delta \mathbf{q}_{\theta'} \Delta \mathbf{q}_\theta \geq \frac{1}{2} |\langle \mathbf{q}_{\theta'}, \mathbf{q}_\theta \rangle| = \frac{1}{2} |i \sin \theta_d|. \quad (4)$$

The scalar product between eigenstates gives the kernel to the FRFT [19],

$$\langle q_{\theta'} | q_\theta \rangle = \sqrt{\frac{ie^{i\theta_d}}{2\pi |\sin \theta_d|}} \exp \left[ i \frac{\cot \theta_d}{2} (q_\theta^2 + q_{\theta'}^2) - i \frac{q_\theta q_{\theta'}}{\sin \theta_d} \right]. \quad (5)$$

One sees immediately that  $|\langle q_{\theta'} | q_\theta \rangle| = (2\pi |\sin(\theta' - \theta)|)^{-1/2}$ , which does not depend on  $q_\theta$  nor  $q_{\theta'}$ , indicating that these two bases are mutually unbiased when  $\sin(\theta' - \theta) \neq 0$  [35].

Recently, Weigert and Wilkinson [15] have shown that one can define three mutually unbiased bases in the phase space of one bosonic mode. For example, consider the dimensionless operators  $x$ ,  $r$ , and  $s$  corresponding to the phase space variables illustrated in Fig. 1. Explicitly,

$$r = \cos \frac{2\pi}{3} x + \sin \frac{2\pi}{3} p, \quad (6)$$

and

$$s = \cos \frac{4\pi}{3} x + \sin \frac{4\pi}{3} p. \quad (7)$$

These operators define a set of MUBs, since their eigenstates satisfy

$$|\langle x | r \rangle| = |\langle x | s \rangle| = |\langle r | s \rangle| = \frac{1}{\sqrt{2\pi \sin \frac{2\pi}{3}}} = \frac{1}{\sqrt{\pi\sqrt{3}}}. \quad (8)$$

Using Eqs. (6) and (7) and (4), these operators satisfy the uncertainty relations

$$\Delta x \Delta r \geq \frac{1}{2} |\langle [x, r] \rangle| = \frac{1}{2} \left| i \sin \frac{2\pi}{3} \right| = \frac{\sqrt{3}}{4}, \quad (9a)$$

$$\Delta x \Delta s \geq \frac{1}{2} |\langle [x, s] \rangle| = \frac{1}{2} \left| i \sin \frac{4\pi}{3} \right| = \frac{\sqrt{3}}{4}, \quad (9b)$$

and

$$\Delta r \Delta s \geq \frac{1}{2} |\langle [r, s] \rangle| = \frac{1}{2} \left| i \sin \frac{2\pi}{3} \right| = \frac{\sqrt{3}}{4}. \quad (9c)$$

We note that, up to a rotation of the entire phase-space,  $x$ ,  $r$  and  $s$  constitute the only possible normalized set of three MUBs. Nevertheless, if we allow for individual scaling of the variables, there are other sets of three variables that satisfy the condition of MUBs [15]. Here we focus on the operators defined in Eqs. (6) and (7), since they can be obtained by simple rotations in phase space, which can be achieved experimentally with relative ease in a number of systems [9, 10, 21, 22].

#### A. Uncertainty relations for the symmetric phase space triple

A number of uncertainty relations can be derived for the variables shown in Fig. 1 a). Some of these follow trivially from the usual uncertainty relations for pairs of operators (9). For example, taking the product of all three pairwise URs (9) gives

$$(\Delta x)^2 (\Delta r)^2 (\Delta s)^2 \geq \frac{3\sqrt{3}}{64} \approx 0.08, \quad (10)$$

which does not appear to be tight, since for the vacuum state the triple product above is  $1/8 = 0.125$ .

A tight uncertainty relation for three phase-space variables was recently demonstrated by Kechrimparis and Weigert [23, 24]. In that work, they use the asymmetric phase-space triple  $x$ ,  $p$  and  $q = -x - p$ , which is equivalent to the one we use here through a symplectic transformation and scaling. While a tight triple product relation for  $(\Delta x)^2 (\Delta r)^2 (\Delta s)^2$  could be obtained directly from the result in Ref. [23], it is illustrative to present the alternative derivation that follows.

First, let us use Eqs. (6) and (7) and the definition of variance to write

$$(\Delta r)^2 = \frac{1}{4} (\Delta x)^2 + \frac{3}{4} (\Delta p)^2 - \frac{\sqrt{3}}{4} \langle \{x, p\} \rangle + \frac{\sqrt{3}}{2} \langle x \rangle \langle p \rangle, \quad (11)$$

and

$$(\Delta s)^2 = \frac{1}{4} (\Delta x)^2 + \frac{3}{4} (\Delta p)^2 + \frac{\sqrt{3}}{4} \langle \{x, p\} \rangle - \frac{\sqrt{3}}{2} \langle x \rangle \langle p \rangle, \quad (12)$$

which multiplied together give

$$(\Delta r)^2(\Delta s)^2 = \frac{1}{16}[(\Delta x)^2 + 3(\Delta p)^2]^2 - \frac{3}{16}((\{x, p\}) - 2\langle x \rangle \langle p \rangle)^2. \quad (13)$$

The Schrödinger-Robertson UR for operators  $x$  and  $p$ , which is a state-dependent generalization of (2), reads [18]

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4} + \frac{1}{4}((\{x, p\}) - 2\langle x \rangle \langle p \rangle)^2. \quad (14)$$

which can be rewritten in the form

$$-((\{x, p\}) - 2\langle x \rangle \langle p \rangle)^2 \geq 1 - 4(\Delta x)^2(\Delta p)^2. \quad (15)$$

Using this expression in Eq. (13), and performing some algebra, we have

$$(\Delta r)^2(\Delta s)^2 \geq \frac{3 + ((\Delta x)^2 - 3(\Delta p)^2)^2}{16}, \quad (16)$$

which means the triple product of the variances is then

$$(\Delta x)^2(\Delta r)^2(\Delta s)^2 \geq (\Delta x)^2 \frac{3 + [(\Delta x)^2 - 3(\Delta p)^2]^2}{16}. \quad (17)$$

Optimizing the right-hand side over all positive values of the  $x$  and  $p$  variances respecting the Heisenberg UR  $(\Delta x)^2(\Delta p)^2 \geq 1/4$  gives (see Appendix A)

$$(\Delta x)^2(\Delta r)^2(\Delta s)^2 \geq \frac{1}{8}. \quad (18)$$

Contrary to the usual Heisenberg uncertainty relation, which can also be saturated by Gaussian squeezed states, inequality (18) is saturated exclusively by the set of coherent states, as we show in Appendix A.

### III. ENTANGLEMENT CRITERION WITH TRIPLES

Quantum mechanical uncertainty relations combined with the positive partial transpose argument [16, 17] can be used to derive entanglement criteria [3, 6, 25]. Let us define the global phase space operators

$$X_{\pm} = x_1 \pm x_2, \quad (19a)$$

$$R_{\pm} = r_1 \pm r_2, \quad (19b)$$

and

$$S_{\pm} = s_1 \pm s_2. \quad (19c)$$

Note that we can also write

$$R_{\pm} = \cos \frac{2\pi}{3} X_{\pm} + \sin \frac{2\pi}{3} P_{\pm}, \quad (20)$$

and

$$S_{\pm} = \cos \frac{4\pi}{3} X_{\pm} + \sin \frac{4\pi}{3} P_{\pm}, \quad (21)$$

where  $P_{\pm} = p_1 \pm p_2$ , and we note that  $[X_{\pm}, P_{\pm}] = 2i$  and  $[X_{\pm}, P_{\mp}] = 0$ . Following the derivations in section II A, it is straightforward to show that the global operators (19) satisfy the uncertainty relation

$$(\Delta X_{\pm})^2(\Delta R_{\pm})^2(\Delta S_{\pm})^2 \geq 1, \quad (22)$$

where either the top row of all plus signs or the bottom row of all minus signs is considered. These URs can be used to develop entanglement criteria following arguments previously given in Refs. [6, 7, 25–27].

Naturally, any bipartite state  $\rho_{12}$  satisfies either of the uncertainty relations (22). Now suppose its partial transpose with respect to –say–system two,  $\rho_{12}^{T_2}$ , is positive. In this case we can say for sure that  $\rho_{12}^{T_2}$  is a bonafide quantum state and also satisfies the uncertainty relations (22). Following this argument, the PPT criterion establishes that if a bipartite state  $\rho_{12}$  has a negative partial transpose, then it cannot be separable and must be entangled [16, 17]. For continuous variables, Simon has shown that the partial transpose is equivalent to a mirror reflection, transforming the phase space variable  $p_2 \rightarrow -p_2$  and leaving all others unchanged [3]. This means taking the global variable  $P_{\mp} \rightarrow P_{\pm}$ .

Let us now define two new global operators

$$U_{\pm} = \cos \frac{2\pi}{3} X_{\pm} + \sin \frac{2\pi}{3} P_{\mp} = r_1 \pm s_2 \quad (23)$$

and

$$V_{\pm} = \cos \frac{4\pi}{3} X_{\pm} + \sin \frac{4\pi}{3} P_{\mp} = s_1 \pm r_2. \quad (24)$$

The standard deviations (square roots of variances) of operators  $X_{\pm}$ ,  $U_{\pm}$  and  $V_{\pm}$  for the original state can be related to the standard deviations of variables  $X_{\pm}$ ,  $R_{\pm}$  and  $S_{\pm}$  on the transposed state as  $\Delta X_{\pm, T} = \Delta X_{\pm}$ ,  $\Delta R_{\pm, T} = \Delta U_{\pm}$ ,  $\Delta S_{\pm, T} = \Delta V_{\pm}$ , where again  $T$  stands for partial transposition. Thus, using the UR (22), any separable state  $\rho_{12}$  will satisfy the inequalities

$$(\Delta X_{\pm})^2(\Delta U_{\pm})^2(\Delta V_{\pm})^2 \geq 1. \quad (25)$$

A state that violates either of the inequalities (25) has a negative partial transpose and is thus entangled. Moreover, the operators  $X_j$ ,  $U_j$  and  $V_j$  ( $j = +$  or  $-$ ) commute. This means they share a common eigenstate, namely, the EPR state [8], for which all of the variances are zero.

Let us now give a real world example of a quantum state that violates criteria (25). Under appropriate conditions, the spatial variables of photon pairs produced from SPDC are well described by the double Gaussian wave function [28–30]

$$\Psi(x_1, x_2) = A \exp\left(-\frac{(x_1 + x_2)^2}{4\sigma_+^2}\right) \exp\left(-\frac{(x_1 - x_2)^2}{4\sigma_-^2}\right), \quad (26)$$

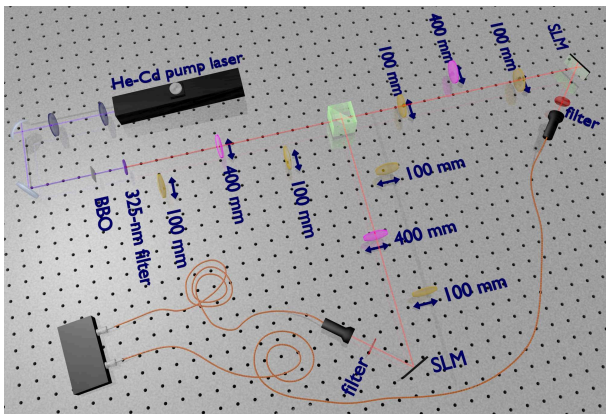


FIG. 2: Experimental setup. Twin photons produced by the process of SPDC are subjected to rotations of  $\pi/3$  (depicted by a single pink lens) and of  $\pi$  (depicted by a pair of yellow lenses). The SLMs are spatial light modulators used to automatically scan a slit across the transverse distribution of the down-converted photons.

where  $A = 1/\sqrt{\pi\sigma_+\sigma_-}$  and the variables  $x$  refer to the transverse position variables at the exit face of the nonlinear crystal. Here we consider the simple case of one spatial dimension. This state is entangled when  $\sigma_- \neq \sigma_+$ . The SPDC state (26) is analogous to the two-mode squeezed state when  $\sigma_- = 1/\sigma_+$ . Though the EPR state mentioned above is unphysical, it is a limiting case of the two-mode squeezed state in the case of infinite squeezing, when  $\sigma_- \rightarrow 0$ .

Under usual experimental conditions, we have  $\sigma_- \ll \sigma_+$  and the two-photon state shows position correlation at the exit face of the crystal. Using lenses or free-propagation, it is possible to observe correlations in other phase space variables. For example, an optical Fourier transform system allows one to observe the momentum anti-correlations of this state [31]. Using other optical systems, correlations in other (rotated) variables can be observed. It has been shown [9, 10] that entangled states of the form (26) with  $\sigma_- < \sigma_+$  display position correlations when the sum of the phase-space rotation angles  $\theta_1$  and  $\theta_2$ , with respect to  $x_1$  and  $x_2$ , respectively, is an even multiple of  $\pi$ , and anti-correlation when the sum is an odd multiple of  $\pi$ . From the definitions (6) and (7), the  $r$  variables are given by a rotation of  $2\pi/3$  and the  $s$  variables by a rotation of  $4\pi/3$ . Noting that  $U_{\pm} = r_1 \pm s_2$  and  $V_{\pm} = s_1 \pm r_2$ , we can see that  $\theta_1 + \theta_2 = 2\pi$  in both cases. Since  $x_1$  and  $x_2$  are also correlated, with  $\theta_1 + \theta_2 = 0$ , the state (26) should show correlations in all three sets of variables, leading to a violation of the entanglement criteria (25).

#### IV. EXPERIMENT AND RESULTS

To test the entanglement criteria derived, we observed the transverse spatial variables of twin photons generated by the process of SPDC [29]. A 2-mm thick beta-barium borate (BBO) crystal cut for type-I phase matching SPDC was pumped by a continuous-wave 325-nm He-Cd laser beam, producing collinear frequency degenerate converted beams at a wavelength of 650 nm. The two down-converted beams were separated at a 50/50 beam splitter and each directed by mirrors and lenses to Holographic Pluto phase-only spatial light modulators (SLM), which were used to perform position correlation measurements by scanning a phase slit in the transverse profile of the beams, as described in Ref. [32].

Both beams were scanned in the horizontal direction of the transverse detection planes over a region of interest of 12 mm. Using slits of  $80 \mu\text{m}$  (equivalent to 10 pixels of our SLM), this procedure totalled  $150 \times 150 = 22500$  data points per measurement. Each data point was sampled for 3 s, leading to the estimated joint detection probabilities. After reflection by the SLMs, the beams were sent through 10-nm FWHM interference filters centered at 650 nm, and were then coupled into multi-mode optical fibers connected to single-photon avalanche diodes (SPAD).

Non-confocal lenses were used in an optical fractional Fourier transform arrangement [9, 10, 19, 33, 34] to achieve the phase-space rotations described in section II. As can be seen in the sketch of our experimental setup displayed in Fig. 2, between the BBO crystal and the beam splitter either two 100-mm confocal lenses were used to achieve a phase-space rotation of  $\pi$ , or a single 400-mm lens was used to perform a  $\pi/3$  rotation. After the beam splitter, similar sets of lenses were used in order to achieve  $2\pi$ ,  $2\pi/3$  or  $4\pi/3$  in each converted beam, depending on the lenses chosen.

In order to interpret the lens systems used as fractional Fourier transforms and therefore as rotations in phase space, dimensionless variables must be used. These variables are achieved through the scaling factor  $d = \sqrt{f \sin(\theta)/k}$ , where  $f$  is the focal length of the lens,  $\theta$  is the phase-space rotation angle and  $k$  is the wavenumber of the down-converted beams [20]. Choosing  $\theta = \pi/3$ , we were able to use the same scaling factor for both the  $\pi/3$  and the  $\pi$  rotations. In this experiment, this corresponds to  $d = 189 \mu\text{m}$ .

The measurement results can be seen in Fig. 3. On the top row the reconstructed joint distributions of the dimensionless variables are shown, from which one can clearly see the intensity correlations in all three pairs of spatial variables. We note that the usual momentum anti-correlations are never observed, though they play a crucial role in the  $r_1, s_2$  and  $s_1, r_2$  correlations, as can be seen in Eqs. (23) and (24). The blue points in the plots on the bottom row are the marginal distributions of the relevant global variables. The error bars correspond to error due to Poissonian count statistics, which attributes



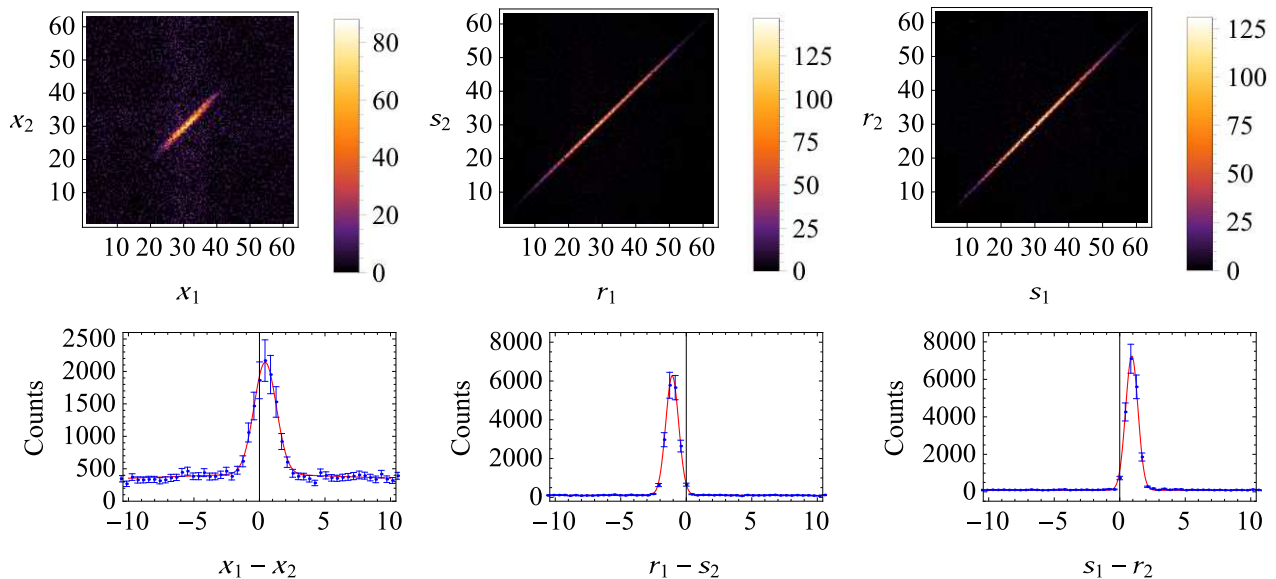


FIG. 3: Measurement results. On the top row we can see the reconstructed joint distributions of the dimensionless variables as obtained experimentally, and on the bottom row are their marginal distributions. The curves shown correspond to Gaussian best fits. The error bars correspond to the square root of the photon counts, due to the fact that the photon-count distribution is Poissonian.

$W$	$(\Delta W_-)^2$	$(\Delta W_+)^2$	$C_W$
$X$	$0.74 \pm 0.02$	$321 \pm 29$	$21 \pm 2$
$U$	$0.2455 \pm 0.0006$	$554 \pm 69$	$55 \pm 3$
$V$	$0.225 \pm 0.001$	$598 \pm 66$	$52 \pm 3$

TABLE I: Variances of global variables  $W_{\pm}$  ( $W = X, U, V$ ) obtained from gaussian fits of marginal distributions of coincidence plots shown in Fig. 3. The last column is the correlation coefficient, defined in the text.

a standard deviation equal to the square root of the count rate. The red curves correspond to Gaussian best fits of the marginal distributions, from which we are able to obtain the variances that are shown in the second column of Table I.

With these results, inequality (25) is

$$(\Delta X_-)^2 (\Delta U_-)^2 (\Delta V_-)^2 = (0.041 \pm 0.001) \not\geq 1, \quad (27)$$

showing that the correlations in the state are sufficient to violate the inequality. As can be seen in the table, the variance  $(\Delta X_-)^2$  is larger than the other two. This is to be expected, as one can see from inspection of Eqs. (23) and (24) that the variances in the rotated global variables in fact depend on both the variance in the position as well as the momentum. Indeed, since our state is not symmetric, meaning  $\sigma_- \neq 1/\sigma_+$  in the wavefunction (26), we do not expect  $(\Delta X_-)^2$  to be equal to  $(\Delta U_-)^2$  nor  $(\Delta V_-)^2$ .

However, we should see the same amount of correlation in each set of measurements, as we will now explain.

Following [30], let us define a correlation coefficient  $C_W$  ( $W = X, U, V$ ) as  $C_W = \Delta W_+ / \Delta W_-$ . Using the double Gaussian wavefunction in Eq. (26), together with the variances defined in Eqs. (23) and (24), and performing a little algebra we find

$$C_U = C_V = \sqrt{\frac{\sigma_+^2 + 3/\sigma_-^2}{\sigma_-^2 + 3/\sigma_+^2}} = \frac{\sigma_+}{\sigma_-}, \quad (28)$$

which is exactly the correlation ratio  $C_X = \Delta X_+ / \Delta X_-$ . The value  $\sigma_+ / \sigma_-$  is related to the Schmidt coefficient of the two-photon state, and thus to the amount of entanglement. The values of the variances in the sum coordinates  $(\Delta W_+)^2$ , as well as the correlation coefficients are shown in Table I. One can see from the table that we observe less correlation in the position  $(x_1, x_2)$  measurements, by roughly a factor of 2.5 compared to the measurements in the other planes. This is clearly observable in the coincidence plots of Fig. 3. The reduced correlation in the position measurements is in most part due to two factors. First, there is extra noise that appears (see Fig. 3) in the near-field measurements, owing to fluorescence of the laser beam on the dichroic filter, which is then imaged onto the detection planes. The rotated variables do not suffer from this noise as drastically, which is an advantage to not working in the image plane of the source [9]. Second, the near-field correlations are governed by a non-Gaussian function that is not separable in the  $x_1 - x_2$  and  $y_1 - y_2$  coordinates [29]. Ignoring the  $y$  component essentially causes mixing of the state describing the correlations in the  $x_1 - x_2$  direction, which leads to broadening of the corresponding probability distribution. This fact

is ignored by the double-gaussian approximation that we have used. The other measurements do not reveal this mixing as much since they are also dependent on the far-field correlations, which are governed by the Gaussian profile of the pump beam, and are thus separable in the  $x$  and  $y$  directions. Experimentally characterizing this non-separability of our source is a subject for future work.

## V. CONCLUSION

We have derived uncertainty relations and entanglement criteria for continuous variables using three mutually unbiased bases. By measuring the spatial correlations between photon pairs generated by SPDC we were able to show that the photons were correlated in three pairs of variables, and that these correlations lead to violation of a separability criterion. Considering entanglement detection in three mutually unbiased bases could be interesting for quantum key distribution, and might improve the sensitivity to an eavesdropper.

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### Appendix A: Derivation of Eq. 18

Let us introduce the shorthand notation:  $\eta = (\Delta x)^2$  and  $\xi = (\Delta p)^2$ . According to (17), we aim to find a minimum of the function

$$g(\eta, \xi) = \frac{\eta}{16} [3 + (\eta - 3\xi)^2], \quad (\text{A1})$$

given the inequality constraint  $\eta\xi \geq 1/4$  coming from the Heisenberg Uncertainty Relation (HUR). The function  $g(\eta, \xi)$  is sketched in Fig. 4.

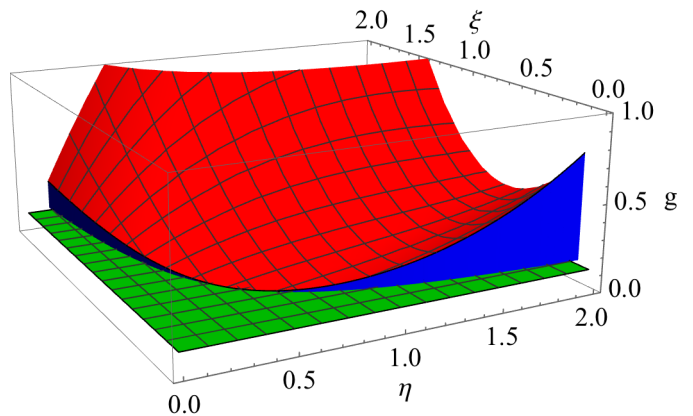


FIG. 4: The function  $g(\eta, \xi)$  in the domain  $\eta\xi \geq 1/4$  (red surface), and the lower bound  $1/8$  (green plane). We can see that the function  $g$  reaches its lowest value  $1/8$  at the point  $(0.5, 0.5)$ .

Consider first the sharp-inequality case  $\eta\xi > 1/4$ . We can calculate the derivatives

$$\frac{\partial g}{\partial \eta} = \frac{3}{16} (1 + \eta^2 - 4\eta\xi + 3\xi^2), \quad (\text{A2})$$

$$\frac{\partial g}{\partial \xi} = \frac{3\eta}{8} (3\xi - \eta), \quad (\text{A3})$$

and easily see that the system of equations

$$\frac{\partial g}{\partial \eta} = 0, \quad \frac{\partial g}{\partial \xi} = 0, \quad (\text{A4})$$

has no solutions.

In the second case, when the HUR is saturated, the function to be minimized becomes

$$g_{\text{sat}}(\eta) \equiv g(\eta, 1/4\eta) = \frac{\eta}{16} \left[ 3 + \left( \eta - \frac{3}{4\eta} \right)^2 \right]. \quad (\text{A5})$$

We find

$$\frac{dg_{\text{sat}}}{d\eta} = \frac{3}{256} \left( 16\eta^2 - \frac{3}{\eta^2} + 8 \right), \quad (\text{A6})$$

and this derivative is equal to 0 when  $\eta = \pm 1/2$  or  $\eta = \pm i\sqrt{3}/2$ . Since the parameter  $\eta$  is real and non-negative, we are left with the single solution  $\eta = 1/2$ . This solution corresponds to the Gaussian coherent state as in this case also  $\xi = 1/2$ . Obviously  $g(1/2, 1/2) = 1/8$ . Finally, we check that the second derivative

$$\frac{d^2 g_{\text{sat}}}{d\eta^2} = \frac{9}{128\eta^3} + \frac{3\eta}{8}, \quad (\text{A7})$$

is always positive, so the solution is a true minimum.

One might imagine some non-gaussian state with appropriate symmetry, or gaussian squeezed state that saturates the triple product UR. However, the solution above dictates explicitly that  $(\Delta x)^2 = (\Delta p)^2 = 1/2$ , which is only attainable by the vacuum state and by displaced vacuum states—the set of coherent states. In other

words, there are no non-gaussian states that saturate the triple product UR. Moreover, it is impossible to apply a squeezing operation such that the variances in both the  $x$  and  $p$  (or any perpendicular directions) remain equal to  $1/2$ .

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