Dirac Hamiltonian and Reissner-Nordström metric: Coulomb interaction in curved space-time

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Dirac Hamiltonian and Reissner–Nordström Metric: Coulomb Interaction in Curved Space–Time

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We investigate the spin-1/2 relativistic quantum dynamics in the curved space-time generated by a central massive charged object (black hole). This necessities a study of the coupling of a Dirac particle to the Reissner–Nordström space-time curvature and the simultaneous covariant coupling to the central electrostatic field. The relativistic Dirac Hamiltonian for the Reissner–Nordström geometry is derived. A Foldy–Wouthuysen transformation reveals the presence of gravitational, and electro-gravitational spin-orbit coupling terms which generalize the Fokker precession terms found for the Dirac–Schwarzschild Hamiltonian, and other electro-gravitational correction terms to the potential proportional to $\alpha^n G$, where α is the fine-structure constant, and G is the gravitational coupling constant. The particle-antiparticle symmetry found for the Dirac–Schwarzschild geometry (and for other geometries which do not include electromagnetic interactions) is shown to be explicitly broken due to the electrostatic coupling. The resulting spectrum of radially symmetric, electrostatically bound systems (with gravitational corrections) is evaluated for example cases.

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I. INTRODUCTION

We continue a series of investigations [1–5] on the coupling of Dirac particles to curved space–time backgrounds. Foundations of the formalism date back to the time of Brill and Wheeler [6], and Greiner, Soffel, Müller, and Boulware [7, 8], who established the formalism of the spin connection matrices. Modern computer algebra [9] makes it possible to perform independent evaluations of spin connection matrices for specific space–time geometries, and the formalism of the Foldy–Wouthuysen transformation facilitates the identification of the nonrelativistic limit, and leads to a consistent interpretation of the nonrelativistic operators [10, 11].

Recently, it has been recalled [1–5] that the Dirac equation provides for an ideal tool to study gravitational interactions of antiparticles. One should recall that the original surprising prediction of the Dirac equation [12, 13] was the existence of positrons, which are the antiparticles of electrons. The Dirac equation describes particles and their antiparticles simultaneously. The mass term in the Dirac equation is first and foremost the inertial mass. However, when coupling the particle to curved space-time and identifying the Hamiltonian, one can establish a connection of the inertial mass to the gravitational mass because the Foldy–Wouthuysen transformed Hamiltonian [2] contains the gravitational potential (plus relativistic corrections, of course). On the basis of this consideration, a symmetry relation was found in Ref. [1] and confirmed in Ref. [2] which established that particles and antiparticles behave identically in the presence of a gravitational field, i.e., both particles and antiparticles are attracted by gravity.

The symmetry relation from Refs. [1, 2] holds for specific classes of metrics. On the one hand, one can show that the Reissner–Nordström metric (which describes a charged gravitational center) belongs to a class of geometries, where a priori, particle–antiparticle symmetry should exist [1, 2], and both particles and antiparticles should be affected identically by the metric. On the other hand, we know that the Dirac equation can be used to describe charged spin–1/2 particles, and that particle-antiparticle symmetry does not hold for electromagnetic interactions [11]. By definition, antiparticles carry the opposite electric charge. How can this apparent contradiction be resolved? The answer is that the presence of the explicit covariant coupling to the electrostatic field, not to the gravitational field, breaks the particle-antiparticle symmetry. It means that we must concern ourselves with the coupling of the Dirac particle to the curved space–time, while at the same time include the electrostatic interaction. The Dirac equation becomes covariant with respect to two gauge groups, the $U(1)$ gauge group of quantum electrodynamics and the $SO(1;3)$ group of local Lorentz transformations. The double-covariant derivative entails the replacement $\mathrm{i}\partial_\mu \to \mathrm{i}(\partial_\mu - \Gamma_\mu) - q\,A_\mu$, where Γ_μ is the spin-connection matrix, q is the charge of the particle, and A_μ is the vector potential [14]. The former covariance is ensured by the four-vector potential A^μ in the Dirac equation, while the latter is described by the spin-connection matrices Γ_μ, both to be discussed in more detail below.

Bound systems featuring both electromagnetic as well as gravitational corrections could be of interest for a number of reasons, not only in the sense of tiny gravitational effects which might be observable in bound systems [15], but also in the context of micro black holes which have been proposed as conceivable candidates for dark matter [16–18] and even classes of novel phenomena at accelerators [19].

The article is organized as follows. In Sec. II, we transform the Reissner–Nordström metric into isotropic coordinates, and transform the electrostatic potential accordingly. These results are then used in Sec. III A in the explicit derivation of the Dirac–Reissner–Nordström Hamiltonian. We then apply the Foldy–Wouthuysen transform to the resulting Hamiltonian in Sec. III B. In Sec. IV, we evaluate the bound–state energies of the transformed Hamiltonian, and consider example cases. Finally, conclusions are drawn in Sec. V. Except where otherwise stated, we use units such that $c = \hbar = \epsilon_0 = 1$ throughout this paper.

II. REISSNER–NORDSTRÖM METRIC AND ELECTROSTATIC POTENTIAL

A. Isotropic Coordinates

In formulating the Dirac equation coupled to the Reissner–Nordström metric we can in large part follow the same steps taken to formulate the gravitationally coupled Dirac Hamiltonian [2]. This also provides the opportunity to check our final result against results previously obtained, when the charge of the gravitational center vanishes ($Q \to 0$) and the transformed Dirac–Reissner–Nordström Hamiltonian reduces to the transformed Dirac–Schwarzschild Hamiltonian found in Eq. (21) of Ref. [2]. We require that the metric be isotropic, in order to ensure that the effective speed of light, expressed in global coordinates does not depend on the spatial direction of the light ray at a given space-time point (note that the speed of light is not constant when expressed in global coordinates, a fact which in particular, allows for the existence of the Shapiro time delay [20–23]). We follow ideas of Eddington [24] and transform the Reissner–Nordström metric to isotropic coordinates. The derivation of the Reissner–Nordström metric is recalled in Appendix A, with the result

$$\begin{aligned} \mathrm{d}s^2 = &\left(1 - \frac{r_\mathrm{s}}{\mathcal{R}} + \frac{r_Q^2}{\mathcal{R}^2}\right) \mathrm{d}t^2 \\ &- \left(1 - \frac{r_\mathrm{s}}{\mathcal{R}} + \frac{r_Q^2}{\mathcal{R}^2}\right)^{-1} \mathrm{d}\mathcal{R}^2 - \mathcal{R}^2 \, \mathrm{d}\Omega^2 \,, \end{aligned} \tag{1}$$

where $r_\mathrm{s} = 2\,G\,M/c^2$ is the Schwarzschild radius and $r_Q^2 = G\,Q^2/(4\pi\epsilon_0 c^4)$ (we temporarily restore SI mksA units for the conversions). In order to convert the Reissner–Nordström metric into a spatially isotropic form, we use the transformation

$$\mathcal{R} = r\left(\left(1 + \frac{r_\mathrm{s}}{4r}\right)^2 - \frac{r_Q^2}{4r^2}\right) = r\,A(r)\,. \tag{2}$$

Under this transform, we find

$$\mathrm{d}\mathcal{R} = \left(\left(1 - \frac{r_\mathrm{s}}{4r}\right)\left(1 + \frac{r_\mathrm{s}}{4r}\right) + \frac{r_Q^2}{4r^2}\right) \mathrm{d}r = B(r)\,\mathrm{d}r\,, \tag{3}$$

and

$$1 - \frac{r_\mathrm{s}}{\mathcal{R}} + \frac{r_Q^2}{\mathcal{R}^2} = \frac{\left(\left(1 - \frac{r_\mathrm{s}}{4r}\right)\left(1 + \frac{r_\mathrm{s}}{4r}\right) + \frac{r_Q^2}{4r^2}\right)^2}{\left(\left(1 + \frac{r_\mathrm{s}}{4r}\right)^2 - \frac{r_Q^2}{4r^2}\right)^2} = \frac{B(r)^2}{A(r)^2}\,. \tag{4}$$

The metric becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{B(r)^2}{A(r)^2}\mathrm{d}t^2 - \frac{A(r)^2}{B(r)^2}B(r)^2\mathrm{d}r^2 - r^2\,A(r)^2\mathrm{d}\Omega \\ &= \frac{B(r)^2}{A(r)^2}\mathrm{d}t^2 - A(r)^2\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right)\,, \end{aligned} \tag{5}$$

i.e.,

$$\mathrm{d}s^2 = w(r)^2\mathrm{d}t^2 - v(r)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\,, \tag{6a}$$

$$w(r) = \frac{\left(1 - \frac{r_\mathrm{s}}{4r}\right)\left(1 + \frac{r_\mathrm{s}}{4r}\right) + \frac{r_Q^2}{4r^2}}{\left(1 + \frac{r_\mathrm{s}}{4r}\right)^2 - \frac{r_Q^2}{4r^2}}\,, \tag{6b}$$

$$v(r) = \left(1 + \frac{r_\mathrm{s}}{4r}\right)^2 - \frac{r_Q^2}{4r^2} \tag{6c}$$

As in Refs. [2, 4, 5], we keep terms only to the first order in G. Both r_s and r_Q^2 are proportional to G; hence, $w(r)$ and $v(r)$ are approximated to

$$w(r) \approx 1 - \frac{r_\mathrm{s}}{2r} + \frac{r_Q^2}{2r^2}\,, \quad v(r) \approx 1 + \frac{r_\mathrm{s}}{2r} - \frac{r_Q^2}{4r^2}\,. \tag{7}$$

In the limit $Q \to 0$, we recover the $w(r)$ and $v(r)$ from the Schwarzschild metric [see Eq. (14) of Ref. [2]].

B. Electrostatic Potential

As shown in Appendix A, the nonzero elements of the field strength tensor are

$$F_{t\mathcal{R}} = -F_{\mathcal{R}t} = \frac{Q}{4\pi\,\mathcal{R}^2}\,. \tag{8}$$

By definition [see Eq. (2.2.28) of Ref. [25]] the field strength tensor is given as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{9}$$

We then find that the resulting equation is solved by

$$A_0 = \frac{Q}{4\pi\,\mathcal{R}}\,, \quad \vec{A} = \vec{0}\,. \tag{10}$$

Applying the isotropic transform [Eq. (2)] to our potential we

$$A_0 = \frac{Q}{4\pi\,r\left(\left(1 + \frac{r_\mathrm{s}}{4r}\right)^2 - \frac{r_Q^2}{4r^2}\right)}\,. \tag{11}$$

Again, we are keeping terms only to the first order in G. Thus, when expressed in terms of the isotropic radial coordinate r, we have

$$A_0 = \frac{Q}{4\pi\,r}\left(1 - \frac{r_\mathrm{s}}{2r} + \frac{r_Q^2}{4r^2}\right) \tag{12}$$

for the electrostatic potential.

III. DIRAC HAMILTONIAN FOR THE REISSNER–NORDSTRÖM METRIC

A. Relativistic Hamiltonian

In order to derive the Dirac Hamiltonian for the Reissner–Nordström metric, one uses the double-covariant coupling prescription $\mathrm{i}\partial_\mu \to \mathrm{i}(\partial_\mu - \Gamma_\mu) - q\,A_\mu$ where Γ_μ is the spin-connection matrix and A_μ is the electrostatic potential, both expressed in isotropic coordinates. Using the form given in Eqs. (6a) and (7) for the Reissner–Nordström metric, one readily evaluates the spin-connection matrices Γ_μ using general formulas and inserts A_μ from Eq. (12). The technical details of the calculation can be found in Appendix B. The Dirac–Reissner–Nordström Hamiltonian, to the first order in G, is finally found as

$$
\begin{aligned}
H_{\rm RN} = & \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p}, \left(1 - \frac{r_{\rm s}}{r} + \frac{3r_Q^2}{4r^2}\right)\right\} \\
& + \frac{Z_Q\,Z_q\,\alpha}{r}\left(1 - \frac{r_{\rm s}}{2r} + \frac{r_Q^2}{4r^2}\right) \\
& + \beta\,m\left(1 - \frac{r_{\rm s}}{2r} + \frac{r_Q^2}{2r^2}\right),
\end{aligned} \tag{13}
$$

where in natural units, we have

$$
q\,Q = 4\pi\,Z_Q\,Z_q\,\alpha\,. \tag{14}
$$

Z_Q and Z_q are the nuclear charge numbers associated with Q and q, respectively. For $Q \to 0$ (which implies $Z_Q \to 0$ and $r_Q \to 0$), we recover the Dirac–Schwarzschild Hamiltonian [2].

B. Foldy–Wouthuysen Transformation

An exact Foldy–Wouthuysen transformation may be used in the case of the free Dirac Hamiltonian [11]. More complicated Hamiltonians require a perturbative approach, expanding about some perturbation parameter. Using an approach similar to the steps taken in Eq. (3) of Ref. [1], we define the dimensionless variable ρ in terms of the fine structure constant, as

$$
\rho = \frac{r}{a_0} \quad a_0 = \frac{\hbar}{\alpha_{\rm eff}\,mc}\,, \quad \alpha_{\rm eff} = \frac{q\,Q}{4\pi\,\epsilon_0\,\hbar\,c} = Z_Q\,Z_q\,\alpha\,, \tag{15}
$$

where we have temporarily implemented SI mkSA units for the sake of clarity (a_0 is a generalized Bohr radius, while $\alpha_{\rm eff}$ is an effective "fine-structure" constant, i.e., coupling constant, for the bound system of charged black hole and test particle). Then, in natural units,

$$
r = \frac{1}{\alpha_{\rm eff}\,m}\rho\,, \quad \vec{\nabla}_r = \alpha_{\rm eff}\,m\,\vec{\nabla}_\rho\,, \tag{16a}
$$

$$
\vec{p} = -\,\mathrm{i}\,\alpha_{\rm eff}\,m\,\vec{\nabla}_\rho\,. \tag{16b}
$$

We then use $\alpha_{\rm eff}$ as our expansion parameter in our calculation, keeping terms up to $\alpha_{\rm eff}^4$, and to the first order in the gravitational interaction (G), i.e., we keep all terms up to order $\alpha_{\rm eff}^4$, and $\alpha_{\rm eff}^4\,G$. E.g., momentum operators contribute one power of $\alpha_{\rm eff}$, according to Eq. (16b). The parameter r_Q^2, where

$$
r_Q^2 = \frac{G\,Q^2}{4\pi} = G\,Z_Q^2\alpha\,, \tag{17}
$$

is counted as a single power of G, because Z_Q^2 may be large, resulting in $Z_Q^2\alpha$ being of order unity. Terms of second order in the gravitational interaction (G^2) are ignored. This is consistent with the approximations made earlier in this article, namely, in Eqs. (7) and (12).

In applying the Foldy–Wouthuysen transformation, we first identify the odd part (in bispinor space) of the Hamiltonian $H_{\rm RN}$

$$
\mathcal{O} = \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p}, \left(1 - \frac{r_{\rm s}}{r} + \frac{3r_Q^2}{4r^2}\right)\right\}. \tag{18}
$$

We now construct the Hermitian operator S and the unitary transform U as

$$
S = -\mathrm{i}\frac{\beta\,\mathcal{O}}{2m}\,, \quad U = \exp\left(\mathrm{i}\,S\right). \tag{19}
$$

We can now apply the first iteration of the Foldy Wouthuysen transform using the approximation

$$
\begin{aligned}
H' = & U\,H_{\rm RN}\,U^+ = \mathrm{e}^{\mathrm{i}\,S}\,H_{\rm RN}\,\mathrm{e}^{-\mathrm{i}\,S} \\
= & H_{\rm RN} + \mathrm{i}\,[S, H_{\rm RN}] + \frac{\mathrm{i}^2}{2!}\,[S,[S,H_{\rm RN}]] + \dots\,.
\end{aligned} \tag{20}
$$

We perform the transformation and calculate

$$
\begin{aligned}
H' = & \beta\left(m + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3}\right) + \frac{Z_Q\,Z_q\,\alpha}{r}\left(1 - \frac{r_{\rm s}}{2r} + \frac{r_Q^2}{4r^2}\right) \\
& - \frac{1}{8m^2}\left[\mathcal{O}, \left[\mathcal{O}, \frac{Z_Q\,Z_q\,\alpha}{r}\right]\right] - \beta\frac{m\,r_{\rm s}}{2r} + \beta\frac{m\,r_Q^2}{2r^2} \\
& + \frac{\beta}{16m}\left\{\mathcal{O}, \left\{\mathcal{O}, \frac{r_{\rm s}}{r} - \frac{r_Q^2}{r^2}\right\}\right\} + \mathcal{O}'\,,
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
\mathcal{O}' = & -\frac{\mathcal{O}^3}{3m^2} + \frac{\beta}{2m}\left[\mathcal{O}, \frac{Z_Q\,Z_q\,\alpha}{r}\left(1 - \frac{r_{\rm s}}{2r}\right)\right] \\
& + \frac{1}{4}\left\{\mathcal{O}, \frac{r_{\rm s}}{r} - \frac{r_Q^2}{r^2}\right\} - \frac{1}{96m^2}\left\{\mathcal{O}, \left\{\mathcal{O}, \left\{\mathcal{O}, \frac{r_{\rm s}}{r}\right\}\right\}\right\}.
\end{aligned} \tag{22}
$$

Notice that the leading-order terms in $\mathcal{O}$ are of order $\alpha_{\rm eff}$ while the leading-order terms in $\mathcal{O}'$ are of order $\alpha_{\rm eff}^3$ and $\alpha_{\rm eff}^2\,G$. Each iteration of the Foldy–Wouthuysen transform eliminates terms up to the leading order of the odd part, but may introduce higher order odd terms. In iterating the procedure the odd terms are eventually eliminated up to a desired order. Applying the transform to H' will give us the Hamiltonian H'' with odd part $\mathcal{O}'' \sim \alpha_{\rm eff}^4 G$. One further iteration will fully eliminate the odd terms up to order $\alpha_{\rm eff}^4$ and first order in G. For an iteration to contribute to the even part of the Hamiltonian the square of the odd part associated with that iteration must be within the desired order. Because the leading terms in $\mathcal{O}'^2$ are of order $\alpha_{\rm eff}^5 G$ and $\alpha_{\rm eff}^6$, they can be ignored within our approximations. Thus our Foldy–Wouthuysen transform of the Dirac–Reissner–Nordström Hamiltonian requires three iterations. The first of which determines the form of the even part, while the final two serve to fully eliminate the odd part (up to our desired order).

The three-fold iterated Foldy–Wouthuysen transformation then gives us

$$
\begin{aligned}
H_{\rm RN}^{(\rm FW)} = & \beta \left(m + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3} \right) \\
& + \frac{Z_Q\, Z_q\, \alpha}{r} \left(1 - \frac{r_{\rm s}}{2r} + \frac{r_Q^2}{4r^2} \right) \\
& - \frac{1}{8m^2} \left[\mathcal{O}, \left[\mathcal{O}, \frac{Z_Q\, Z_q\, \alpha}{r} \right] \right] - \beta \frac{m\, r_{\rm s}}{2r} + \beta \frac{m\, r_Q^2}{2r^2} \\
& + \frac{\beta}{16m} \left\{ \mathcal{O}, \left\{ \mathcal{O}, \frac{r_{\rm s}}{r} - \frac{r_Q^2}{r^2} \right\} \right\} .
\end{aligned}
\tag{23}
$$

Finally, we calculate all the terms involving the original odd part $\mathcal{O}$, giving the final result

$$
\begin{aligned}
H_{\rm RN}^{(\rm FW)} = & \beta \left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} \right) \\
& + \frac{Z_Q\, Z_q\, \alpha}{r} \left(1 - \frac{r_{\rm s}}{2r} + \frac{r_Q^2}{4r^2} \right) \\
& - \frac{Z_Q\, Z_q\, \alpha\, \pi}{2m^2} \delta^{(3)}(\vec{r}) - \frac{Z_Q\, Z_q\, \alpha}{4m^2} \frac{\vec{\Sigma} \cdot \vec{L}}{r^3} \\
& - \beta \frac{m}{2} \left(\frac{r_{\rm s}}{r} - \frac{r_Q^2}{r^2} \right) - \beta \frac{3}{8m} \left\{ \vec{p}^{\,2}, \frac{r_{\rm s}}{r} - \frac{2r_Q^2}{3r^2} \right\} \\
& + \beta \frac{3\pi\, r_{\rm s}}{4m} \delta^{(3)}(\vec{r}) + \beta \frac{3}{8m} \frac{\vec{\Sigma} \cdot \vec{L}}{r^2} \left(\frac{r_{\rm s}}{r} - \frac{4r_Q^2}{3r^2} \right) \\
& + \beta \frac{r_Q^2}{4\, m\, r^4} - \beta \frac{\pi\, r_Q^2}{m\, r} \delta^{(3)}(\vec{r}) .
\end{aligned}
\tag{24}
$$

A few remarks are in order. For a reference S state, the expectation values of the operators $\{\vec{p}^{\,2}, 1/r^2\}$, $1/r^4$, and $\delta^{(3)}(\vec{r})/r$ diverge. In this article, we shall explicitly exclude S states from the analysis and concentrate on highly excited Rydberg states for which the expectation value of $\delta^{(3)}(\vec{r})/r$ vanishes [3]. The emergence of this operator is a consequence of the point nucleus approximation inherent to the Coulomb potential, which is manifest in the divergence of the scalar potential A_0 given in Eq. (11) for $r \to 0$. For a realistic nucleus (a realistic central charged black hole), this divergence is cut off due to the nuclear finite-size effect, and the operator $\delta^{(3)}(\vec{r})/r$ would need to be replaced by a term proportional to $V_n(r) \vec{\nabla}^2 V_n(r)$, where $V_n(r)$ is the nuclear potential including the finite-size effect [26, 27]. We now return to the analysis of the result given in Eq. (24). For $Q \to 0$ ($Z_Q \to 0$), we recover the Foldy–Wouthuysen transformed Dirac–Schwarzschild Hamiltonian found in Eq. (21) of Ref. [2]. Alternatively, we can rewrite the transformed Hamiltonian as

$$
H_{\rm RN}^{(\rm FW)} = H_{\rm F}^{(\rm FW)} + H'_{\rm DC} + H'_{\rm DS} \,, \tag{25}
$$

where $H_F^{(\rm FW)}$ is the free Hamiltonian,

$$
H_F^{(\rm FW)} = \beta \left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} \right) . \tag{26}
$$

$H'_{\rm DC}$ is a gravitationally modified Dirac–Coulomb Hamiltonian without the kinetic terms which are summarized in $H_F^{(\rm FW)}$,

$$
\begin{aligned}
H'_{\rm DC} = & \frac{Z_Q\, Z_q\, \alpha}{r} \left(1 - \frac{r_{\rm s}}{2r} + \frac{r_Q^2}{4r^2} \right) \\
& - \frac{Z_Q\, Z_q\, \alpha \pi}{2m^2} \delta^{(3)}(\vec{r}) - \frac{Z_Q\, Z_q\, \alpha}{4m^2} \frac{\vec{\Sigma} \cdot \vec{L}}{r^3} ,
\end{aligned}
\tag{27}
$$

Moreover, $H'_{\rm DS}$ is an electromagnetically modified Dirac–Schwarzschild Hamiltonian, again without the kinetic terms which are found in $H_F^{(\rm FW)}$,

$$
\begin{aligned}
H'_{\rm DS} = & - \beta \frac{m}{2} \left(\frac{r_{\rm s}}{r} - \frac{r_Q^2}{r^2} \right) - \beta \frac{3}{8m} \left\{ \vec{p}^{\,2}, \frac{r_{\rm s}}{r} - \frac{2r_Q^2}{3r^2} \right\} \\
& + \beta \frac{3\pi\, r_{\rm s}}{4m} \delta^{(3)}(\vec{r}) + \beta \frac{3}{8m} \frac{\vec{\Sigma} \cdot \vec{L}}{r^2} \left(\frac{r_{\rm s}}{r} - \frac{4r_Q^2}{3r^2} \right) \\
& + \beta \frac{r_Q^2}{4\, m\, r^4} - \beta \frac{\pi\, r_Q^2}{m\, r} \delta^{(3)}(\vec{r}) .
\end{aligned}
\tag{28}
$$

Up to the electromagnetic modifications of the gravitational terms in the Dirac–Schwarzschild Hamiltonian, and up to the gravitational modifications of the Dirac–Coulomb Hamiltonian, we thus have $H'_{\rm DC} \approx H_{\rm DC}^{(\rm FW)} - H_F^{(\rm FW)}$ and $H'_{\rm DS} \approx H_{\rm DS}^{(\rm FW)} - H_F^{(\rm FW)}$, where $H_{\rm DC}^{(\rm FW)}$ and $H_{\rm DS}^{(\rm FW)}$ are given in Eqs. (30) and (47) of Ref. [4]. The relativistic corrections found in the transformed Reissner–Nordström Hamiltonian are approximately equal to a sum of the corrections found for the Dirac–Coulomb and the Dirac–Schwarzschild Hamiltonians, with additional electro-gravitational mixing terms (the latter are proportional to the product of the gravitational coupling constant, and a power of the fine-structure constant). Both $H_F^{(\rm FW)}$ and $H'_{\rm DS}$ exhibit particle–antiparticle symmetry (all terms have a β prefactor), while $H'_{\rm DC}$ changes sign under particle–antiparticle interchange (no β prefactor).

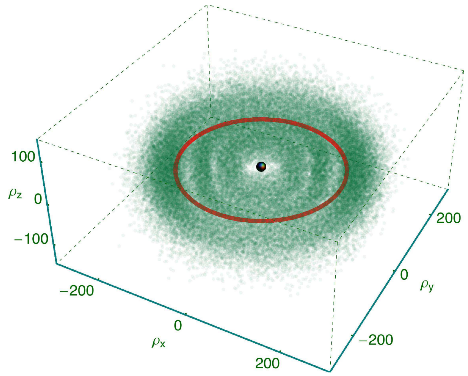

FIG. 1. (Color online.) Scatter plot of the probability density of finding a bound particle (electron) in a state of $n = 12$, $\ell = 9$ and $m = |\ell| = 9$ in the field of a charged heavy black hole with mass 10^{-12} times the mass of the earth, and charge number $Z_Q = 10$. The points are distributed randomly, with the number of scattered points in a reference volume being proportional to the probability of finding the bound electron in the volume. The two radial minima of the probability density are clearly visible. The gravitational center is depicted as a black dot. For reference, the classical trajectory at $\langle\rho\rangle = \int \rho\, |\psi(\vec\rho)|^2 \, \mathrm{d}^3 r = 171$ is also shown. Note that the scaled radial coordinate $\vec\rho$ given in Eq. (30) is dimensionless, as reflected in the labeling of the axes.

IV. BOUND–STATE ENERGIES

It remains to evaluate and discuss the bound-state energies in the potential described by Eq. (24), and to consider an example case. First, we observe that the product $Z_Q\, Z_q$ has to be negative for the electrostatic interaction to be attractive and bound states to exist. We thus define the coupling constants,

$$\alpha_{\rm eff} = -\, Z_Q\, Z_q\, \alpha > 0\,, \tag{29a}$$

$$\alpha_G = \frac{GmM}{\hbar c} = GmM\,, \tag{29b}$$

$$\alpha_Q = r_Q^2 \left(\frac{mc}{\hbar}\right)^2 = \frac{Z_Q^2\, e^2\, G m^2}{4\pi\epsilon_0 \hbar^2 c^2} = Z_Q^2\, \alpha\, G m^2\,, \tag{29c}$$

where in the intermediate steps we temporarily restore full SI mksA units. Following Ref. [3], it is advantageous to scale the coordinate variable according to

$$\vec\rho = \alpha_{\rm eff}\, m\, \vec r\,, \qquad \vec\nabla \equiv \vec\nabla_r = \alpha_{\rm eff}\, m\, \vec\nabla_\rho\,, \tag{30}$$

where $\vec\rho$ is the coordinate in "atomic units"; the "Bohr radius" is $(\alpha_{\rm eff}\, m)^{-1}$.

From Eq. (24), we first extract the effective Hamiltonian applicable to particle (as opposed to antiparticle) states [hence denoted with a superscript (+)], and scale the expression according to Eq. (30),

$$H^{(+)}_{\rm RN} = m + \alpha_{\rm eff}^2\, m \left(-\frac{1}{2}\vec\nabla_\rho^2 - \frac{1}{\rho}\right) + \alpha_{\rm eff}^4\, m \left(-\frac{1}{8}\vec\nabla_\rho^4 + \frac{\pi}{2}\,\delta^{(3)}(\vec\rho) + \frac{\vec\sigma\cdot\vec L}{4\rho^3}\right) - \frac{\alpha_G\, \alpha_{\rm eff}\, m}{\rho} + \alpha_G\, \alpha_{\rm eff}^3\, m \left(\frac{3}{4}\left\{\vec\nabla_\rho^2, \frac{1}{\rho}\right\} \right.$$
$$\left. + \frac{3\pi}{2}\delta^{(3)}(\vec\rho) + \frac{3\,\vec\sigma\cdot\vec L}{4\rho^3} + \frac{1}{\rho^2}\right) + \frac{\alpha_Q\, \alpha_{\rm eff}^2\, m}{2\rho^2} + \alpha_Q\, \alpha_{\rm eff}^4\, m \left(-\frac{1}{4\rho^3} - \frac{1}{4}\left\{\vec\nabla_\rho^2, \frac{1}{\rho^2}\right\} - \frac{\vec\sigma\cdot\vec L}{2\,\rho^4} + \frac{1}{4\,\rho^4} - \frac{\pi}{8\rho}\delta^{(3)}(\vec\rho)\right)\,. \tag{31}$$

Let us break down the matrix elements for the energy corrections in an unperturbed Dirac–Coulomb state with quantum numbers n, ℓ and j according to the Hamiltonians in Eqs. (26), (27) and (28). An evaluation using formulas given in Ref. [28] leads to the results

$$\langle H_F^{(+)}\rangle = m\left\{1 + \frac{\alpha_{\rm eff}^2}{2n^2} + \alpha_{\rm eff}^4\left(\frac{3}{8n^4} - \frac{1}{n^3(2\ell+1)}\right)\right\}\,, \tag{32a}$$

$$\langle {H'}^{(+)}_{\rm DC}\rangle = m\left\{-\frac{\alpha_{\rm eff}^2}{n^2} + \alpha_{\rm eff}^4\left(-\frac{\delta_{j,\ell-1/2}}{2n^3\ell(2\ell+1)} + \frac{\delta_{j,\ell+1/2}}{2n^3\ell(\ell+1)\,(2\ell+1)}\right) + \frac{2\alpha_G\, \alpha_{\rm eff}^3}{n^3\,(2\ell+1)} - \frac{\alpha_Q\, \alpha_{\rm eff}^4}{2\,n^3\,\ell\,(\ell+1)\,(2\ell+1)}\right\}\,, \tag{32b}$$

$$\langle {H'}^{(+)}_{\rm DS}\rangle = m\left\{\alpha_G\left(-\frac{\alpha_{\rm eff}}{n^2} + \frac{\alpha_{\rm eff}^3}{n^2}\left[\delta_{j,\ell-1/2}\left(\frac{3}{2n^4} - \frac{3\,(4\ell+1)}{2n^3\ell(2\ell+1)}\right) + \delta_{j,\ell+1/2}\left(\frac{3}{2n^4} - \frac{3\,(4\ell+3)}{2n^3(\ell+1)(2\ell+1)}\right)\right]\right)\right.$$
$$+ \alpha_Q\left(\frac{\alpha_{\rm eff}^2}{n^3\,(2\ell+1)} + \alpha_{\rm eff}^4\left[\delta_{j,\ell-1/2}\left(-\frac{2\ell}{n^5(2\ell-1)\,(2\ell+1)} + \frac{4\ell+1}{n^3\,\ell\,(\ell+1)\,(2\ell-1)\,(2\ell+1)}\right)\right.\right.$$
$$\left.\left.\left. + \;\delta_{j,\ell+1/2}\left(-\frac{2(\ell+1)}{n^5(2\ell-1)\,(2\ell+1)} + \frac{4\ell+3}{n^3\,\ell\,(\ell+1)\,(2\ell-1)\,(2\ell+1)}\right)\right]\right)\right\}\,, \tag{32c}$$

where the functional form is seen to depend on the relative orientation of orbital angular momentum and spin. Formulas become more compact when expressed in terms of the Dirac angular quantum number $\varkappa = (-1)^{\ell+j+1/2}\,(j+\frac{1}{2})$, which is defined as the negative of the eigenvalue of the Dirac angular operator $K = \beta\,(\vec\Sigma\cdot\vec L + 1)$, i.e., $K\,\psi = -\varkappa\,\psi$ (see

Refs. [29] for further discussion). The results read as follows,

$$\langle H_F^{(+)}\rangle = m\left\{1+\frac{\alpha_{\rm eff}^2}{2n^2}+\alpha_{\rm eff}^4\left(\frac{3}{8n^4}-\frac{\varkappa}{|\varkappa|\,n^3(2\varkappa+1)}\right)\right\}, \tag{33a}$$

$$\langle {H'}_{\rm DC}^{(+)}\rangle = m\left\{-\frac{\alpha_{\rm eff}^2}{n^2}-\frac{\alpha_{\rm eff}^4}{2|\varkappa|n^3\,(2\varkappa+1)}+\frac{2\alpha_G\,\alpha_{\rm eff}^3\,\varkappa}{|\varkappa|\,n^3\,(2\varkappa+1)}-\frac{\alpha_Q\,\alpha_{\rm eff}^4}{2\,|\varkappa|\,n^3\,(\varkappa+1)\,(2\varkappa+1)}\right\}, \tag{33b}$$

$$\langle {H'}_{\rm DS}^{(+)}\rangle = m\left\{\alpha_G\left(-\frac{\alpha_{\rm eff}}{n^2}+\alpha_{\rm eff}^3\left[\frac{3}{2n^4}-\frac{3\,(4\varkappa+1)}{2|\varkappa|\,n^3\,(2\varkappa+1)}\right]\right)+\alpha_Q\left[\frac{\alpha_{\rm eff}^2\,\varkappa}{|\varkappa|\,n^3\,(2\varkappa+1)}\right.\right.$$
$$\left.\left.+\alpha_{\rm eff}^4\left(-\frac{2\varkappa^2}{|\varkappa|\,n^5\,(2\varkappa-1)\,(2\varkappa+1)}+\frac{\varkappa\,(4\varkappa+1)}{|\varkappa|\,n^3\,(\varkappa+1)\,(2\varkappa-1)\,(2\varkappa+1)}\right)\right]\right\}. \tag{33c}$$

An important check consists in the verification of the Dirac–Coulomb energy, which is obtained as the sum of the $\alpha_{\rm eff}^4$ term from $\langle H_F^{(+)}\rangle$ and the $\alpha_{\rm eff}^4$ term from $\langle {H'}_{\rm DC}^{(+)}\rangle$,

$$\langle H_F^{(+)}\rangle\Big|_{\alpha_{\rm eff}^4}+\langle {H'}_{\rm DC}^{(+)}\rangle\Big|_{\alpha_{\rm eff}^4}=\left(\frac{3}{8n^4}-\frac{\varkappa}{|\varkappa|\,n^3(2\varkappa+1)}\right)-\left(\frac{1}{2|\varkappa|\,n^3\,(2\varkappa+1)}\right)=\frac{3}{8n^4}-\frac{1}{2|\varkappa|n^3}. \tag{34}$$

The latter result is in agreement with the literature [see, e.g., Eq. (2.87) of Ref. [30]]. The relativistic corrections to the Dirac–Schwarzschild energy are obtained as a sum of the $\alpha_{\rm eff}^4$ coefficient of $\langle H_F\rangle$ and the $\alpha_G\,\alpha_{\rm eff}^3$ term from $\langle H'_{\rm DS}\rangle$,

$$\langle H_F^{(+)}\rangle\Big|_{\alpha_{\rm eff}^4}+\langle {H'}_{\rm DS}^{(+)}\rangle\Big|_{\alpha_G\,\alpha_{\rm eff}^3}=\left(\frac{3}{8n^4}-\frac{\varkappa}{|\varkappa|\,n^3(2\varkappa+1)}\right)+\left(\frac{3}{2n^4}-\frac{3\,(4\varkappa+1)}{2|\varkappa|\,n^3\,(2\varkappa+1)}\right)=\frac{15}{8n^4}-\frac{14\varkappa+3}{2|\varkappa|\,n^3\,(2\varkappa+1)}. \tag{35}$$

[We note that in the absence of the electrostatic potential, we would scale the radial variable differently, namely, $\vec r=\alpha_G m\,\vec\rho$, leading to the sum of coefficients in the final α_G^4 term as indicated in Eqs. (3a), (3b) and (12) of Ref. [3].] The final result given in Eq. (35) is just the Dirac–Schwarzschild formula [Eq. (12) of Ref. [3]].

For Rydberg states with a vanishing probability density at the origin (see Figs. 1 and 2), the influence of the black hole at the center can be described by a small (complex rather than real) energy correction (see Sec. IV of Ref. [3]). We have performed the scaling of the radial variable according to Eq. (30), implicitly assuming that the electrostatic interaction corresponding to the coupling constant $\alpha_{\rm eff}$ dominates over the gravitational terms (coupling constant α_G); or, otherwise we would have had to perform the initial scaling to the "Bohr radius" of the system differently. One might think that, for the original expansion in powers of $\alpha_{\rm eff}$ and α_G to be valid, $\alpha_{\rm eff}$ should not be too small in comparison to α_G, or else, before encountering any gravitational effect, we should first include higher-order terms in the $\alpha_{\rm eff}$ expansion. Fortunately, the bound-state theory of atoms permits us to sum all the relativistic corrections due to the electrostatic central potential into a convenient all-order (in $\alpha_{\rm eff}$) formula, which reads as

$$E_{\rm RN}=m\left\{f(n,\varkappa)+\alpha_G\left[-\frac{\alpha_{\rm eff}}{n^2}+\alpha_{\rm eff}^3\left(\frac{3}{2n^4}-\frac{8\varkappa+3}{2|\varkappa|\,n^3\,(2\varkappa+1)}\right)\right]+\alpha_Q\left[\alpha_{\rm eff}^2\,\frac{\varkappa}{|\varkappa|\,n^3\,(2\varkappa+1)}\right.\right.$$
$$\left.\left.+\alpha_{\rm eff}^4\left(-\frac{2\varkappa^2}{|\varkappa|\,n^5\,(2\varkappa-1)\,(2\varkappa+1)}+\frac{3}{2|\varkappa|\,n^3\,(\varkappa+1)\,(2\varkappa-1)}\right)\right]\right\},$$
$$f(n,\varkappa)=\left(1+\frac{\alpha_{\rm eff}^2}{(n-|\varkappa|+\sqrt{\varkappa^2-\alpha_{\rm eff}^2})^2}\right)^{-1/2}=1-\frac{\alpha_{\rm eff}^2}{2n^2}+\alpha_{\rm eff}^4\left(\frac{3}{8n^4}-\frac{1}{2|\varkappa|n^3}\right)+\mathcal{O}(\alpha_{\rm eff}^6). \tag{36}$$

Here, $f(n,\varkappa)$ is the dimensionless Dirac energy [29].

We now consider a numerical example. Because Z_Q and Z_q are the nuclear charge numbers associated with the atoms, we have $Z_Q=+10$ and $Z_q=-1$ for an orbiting electron around a positively charged small black hole. Following Ref. [3], we thus consider a charged black hole with a mass M equal to 10^{-13} times the mass of the earth,

$$M=10^{-13}M_\oplus=5.9742\times10^{11}\,{\rm kg}, \tag{37}$$

and assume that $m=m_e$ (electron mass). In the numerical calculations, we thus use the following coupling constants

$$\alpha_{\rm eff}=10\,\alpha=7.297\,352\times10^{-2}, \tag{38a}$$
$$\alpha_G=1.148\,884\times10^{-3}, \tag{38b}$$
$$\alpha_Q=1.278\,353\times10^{-45}. \tag{38c}$$

Here, in order to ensure the reproducibility of the results given below, we assume all decimal places given assumed

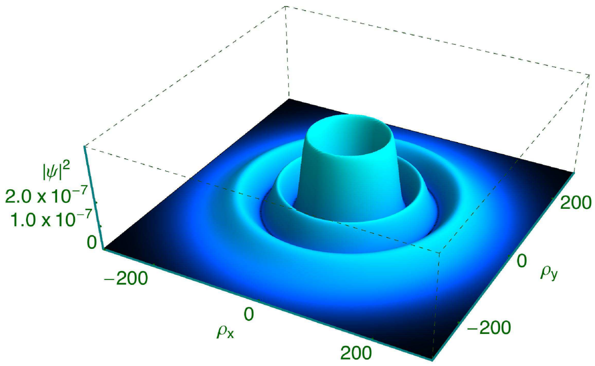

FIG. 2. (Color online.) Plot of the probability density $|\psi(\vec{\rho})|^2$ for a bound electron in a state with quantum numbers $n = 12$, $\ell = 9$ and $m = 9$, in the field of the same charged black hole as given in Fig. 1. The quantum numbers are $n = 12$, $\ell = 9$ and $m = |\ell| = 9$, and the polar angle is $\theta = 90^\circ$, i.e., the plot pertains to the (ρ_x, ρ_y) plane, with the dimensionless vector-valued variable $\vec{\rho}$ being defined in Eq. (30).

to be exact, even if the Newton's gravitational constant is currently known only to one part in 10^4 (see Table XL of Ref. [31]). We consider two atomic states with quantum numbers

$$n_1 = 12\,, \quad \ell_1 = 9\,, \quad j_1 = \frac{19}{2}\,, \quad \varkappa_1 = -10\,, \tag{39a}$$

$$n_2 = 12\,, \quad \ell_2 = 9\,, \quad j_2 = \frac{17}{2}\,, \quad \varkappa_2 = 9\,, \tag{39b}$$

The energy formula (36) evaluates to

$$E_1 = m \left(\left(1 + \frac{\alpha_{\rm eff}^2}{(2 + \sqrt{100 - \alpha_{\rm eff}^2})^2} \right)^{-1/2} - \frac{\alpha_G\,\alpha_{\rm eff}}{144} - \frac{59\alpha_G\,\alpha_{\rm eff}^3}{1\,313\,280} + \frac{\alpha_Q\,\alpha_{\rm eff}^2}{32\,832} + \frac{\alpha_Q\,\alpha_{\rm eff}^4}{3\,878\,280} \right)\,, \tag{40}$$

which translates into SI mksA units as follows (the coefficient term which generates the contribution is given separately as a subscript of each item),

$$\begin{aligned} E_1 = m_e\,c^2 + \big(& -9.44855|_{\rm Dirac-Coulomb} - 0.29751|_{\alpha_G\,\alpha_{\rm eff}} \\ & - 1.02491 \times 10^{-5}\big|_{\alpha_G\,\alpha_{\rm eff}^3} \\ & + 1.05951 \times 10^{-46}\big|_{\alpha_Q\,\alpha_{\rm eff}^2} \\ & + 4.77631 \times 10^{-51}\big|_{\alpha_Q\,\alpha_{\rm eff}^4} \big)\ {\rm eV}\,. \end{aligned} \tag{41}$$

It becomes clear that the backaction effect of the space-time curvature induced by the charged particle, parameterized by α_Q, only is a small effect in our example case, but still constitutes a conceptually interesting correction. The energy of the second bound state considered for our example calculation is

$$E_2 = m \left(\left(1 + \frac{\alpha_{\rm eff}^2}{(3 + \sqrt{81 - \alpha_{\rm eff}^2})^2} \right)^{-1/2} - \frac{\alpha_G\,\alpha_{\rm eff}}{144} - \frac{43\alpha_G\,\alpha_{\rm eff}^3}{787\,968} + \frac{\alpha_Q\,\alpha_{\rm eff}^2}{32\,832} + \frac{23\,\alpha_Q\,\alpha_{\rm eff}^4}{66\,977\,280} \right)\,. \tag{42}$$

Here, the breakdown of contributions is as follows,

$$\begin{aligned} E_2 = m_e\,c^2 + \big(& -9.44860|_{\rm Dirac-Coulomb} - 0.29751|_{\alpha_G\,\alpha_{\rm eff}} \\ & - 1.24495 \times 10^{-5}\big|_{\alpha_G\,\alpha_{\rm eff}^3} \\ & + 1.05951 \times 10^{-46}\big|_{\alpha_Q\,\alpha_{\rm eff}^2} \\ & + 6.36110 \times 10^{-51}\big|_{\alpha_Q\,\alpha_{\rm eff}^4} \big)\ {\rm eV}\,. \end{aligned} \tag{43}$$

The first terms on the right-hand sides of Eqs. (41) and (43) contain the electron rest mass. An expansion of the energies given in Eqs. (40) and (42) in powers of $\alpha_{\rm eff}$ implies that they differ only by the fine structure (i.e., at order $\alpha_{\rm eff}^4$ for terms free of α_G and α_Q).

V. CONCLUSIONS

In this paper, we find the nonrelativistic limit of the Dirac equation coupled to a statically charged gravitational center. To carry out this calculation we first have to derive the Dirac–Reissner–Nordström Hamiltonian. The derivation requires that we first transform the metric, and consequently the potential, into isotropic coordinates, and then couple the Dirac equation to both the curved space–time and the electrostatic potential (see Sec. II A and Appendix A). The derivation of the Reissner–Nordström metric in isotropic coordinates could be of interest in a wider context. Starting from generalized Dirac Hamiltonian, we find the nonrelativistic limit by applying the Foldy–Wouthuysen program (Secs. II B and III). We carry out the transformation keeping terms up to the fourth order in $\alpha_{\rm eff}$, where $\alpha_{\rm eff} = -Z_Q\,Z_q\,\alpha$ is an effective coupling constant for the bound system [see Eq. (15)]. Furthermore, we keep terms of first order in the gravitational constant G, i.e., first order in the effective coupling constants α_G and α_Q defined in Eqs. (29b) and (29c).

The Foldy–Wouthuysen transformation of the Dirac–Reissner–Nordström Hamiltonian is carried out in Sec. III (see also Appendix B), and the structure of the resulting bound-state spectrum is analyzed in Sec. IV. The final result for the Foldy–Wouthuysen transformed Hamiltonian is given in Eq. (24); it contains a number of familiar terms. As should be expected, when we remove the charge of the gravitational center ($\alpha_{\rm eff} = \alpha_Q = 0$, but $\alpha_G \neq 0$) we recover the nonrelativistic limit of Dirac–Schwarzschild Hamiltonian [2, 3]. Additionally, if we were to neglect the gravitational terms ($\alpha_G = \alpha_Q = 0$), the nonrelativistic limit of the Dirac–Coulomb Hamiltonian is recovered. We also find terms additional terms which are due to the presence of the center charge which curves space–time, as manifest in the differences of the Schwarzschild and the Reissner–Nordström metric. After the Foldy–Wouthuysen transformation, one recognizes these terms as mixing terms, proportional to a the product of a gravitational coupling (α_G or α_Q) and a power of the effective electromagnetic coupling constant $\alpha_{\rm eff}$.

The result for the Dirac–Reissner–Nordström Hamiltonian given in Eq. (24) can naturally be written as the sum of three contributions, a free Hamiltonian H_F (with relativistic corrections), a modified Dirac–Coulomb Hamiltonian $H'_{\rm DC}$, and a modified Dirac–Schwarzschild gravitational potential term $H'_{\rm DS}$ [see Eqs. (27) and (28)]. There are perturbations to the Coulomb potential due to the curvature in space–time, resulting from both the mass and the charge of the gravitational center, i.e., proportional to both $r_{\rm s}$ as well as r_Q^2 [see Eq. (27)]. The leading gravitational term (proportional to $r_{\rm s}$) and the leading electro–gravitational mixing term (proportional to r_Q^2) enter with opposite sign in both $H'_{\rm DC}$ and $H'_{\rm DS}$. $H'_{\rm DC}$ breaks the particle-antiparticle symmetry, while $H'_{\rm DS}$ conserves it. The electric field corresponding to the Coulomb potential leads to a nonvanishing energy-momentum tensor, which modifies the space-time curvature around the charged black hole; hence the differences of the Schwarzschild and Reissner–Nordström metrics. In turn, the metric enters the formulation of the generalized Dirac Hamiltonian, which naturally contains terms due to the modified space-time curvature, i.e., proportional to r_Q^2. These higher-order (in $\alpha_{\rm eff}$) correction terms for bound states resulting from these terms are clearly identified after a Foldy–Wouthuysen transformation.

The gravitational corrections to a Coulomb bound system, which are derived here using a rigorous approach, are of interest for a number of reasons. Space-time noncommutativity in a concept inspired by string theory (see Ref. [32]), which could be of relevance in the unification of gravity with the other fundamental interactions. As shown in Ref. [15], space-time noncommutativity may ultimately induce certain shifts of energy levels in atomic systems which may be detectable in the future. Here, the leading fully relativistic gravitational corrections are derived using a rigorous approach which does not require space-time quantization. Second, micro black holes have been proposed in various contexts of physics, including candidates for dark matter [16–18] and even classes of novel phenomena at accelerators [19]. For charged micro black holes, a quantum mechanical description requires the use of the Reissner–Nordström metric.

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Appendix A: The Reissner–Nordström Metric

The derivation of the Reissner–Nordström metric, which describes the curvature of space–time due to a charged gravitational center, is generally left as an exercise in textbooks (problem 6.3 in [25] for example). It is also possible to find unpublished works which detail the derivation [33]. Here we briefly outline the derivation, and the assumptions made.

Here we are interested in a spherically symmetric, statically charged, stationary black hole. Such a black hole will result in a static, spherically symmetric space–time. From Eq. (6.1.5) of [25] we know that the metric of such a space–time is of the form

$$\mathrm{d}s^2 = f(\mathcal{R})\,\mathrm{d}t^2 - h(\mathcal{R})\,\mathrm{d}\mathcal{R}^2 - \mathcal{R}^2\,\mathrm{d}\Omega\,, \tag{A1}$$

$$\mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\,, \tag{A2}$$

where we have adjusted from the "East–coast" convention used in [25] to the "West-coast" convention we use in this paper. To derive the Reissner–Nordström metric we use the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,, \tag{A3}$$

which is equivalent to

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)\,, \tag{A4}$$

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ is the electromagnetic stress–energy tensor. Using the "West–coast" convention, the electromagnetic stress–energy tensor is

$$T_{\mu\nu} = -\left(F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F^{\kappa\lambda} F_{\kappa\lambda} \right)\,. \tag{A5}$$

Notice that this equation has the opposite sign as compared to the stress–energy tensor in the "East–coast" convention [see Eq. (5.22) of [34]]. With this definition it is trivial to show that $T = T_\mu{}^\mu = 0$, and Eq. (A4) becomes

$$R_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,. \tag{A6}$$

We now need to calculate for the components of the Ricci tensor and the components of the electromagnetic stress–energy tensor. By definition, the Ricci tensor is

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda}\,, \tag{A7}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is the Christoffel symbol. The non–zero components of the Ricci tensor are then found to be

$$R_{tt} = \frac{f'}{\mathcal{R}\,h} - \frac{f'^{\,2}}{4\,f h} - \frac{f'\,h'}{4\,h^2} + \frac{f''}{2\,h}\,, \tag{A8a}$$

$$R_{\mathcal{R}\mathcal{R}} = \frac{f'^{\,2}}{4\,f^2} + \frac{h'}{\mathcal{R}\,h} + \frac{f'\,h'}{4\,f h} - \frac{f''}{2\,f}\,, \tag{A8b}$$

$$R_{\theta\theta} = 1 - \frac{1}{h} - \frac{\mathcal{R}\,f'}{2\,f h} + \frac{\mathcal{R}\,h'}{2\,h^2}\,, \quad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta\,, \tag{A8c}$$

where prime indicates differentiation with respect to $\mathcal{R}$.

We now consider the electromagnetic stress–energy tensor. As we are considering a static spherically symmetric electric field (without currents or magnetic fields), the only non–zero components of the field strength tensor are

$$F_{t\mathcal{R}} = -F_{\mathcal{R}t} = E(\mathcal{R})\,. \tag{A9}$$

The electric field outside of a uniformly charged sphere is given as

$$E(\mathcal{R}) = \frac{Q}{4\pi\,\mathcal{R}^2}\,. \tag{A10}$$

Using Eq. (A5) we can now calculate the components of the electromagnetic stress–energy tensor. The non–zero components are

$$T_{tt} = \frac{1}{h}\frac{Q^2}{32\pi^2\,\mathcal{R}^4}\,, \tag{A11a}$$

$$T_{\mathcal{R}\mathcal{R}} = -\,\frac{1}{f}\frac{Q^2}{32\pi^2\,\mathcal{R}^4}\,, \tag{A11b}$$

$$T_{\theta\theta} = \frac{1}{f\,h}\frac{Q^2}{32\pi^2\,\mathcal{R}^2}\,, \quad T_{\phi\phi} = T_{\theta\theta}\sin^2\theta\,. \tag{A11c}$$

Notice that $f^{-1}T_{tt} + h^{-1}T_{\mathcal{R}\mathcal{R}} = 0$, therefore

$$\frac{1}{f}R_{tt} + \frac{1}{h}R_{\mathcal{R}\mathcal{R}} = \frac{1}{r\,fh^2}(fh)' = 0\,, \tag{A12}$$

from which we conclude that

$$f = K\,h^{-1}\,, \tag{A13}$$

where K is a constant. As was done in Sec. 6.1 of [25], we can gauge away K by re–scaling the time coordinate as $dt \to \sqrt{K}dt$. Thus $h = f^{-1}$, and Eqs. (A8c) and (A11c) become

$$R_{\theta\theta} = 1 - \partial_{\mathcal{R}}\left(\mathcal{R}\,f\right)\,, \quad T_{\theta\theta} = \frac{Q^2}{32\pi^2\,\mathcal{R}^2}\,, \tag{A14}$$

respectively. Applying these equations to Eq. (A6) we find

$$1 - \partial_{\mathcal{R}}\left(\mathcal{R}\,f\right) = \frac{r_Q^2}{8\pi\,\mathcal{R}^2}\,, \qquad r_{\rm Q}^2 = \frac{G\,Q^2}{4\pi}\,. \tag{A15}$$

This equation is solved by

$$f = 1 + \frac{C}{\mathcal{R}} + \frac{r_Q^2}{\mathcal{R}^2}\,. \tag{A16}$$

If we set $Q = 0$ then we should recover the Schwarzschild metric. Thus $C = -r_{\rm s} = 2GM$, and the Reissner–Nordström metric is found to be

$$\begin{aligned} {\rm d}s^2 = {} & \left(1 - \frac{r_{\rm s}}{\mathcal{R}} + \frac{r_Q^2}{\mathcal{R}^2}\right){\rm d}t^2 \\ & - \left(1 - \frac{r_{\rm s}}{\mathcal{R}} + \frac{r_Q^2}{\mathcal{R}^2}\right)^{-1}{\rm d}\mathcal{R}^2 - r^2\,{\rm d}\Omega^2\,. \end{aligned} \tag{A17}$$

The simplified approach to the derivation of the metric taken here, makes extensive use of the known solution for the Schwarzschild geometry and leads to a streamlined derivation.

Appendix B: Derivation of the Hamiltonian

Here we follow the notation utilized in Refs. [2, 5]. The flat–space–time Dirac gamma matrices are denoted with a tilde ($\widetilde{\gamma}$) while the curved–space–time Dirac gamma matrices are written with an overline ($\overline{\gamma}$). The notation for the curved-space–time matrices is inspired by the covariant structure of their anticommutator, expressed in Eq. (B4), and the tensor ("vector") structure is denoted by the overline. By contrast, from the point of view of general relativity, the flat-space matrices can be regarded as "modified" γ matrices, hence the tilde. The flat-space matrices are used in the Dirac representation,

$$\widetilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}\,, \quad \widetilde{\gamma}^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\,, \tag{B1a}$$

$$\widetilde{\gamma}^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\,, \qquad \widetilde{\gamma}^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\,, \tag{B1b}$$

$$\widetilde{\gamma}^5 = {\rm i}\,\widetilde{\gamma}^0\,\widetilde{\gamma}^1\,\widetilde{\gamma}^2\,\widetilde{\gamma}^3 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}\,, \tag{B1c}$$

where the σ^i are the (2×2) Pauli matrices [11].

As in Appendix C of Ref. [5] we draw inspiration from Ref. [35] and use the capital Latin indices $A, B, C, ... = 0, 1, 2, 3$ to denote the local Lorentz frame ("anholonomic basis") and $I, J, K, ... = 1, 2, 3$ for spatial coordinates in the anholonomic basis. The Greek indices $\mu, \nu, \lambda, ... = 0, 1, 2, 3$ denote the global coordinates, while the lower case Latin indices $i, j, k, ... = 1, 2, 3$ are used for the global spatial coordinates. As in Refs. [2, 5] we use the "West Coast" convention for the flat–space–time metric, denoted as $\tilde{g}_{AB} = \eta_{AB} = \eta^{AB} = {\rm diag}(1, -1, -1, -1)$. The curved–space–time metric is denoted using $\overline{g}_{\mu\nu} = g_{\mu\nu}$, without the need for an overline, as η denote the flat–space–time metric. From Eq. (6) we know that the curved–space–time metric is

$$g_{\mu\nu} = {\rm diag}\left(w^2, -v^2, -v^2, -v^2\right)\,, \tag{B2a}$$

$$g^{\mu\nu} = {\rm diag}\left(w^{-2}, -v^{-2}, -v^{-2}, -v^{-2}\right)\,. \tag{B2b}$$

This metric has the same structure as the isotropic Schwarzschild metric in the Eddington reparameterization [see Ref. [24] and Eq. (8) of Ref. [2], as well as Eqs. (C6) and (C7) of Ref. [5]]. However, the functions $w = w(r)$ and $v = v(r)$ are different for the Reissner–Nordström geometry. The curved–space–time Dirac gamma matrices can be expressed in terms of the flat–space–time Dirac gamma matrices as

$$\overline{\gamma}^\mu(x) = e^\mu_A(x)\widetilde{\gamma}^A\,, \quad \overline{\gamma}_\mu(x) = e^A_\mu(x)\widetilde{\gamma}_A\,, \tag{B3}$$

where $e^\mu_A(x)$ are the vierbein coefficients. By definition, the curved–space–time Dirac gamma matrices must satisfy the condition

$$\left\{\overline{\gamma}_\mu(x), \overline{\gamma}_\nu(x)\right\} = 2g_{\mu\nu}\,, \tag{B4}$$

from which we find

$$g_{\mu\nu} = \frac{1}{2}\left\{\overline{\gamma}_\mu(x), \overline{\gamma}_\nu(x)\right\} = e^A_\mu(x)\, e^B_\mu(x)\, \eta_{AB}\,, \tag{B5a}$$

$$g^{\mu\nu} = \frac{1}{2}\left\{\overline{\gamma}^\mu(x), \overline{\gamma}^\nu(x)\right\} = e^\mu_A(x)\, e^\mu_B(x)\, \eta^{AB}\,. \tag{B5b}$$

The vierbein coefficients that satisfy these equations are

$$e^0_\mu = \delta^0_\mu\, w\,, \quad e^I_\mu = \delta^I_\mu\, v\,, \tag{B6a}$$

$$e^\mu_0 = \delta^\mu_0\, \frac{1}{w}\,, \quad e^\mu_I = \delta^\mu_I\, \frac{1}{v}\,. \tag{B6b}$$

Here δ^μ_A and δ^A_μ denote the Kronecker delta.

As is well known, when formulating the Dirac equation in curved–space–time, one replaces the $\widetilde{\gamma}^A \to \overline{\gamma}^\mu$ and $\partial_A \to \nabla_\mu$ (see Refs. [1, 2, 4, 5, 36–40]), where

$$\nabla_\mu = \partial_\mu - \Gamma_\mu\,, \tag{B7}$$

$$\Gamma_\mu = \frac{\mathrm{i}}{4}\omega^{AB}_\mu\, \sigma_{AB}\,, \quad \sigma_{AB} = \frac{\mathrm{i}}{2}\left[\widetilde{\gamma}_A, \widetilde{\gamma}_B\right]\,, \tag{B8}$$

$$\omega^{AB}_\nu = e^A_\mu \nabla_\nu\, e^{\mu B} = e^A_\mu \partial_\nu\, e^{\mu B} + e^A_\mu \Gamma^\mu_{\nu\lambda}\, e^{\lambda B}\,. \tag{B9}$$

For absolute clarity, we emphasize that the covariant derivative "∇" in Eq. (B7) acts on a spinor, while the "∇" in Eq. (B9) is the holonomic covariant derivative acting on a vector, defined as $\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda$. Furthermore, Γ_μ is the affine spin–connection matrix, while $\Gamma^\mu_{\nu\lambda}$ are the Christoffel symbols and ω^{AB}_ν are the Ricci rotation coefficients. This transformation ensures that the curved–space–time Dirac equation is invariant under a Lorentz transformation.

When coupling the electrostatic potential to the Dirac equation in flat–space–time one replaces $\mathrm{i}\partial_B \to \mathrm{i}\partial_B - q\, A_B$, where q is the charge of the particle [11]. This is easily generalized to curved–space–time as $\mathrm{i}\nabla_\mu \to \mathrm{i}\nabla_\mu - q\, A_\mu$. Thus the Dirac equation in curved space–time, coupled to an electrostatic potential, is

$$\left(\overline{\gamma}^\mu \left(\mathrm{i}\nabla_\mu - q\, A_\mu\right) - m\right)\psi = 0\,. \tag{B10}$$

We know that the only non–zero term of the electrostatic potential is A_0 (Eq. (12)), thus we can somewhat simplify the Dirac equation to

$$\left(\mathrm{i}\overline{\gamma}^\mu \partial_\mu - \mathrm{i}\overline{\gamma}^\mu \Gamma_\mu - \overline{\gamma}^0 q\, A_0 - m\right)\psi = 0\,. \tag{B11}$$

Multiplying by $\overline{\gamma}^0$ on the left, and rearranging the equation we obtain

$$\mathrm{i}\left(\overline{\gamma}^0\right)^2 \partial_0 \psi = \left(-\mathrm{i}\,\overline{\gamma}^0\overline{\gamma}^i\partial_i + \mathrm{i}\,\overline{\gamma}^0\overline{\gamma}^\mu\Gamma_\mu + \left(\overline{\gamma}^0\right)^2 q\, A_0 + \overline{\gamma}^0 m\right)\psi\,. \tag{B12}$$

We now utilize our vierbein, along with an explicit calculation of the affine spin–connection matrix to find

$$\overline{\gamma}^0 = \frac{1}{w}\widetilde{\gamma}^0\,, \quad \left(\overline{\gamma}^0\right)^2 = \frac{1}{w^2}\,, \quad \overline{\gamma}^0\overline{\gamma}^i = \frac{1}{vw}\widetilde{\gamma}^i\,, \tag{B13a}$$

$$\overline{\gamma}^0\overline{\gamma}^\mu\Gamma_\mu = -\frac{\vec{\alpha}\cdot\vec{\nabla}\, w}{2vw^2} - \frac{\vec{\alpha}\cdot\vec{\nabla}\, v}{v^2 w}\,. \tag{B13b}$$

The form of these results are in full agreement with Refs. [2, 5], the only differences coming from the definitions of the functions w and v. Here $\vec{\alpha} = \widetilde{\gamma}^0\vec{\widetilde{\gamma}}$. Applying the results found in Eq. (B13) to Eq. (B12) and multiplying by w^2 on the left we find $\mathrm{i}\,\partial_t\,\psi = H\,\psi$ where

$$H = \frac{w}{v}\vec{\alpha}\cdot\vec{p} + \frac{\vec{\alpha}\cdot[\vec{p}, w]}{2v} + \frac{w}{v}\frac{\vec{\alpha}\cdot[\vec{p}, v]}{v} + q\, A_0 + \beta\, m\, w\,, \tag{B14}$$

and $\beta = \widetilde{\gamma}^0$. As done in [2], we rescale the spatial coordinates according to $\psi' = v^{3/2}\psi$, and $H' = v^{3/2} H\, v^{-3/2}$, to find the Hermitian Hamiltonian

$$H' = \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p}, \frac{w}{v}\right\} + q\, A_0 + \beta\, m\, w\,. \tag{B15}$$

Finally we use our approximations from Eq. (7) to find

$$\frac{w}{v} \approx 1 - \frac{r_{\mathrm{s}}}{r} + \frac{3r_Q^2}{4r^2}\,, \tag{B16}$$

and apply them, along with Eq. (12) to the Hamiltonian to find the Dirac–Reissner–Nordström Hamiltonian, to the first order in G, as

$$H_{\mathrm{RN}} = \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p}, \left(1 - \frac{r_{\mathrm{s}}}{r} + \frac{3r_Q^2}{4r^2}\right)\right\} + \frac{Z_Q\, Z_q\, \alpha}{r}\left(1 - \frac{r_{\mathrm{s}}}{2r} + \frac{r_Q^2}{4r^2}\right) + \beta\, m\left(1 - \frac{r_{\mathrm{s}}}{2r} + \frac{r_Q^2}{2r^2}\right)\,, \tag{B17}$$

where we use $q\,Q = 4\pi\, Z_Q\, Z_q\, \alpha$. Here, Z_Q is the nuclear charge number of the central gravitational object (charge Q), while Z_q is the nuclear charge number associated with the test charge q, and α is the fine–structure constant.

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