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Dynamical invariants for pseudo-hermitian Hamiltonians

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We derive the dynamical invariants for a general N-state quantum system described by a pseudo-hermitian Hamiltonian. Explicit expressions are presented for two- and three-state systems, which are exemplified by explicit analytic solutions for constant couplings. In the two-state case, we derive non-hermitian analogs of the Bloch vector and the Bloch equation, customary for hermitian quantum systems. We suggest possible physical implementations of the dynamical invariants in waveguide optics and frequency conversion.

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I. INTRODUCTION

Non-hermitian Hamiltonians are an interesting and useful concept in quantum physics. Over half a century ago [1, 2], Feshbach provided a systematic procedure for deriving the effective non-hermitian Hamiltonian for a quantum system with a discrete spectrum coupled to a continuum. This effective Hamiltonian has either real eigenvalues or pairs of complex-conjugated eigenvalues, which is a result of time invariance arguments [3]. Furthermore, it was proved [4] that the class of all diagonalizable operators with discrete spectra is pseudo-hermitian if and only if its eigenvalues are either real or grouped in complex-conjugated pairs. This result triggered significant research on the so-called pseudo-hermitian Hamiltonians [11]. Moreover, a possible extension of quantum mechanics was introduced [9, 12, 13], which uses complex Hamiltonians with space-time reflection (PT) symmetry, such that the Hamiltonian has real eigenvalues.

The operator $\mathbf{H}$ is called pseudo-hermitian if there exists a hermitian operator $\boldsymbol{\eta}$ ($\boldsymbol{\eta}^\dagger = \boldsymbol{\eta}$), such that

$$\boldsymbol{\eta}\mathbf{H}\boldsymbol{\eta}^{-1} = \mathbf{H}^\dagger. \tag{1}$$

It follows that under quite general conditions of discrete spectra and locality, many effective non-hermitian Hamiltonians can be considered as pseudo-hermitian. Obviously, the hermitian Hamiltonians are a special case when $\boldsymbol{\eta} = \mathbf{I}$, the identity operator.

In addition to these very general results, pseudo-Hermiticity is connected to numerous practical applications. Examples include description of spinor fields in gravitational Kerr fields [14], optical microspiral cavities [15], microcavities, perturbed by particles [16], modelling a possible discrepancy between experiment and the Standard Model value of the muon's anomalous g-factor [17], describing Maxwell's equations in pseudo-hermitian form [18], describing a weak backscattering between counter-propagating travelling waves in a general open quantum system [19], modelling the propagation of light in perturbed medium [20, 21], etc. For many other applications, which include quantum cosmology, magnetohydrodynamics, quantum chaos, etc. see Ref. [8].

In this paper, we derive conservation laws for general pseudo-hermitian quantum systems, which require knowledge of the matrix $\boldsymbol{\eta}$. We calculate these conservation laws for the special cases of two- and three-level systems. The procedure can be easily generalized and applied to systems with an arbitrary number of quantum levels.

The paper is organized as follows. In Sec. II we derive the dynamical invariants for a general N-dimensional pseudo-hermitian Hamiltonian. In Sec. III we consider a two-level pseudo-hermitian system and derive the matrix $\boldsymbol{\eta}$. We write down the two-level dynamical invariants and exemplify them with the explicit analytic solution for a pseudo-hermitian Rabi model. We show that an analogue of the Bloch vector and the Bloch equations can be obtained. In Sec. IV we investigate three-level pseudo-hermitian quantum systems. Finally, in Sec. V we summarise the conclusions.

II. DYNAMICAL INVARIANTS

Consider an N-level quantum system in a state with a density matrix $\boldsymbol{\rho}$, which evolves according to the quantum Liouville equation ($\hbar = 1$)

$$\mathrm{i}\,\frac{d}{dt}\boldsymbol{\rho} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H}^\dagger. \tag{2}$$

We assume that $\mathbf{H}$ is pseudo-hermitian, Eq. (1). We shall obtain conservation laws for the case when all matrix elements of $\boldsymbol{\eta}$ are time-independent. We shall show later that this conclusion does not lead to severe restrictions on $\mathbf{H}$.

We shall show that the invariants of Eq. (2) are

$$\mathrm{Tr}\,(\boldsymbol{\eta}\boldsymbol{\rho})^n = \mathrm{const}, \quad (n = 1, 2, \ldots, N-1). \tag{3}$$

When $\mathbf{H}$ is hermitian, we have $\boldsymbol{\eta} = \mathbf{I}$ and we recover the well known dynamical invariants $\mathrm{Tr}\boldsymbol{\rho}^n$ in this case. The proof of Eq. (3) follows.

First, we have

$$\frac{d}{dt}\mathrm{Tr}\,(\boldsymbol{\eta\rho})^n = \sum_k \langle k|\boldsymbol{\eta\dot{\rho}\eta\rho}\cdots\boldsymbol{\eta\rho}|k\rangle + \sum_k \langle k|\boldsymbol{\eta\rho\eta\dot{\rho}}\cdots\boldsymbol{\eta\rho}|k\rangle + \cdots + \sum_k \langle k|\boldsymbol{\eta\rho\eta\rho}\cdots\boldsymbol{\eta\dot{\rho}}|k\rangle, \tag{4}$$

where a dot denotes a time derivative. Next, the cyclic property of the trace,

$$\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA}), \tag{5}$$

yields

$$\begin{aligned}\frac{d}{dt}\mathrm{Tr}\,(\boldsymbol{\eta\rho})^n &= n\mathrm{Tr}(\boldsymbol{\eta\rho\eta}\cdots\boldsymbol{\eta\dot{\rho}})\\ &= n\mathrm{Tr}\left[\boldsymbol{\eta\rho\eta}\cdots\boldsymbol{\eta}(\mathrm{i}\,\boldsymbol{\rho}\mathbf{H}^\dagger - \mathrm{i}\,\mathbf{H}\boldsymbol{\rho})\right]\\ &= \mathrm{i}\,n[\mathrm{Tr}(\mathbf{H}^\dagger\boldsymbol{\eta\rho}\cdots\boldsymbol{\eta\rho}) - \mathrm{Tr}(\boldsymbol{\eta\rho}\cdots\boldsymbol{\eta}\mathbf{H}\boldsymbol{\rho})]\\ &= \mathrm{i}\,n[\mathrm{Tr}(\boldsymbol{\eta}\mathbf{H}\boldsymbol{\rho}\cdots\boldsymbol{\eta\rho}) - \mathrm{Tr}(\boldsymbol{\eta\rho}\cdots\boldsymbol{\eta}\mathbf{H}\boldsymbol{\rho}) \;= 0.\end{aligned} \tag{6}$$

From here Eq. (3) follows immediately. In the derivation we have used Eqs. (2), (5) and the relation $\boldsymbol{\eta}\mathbf{H} = \mathbf{H}^\dagger\boldsymbol{\eta}$, which follows from Eq. (1).

According to the Cayley-Hamilton theorem each square matrix satisfies its characteristic equation, and hence the Nth power of any N-dimensional square matrix can be expressed by its powers $n = 1, 2, \ldots, N-1$. The implication is that only $N-1$ constants of motion (3) can be independent, e.g. for $n = 1, 2, \ldots, N-1$.

It follows from the above discussion that, for any given Hamiltonian $\mathbf{H}$, the problem of finding the invariants (3) is reduced to the one of finding the most general form of the matrix $\boldsymbol{\eta}$. We consider below explicitly the cases of two and three levels, which are of most practical interest.

In Ref. [5] it is shown that for a positive-definite $\boldsymbol{\eta}$ (i.e., all eigenvalues of $\boldsymbol{\eta}$ are positive), there exists a unitary transformation which maps the problem into one, described by Hermitian Hamiltonian. We should stress that in our case, $\boldsymbol{\eta}$ needs not be positive-definite.

III. TWO-LEVEL PSEUDO-HERMITIAN SYSTEM

A. General pseudo-hermitian hamiltonian for a two-level system

We seek the general form of $\mathbf{H}$. We write the Hamiltonian as the most general 2×2 complex matrix,

$$\mathbf{H} = \begin{bmatrix} \Delta_1 + \mathrm{i}\,\Gamma_1 & \Omega_1 e^{\mathrm{i}\,(\phi+\psi)} \\ \Omega_2 e^{-\mathrm{i}\,\phi} & \Delta_2 + \mathrm{i}\,\Gamma_2 \end{bmatrix}, \tag{7}$$

with all parameters being real. Using condition (1) we arrive at (cf. Appendices A and B)

$$\Gamma_1 = -\Gamma_2 \equiv \Gamma, \tag{8}$$

and

$$\psi(\Gamma) = \arcsin\frac{(\Delta_2-\Delta_1)\Gamma}{\Omega_1\Omega_2}. \tag{9}$$

Hence

$$\mathbf{H} = \begin{bmatrix} \Delta_1 + \mathrm{i}\,\Gamma & \Omega_1 e^{\mathrm{i}\,[\psi(\Gamma)+\phi]} \\ \Omega_2 e^{-\mathrm{i}\,\phi} & \Delta_2 - \mathrm{i}\,\Gamma \end{bmatrix}. \tag{10}$$

In Appendix A we find that

$$\boldsymbol{\eta} = \begin{bmatrix} \dfrac{\Omega_2}{\Omega_1}\sin(\theta-\phi) & -\Gamma e^{\mathrm{i}\,\theta} \\ -\Gamma e^{-\mathrm{i}\,\theta} & \sin(\theta-\phi-\psi(\Gamma)) \end{bmatrix}, \tag{11}$$

where θ is an arbitrary angle. It is obvious that the matrix $\boldsymbol{\eta}$ is defined up to a factor, which can be any real function of time $f(t)$, i.e., $f(t)\boldsymbol{\eta}$ can replace $\boldsymbol{\eta}$. There are infinite number of matrices $\boldsymbol{\eta}$, which are not connected trivially [22], i.e., by a factor $f(t)$. They can be obtained by choosing different magnitudes of the angle θ.

Next, we set $\Gamma = 0$. Such a situation occurs quite often in applications [1–3]. In addition, in quantum mechanics, ρ_{kk} gives the probability for the system to be in state $|k\rangle$ and the condition $\Gamma = 0$ becomes mandatory. Indeed, in the opposite case one of the quantum mechanical levels will have a negative decay rate, which is not physical and could lead to probabilities greater than 1. We note, however, that in classical optics Γ may be nonzero. With these considerations, we have

$$\mathbf{H} = \begin{bmatrix} \Delta_1 & \Omega_1 e^{\mathrm{i}\,\phi} \\ \Omega_2 e^{-\mathrm{i}\,\phi} & \Delta_2 \end{bmatrix}, \tag{12a}$$

$$\boldsymbol{\eta} = \begin{bmatrix} \Omega_2/\Omega_1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{12b}$$

where we have removed the common factor $\sin(\theta-\phi)$ in front of $\boldsymbol{\eta}$. Obviously when $\Omega_1 = \Omega_2$, the Hamiltonian is hermitian and $\boldsymbol{\eta} = \mathbf{I}_2$, the unit matrix.

The problem of finding the form of a most general 2×2 $\boldsymbol{\eta}$ is also addressed in Ref. [6].

B. Constants of motion

Having derived the constants of motion (3) in the general case, and the matrix $\boldsymbol{\eta}$, Eq. (12b), it is straightforward to write down the constants of motion for a pseudo-hermitian two-level Hamiltonian,

$$\mathrm{Tr}(\boldsymbol{\eta\rho}) = \rho_{11}\Omega_2/\Omega_1 + \rho_{22} = \mathrm{const}, \tag{13a}$$

$$\mathrm{Tr}(\boldsymbol{\eta\rho})^2 = \rho_{11}^2\Omega_2^2/\Omega_1^2 + \rho_{22}^2 + 2\rho_{12}\rho_{21}\Omega_2/\Omega_1 = \mathrm{const}. \tag{13b}$$

By using the Cayley-Hamilton theorem we can express the second invariant by the first one: $\mathrm{Tr}(\boldsymbol{\eta\rho})^2 = [\mathrm{Tr}(\boldsymbol{\eta\rho})]^2 - \det(\boldsymbol{\eta\rho})$. In Appendix C, we prove that $\det(\boldsymbol{\eta\rho})$ is an invariant for a general N-level system, as long as $\boldsymbol{\eta} = \text{const}$. Thus, only Eq. (13a) is an independent invariant.

During the derivation of these conservation laws, we have only used that $\boldsymbol{\eta}$ has time-independent elements. This means that we require $\Omega_1(t)/\Omega_2(t) = \text{const}$, i.e. $\Omega_1(t)$ and $\Omega_2(t)$ must have the same time dependence. In other words, if

$$\mathbf{H} = \begin{bmatrix} \Delta_1(t) & Ag(t)e^{\mathrm{i}\,\phi(t)} \\ Bg(t)e^{-\mathrm{i}\,\phi(t)} & \Delta_2(t) \end{bmatrix}, \tag{14}$$

for *any* functions of time $\Delta_1(t)$, $\Delta_2(t)$, $\phi(t)$ and $g(t)$ (A and B are real constants not necessarily equal), the matrix $\boldsymbol{\eta}$ is time-independent and the conservation laws (13a) and (13b) can be used. Thus the condition $\boldsymbol{\eta} = \text{const}$ is not overly restrictive.

The problem of time-dependent, positive-definite $\boldsymbol{\eta}$ is addressed in Refs. [7] and [10].

C. Bloch vector

In Appendix D we derive analogues of the Bloch vector and the Bloch equation in a general N-state pseudo-hermitian system with $\boldsymbol{\eta} = \text{const}$. For $N = 2$, we have

$$\dot{\mathbf{S}} = \boldsymbol{\Upsilon} \times \mathbf{S}, \tag{15}$$

with the "torque vector" $\boldsymbol{\Upsilon}$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \Omega_1 e^{\mathrm{i}\,\phi} + \Omega_2 e^{-\mathrm{i}\,\phi} \\ \mathrm{i}\,\Omega_1 e^{\mathrm{i}\,\phi} - \mathrm{i}\,\Omega_2 e^{-\mathrm{i}\,\phi} \\ \Delta_1 - \Delta_2 \end{bmatrix}, \tag{16}$$

and the "Bloch vector"

$$\mathbf{S} = \begin{bmatrix} \rho_{21} + \Omega_2\rho_{12}/\Omega_1 \\ -\mathrm{i}\,\rho_{21} + \mathrm{i}\,\Omega_2\rho_{21}/\Omega_1 \\ \Omega_2\rho_{11}/\Omega_1 - \rho_{22} \end{bmatrix}. \tag{17}$$

Because neither the Bloch vector $\mathbf{S}(t)$ nor the torque vector $\boldsymbol{\Upsilon}(t)$ are real, the simple geometrical picture in the hermitian limit of a Bloch vector rotating upon a Bloch sphere cannot be applied here.

D. Implementations in physical systems

As we mentioned in the Introduction, non-hermitian Hamilonians are encountered in a number of physical platforms. Here we scrutinize a few of these.

1. *Guided Wave Optics*

One intriguing application of our dynamical invariants is in the area of guided wave optics [20]. Let us consider two electromagnetic modes with complex amplitudes A and B, which propagate in z direction,

$$a(z,x,t) = Ae^{\mathrm{i}\,(\omega_a t - \beta_a z)} f_a(x), \tag{18a}$$

$$b(z,x,t) = Be^{\mathrm{i}\,(\omega_b t - \beta_b z)} f_b(x). \tag{18b}$$

If the modes propagate in an unperturbed medium, then A and B are constants. However for a perturbed medium (electric field, a sound wave, surface corrugation, etc.) A and B depend on the propagation direction z. If the two modes propagate in opposite directions (contra-directional case) they obey the two coupled equations [20]

$$\mathrm{i}\,\frac{dA}{dz} = \kappa e^{\mathrm{i}\,\phi} e^{-\mathrm{i}\,\Delta z} B, \tag{19a}$$

$$\mathrm{i}\,\frac{dB}{dz} = -\kappa e^{-\mathrm{i}\,\phi} e^{\mathrm{i}\,\Delta z} A. \tag{19b}$$

Here, the coupling $\kappa(z)$ is a real function of z, $\Delta(z)$ is a (real) phase mismatch and $\phi(z)$ is a real phase. We substitute

$$A_1 = Ae^{\mathrm{i}\,\Delta z/2}, \quad A_2 = Be^{\mathrm{i}\,\Delta z/2} \tag{20}$$

into Eqs. (19). This gives

$$\mathrm{i}\,\frac{d}{dz}\mathbf{A} = \mathbf{H}\mathbf{A}, \tag{21}$$

where $\mathbf{A} = [A_1, A_2]^T$, and

$$\mathbf{H} = \begin{bmatrix} -\Delta/2 & \kappa(z)e^{\mathrm{i}\,\phi(z)} \\ -\kappa(z)e^{-\mathrm{i}\,\phi(z)} & \Delta/2 \end{bmatrix}. \tag{22}$$

Obviously this "Hamiltonian" is pseudo-hermitian. Indeed, upon comparison with Eq. (12), we find the correspondence $\Omega_2 = -\Omega_1$ and $t \to z$.

If we consider propagation of partially polarized light then, in general, A_1 and A_2 are functions of time [25]. Then if we define $\rho_{jk} = \overline{A_j^* A_k}$, we obtain the Liouville equation (2) for $\boldsymbol{\rho}$. Here the overline means a time average. Using Eqs. (12) and (22) it becomes obvious that $\boldsymbol{\eta} = \text{const}$. Then the dynamical invariants are

$$\mathrm{Tr}(\boldsymbol{\eta\rho}) = \rho_{11} - \rho_{22}, \tag{23a}$$

$$\mathrm{Tr}(\boldsymbol{\eta\rho})^2 = \rho_{11}^2 + \rho_{22}^2 - 2\rho_{12}\rho_{21}. \tag{23b}$$

The constant of motion (23a) is already known: it implies the conservation of energy.

2. *Sum-frequency generation*

Another interesting application is in the area of sum-frequency generation [21]. Let us consider a crystal with non-linear susceptibility $\chi^{(2)} \neq 0$ (non-centrosymmetric crystal). Let us mix a weak signal with frequency ω_1 with a high-intensity signal with frequency ω_2. Then the signal with frequency ω_1 will be converted into a signal with frequency $\omega_3 = \omega_1 + \omega_2$. Let us denote the amplitudes of the signals with frequencies ω_1 and ω_3 by A_1 and A_3, respectively, and introduce the vector $\mathbf{A} = [A_1, A_3]^T$. Then, the amplitudes satisfy the following equation [21]:

$$\mathrm{i}\,\frac{d}{dz}\mathbf{A} = \mathbf{H}\mathbf{A}, \tag{24}$$

where

$$\mathbf{H} = \begin{bmatrix} -\Delta/2 & K_1(z) \\ K_3(z) & \Delta/2 \end{bmatrix}. \tag{25}$$

Here Δ is the phase mismatch and $K_j \propto \omega_j^2 \chi^{(2)}(z)$, $j = 1, 3$. Obviously, for $\omega_1 \neq \omega_3$, the couplings are different in magnitude. If $\chi^{(2)}$ is z-dependent, then K_1 and K_2 have the *same* z-dependence. As a consequence $\boldsymbol{\eta} = $ const and the dynamical invariants (13) can be applied.

E. Pseudo-hermitian Rabi model

Let us assume that all matrix elements of the Hamiltonian (12) are time-independent. We call this *pseudo-hermitian Rabi model* in analogy to the hermitian case [26]. The solution of Eq. (2) reads

$$\boldsymbol{\rho}(t) = \mathrm{e}^{-\mathrm{i}\,\mathbf{H}t}\boldsymbol{\rho}(0)\mathrm{e}^{\,\mathrm{i}\,\mathbf{H}^\dagger t}, \tag{26}$$

with

$$\mathrm{e}^{-\mathrm{i}\,\mathbf{H}t} = \begin{bmatrix} \cos\beta t - \mathrm{i}\,\frac{\Delta}{\beta}\sin\beta t & -\mathrm{i}\,\frac{\Omega_1}{\beta}e^{\mathrm{i}\,\phi}\sin\beta t \\ -\mathrm{i}\,\frac{\Omega_2}{\beta}e^{-\mathrm{i}\,\phi}\sin\beta t & \cos\beta t + \frac{\Delta}{\beta}\sin\beta t \end{bmatrix}, \tag{27}$$

where

$$\beta = \sqrt{\Omega_1\Omega_2 + \Delta^2}. \tag{28}$$

Obviously, when $\Omega_1 = \Omega_2$, Eq. (27) coincides with the propagator for the Rabi model [26]. However, for $\Omega_1 \neq \Omega_2$ it is not unitary, since $\mathbf{H}$ is not hermitian.

Let us assume that the system is initially, at time $t = 0$, in state $|1\rangle$. This implies that $\rho_{11}(t = 0) = 1$ and all other matrix elements of $\boldsymbol{\rho}(t = 0)$ are 0. Then upon substitution of Eq. (27) into Eq. (26), we derive

$$\rho_{11}(t) = \cos^2\beta t + \frac{\Delta^2}{\beta^2}\sin^2\beta t, \tag{29a}$$

$$\rho_{22}(t) = \frac{\Omega_2^2}{\beta^2}\sin^2\beta t. \tag{29b}$$

Obviously $\rho_{11}(t) + \rho_{22}(t) \neq 1$, unless $\Omega_1 = \Omega_2$, due to the fact that the "propagator" (27) is not unitary. Instead, as it is easily verified, Eqs. (29) satisfy the pseudo-hermitian probability invariant (13a).

In Fig. 1 we plot $\rho_{11}(t)$, $\rho_{22}(t)$ as well as the total population $\rho_{11}(t) + \rho_{22}(t)$ for different values of Ω_1 and Ω_2. When $\Omega_1 = \Omega_2$, we recover the familiar case of conserved total probability, as it is the case in hermitian quantum mechanics. In Fig. 1(b) we show the resonance Rabi case for the pseudo-hermitian Hamiltonian. The sum of probabilities $\rho_{11}(t) + \rho_{22}(t)$ is conserved only in the hermitian case (lower right frame).

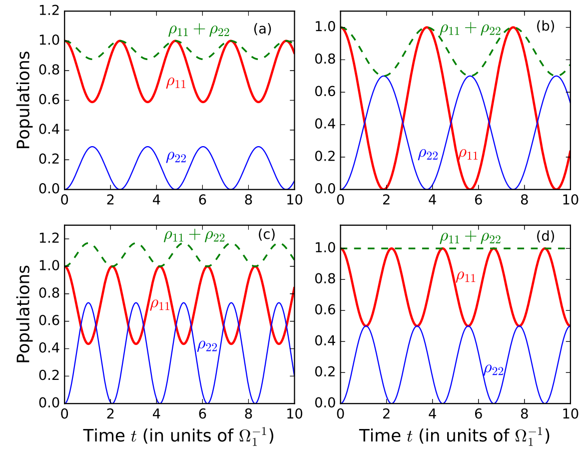

FIG. 1: (Color online) Time evolution of the transition probabilities in the pseudo-hermitian Rabi model in a two-state system, for (a) $\Omega_2/\Omega_1 = 0.7$, $\Delta = \Omega_1$; (b) $\Omega_2/\Omega_1 = 0.7$, $\Delta = 0$; (c) $\Omega_2/\Omega_1 = 1.3$, $\Delta = \Omega_1$; (d) $\Omega_1 = \Omega_2$, $\Delta = \Omega_1$. For $\Omega_2 > \Omega_1$ (frame (c)), we have $\rho_{11}(t) + \rho_{22}(t) \geqq 1$.

The case $\Omega_2 > \Omega_1$ (lower left plot)is of particular interest in sum-frequency generation [21] discussed in Sec. III D 2. There ρ_{jj} gives the energy stored in electromagnetic mode j, and $\rho_{11} + \rho_{22}$ gives the total energy stored in the input *and* output field. The total energy in these two modes is *not* conserved, since an additional mode exists — one of the strong input modes [21].

In Ref. [5], it is shown that for a positive-definite matrix $\boldsymbol{\eta}$, the problem can be reduced to one studying ordinary Hermitian systems. In our case, $\boldsymbol{\eta}$ needs not be positive-definite. However, it becomes such, if Ω_1 and Ω_2 have the same signs, cf. Eq. (12b). Indeed, only then both of the eigenvalues of $\boldsymbol{\eta}$ are positive. We make two observations, regarding non-positive $\boldsymbol{\eta}$'s. First, it is easy to prove that when Ω_1 and Ω_2 have different signs, the total population $\rho_{11} + \rho_{22}$ becomes greater than 1, for some intervals of time, regardless of the magnitude of Ω_1 and Ω_2. This is easily seen from Eqs. (29). Thus, we do not expect quantum-mechanical applications. Only electromagnetic applications are possible, where ρ_{jj} are not probabilities. Second, when Ω_1 and Ω_2 have different signs, β may becomes imaginary, cf. Eq. (28). In this case, the populations ρ_{11} and ρ_{22} no longer oscillate but increase exponentially in time. In other words, qualitatively different behaviour is observed.

IV. PSEUDO-HERMITIAN THREE-STATE SYSTEM

A. Pseudo-hermitian three-state Hamiltonian

Consider a general complex three-state Hamiltonian, with $H_{13} = H_{31} = 0$,

$$\mathbf{H} = \begin{bmatrix} \Delta_1 + \mathrm{i}\,\Gamma_1 & \Omega_1 e^{\mathrm{i}\,(\phi_1+\psi_1)} & 0 \\ \Omega_2 e^{-\mathrm{i}\,\phi_1} & \Delta_2 + \mathrm{i}\,\Gamma_2 & \Omega_3 e^{\mathrm{i}\,(\phi_2+\psi_2)} \\ 0 & \Omega_4 e^{-\mathrm{i}\,\phi_2} & \Delta_3 + \mathrm{i}\,\Gamma_3 \end{bmatrix}. \quad (30)$$

The condition $H_{13} = H_{31} = 0$ corresponds to Λ (lambda), Ξ (ladder) or V quantum systems in the hermitian case [27, 28]. As in the two-level system, $\Delta_k(t)$, $\Gamma_k(t)$, $\phi_k(t)$, $\psi_k(t)$ and $\Omega_k(t)$ are arbitrary real functions of time.

We proceed with condition (1) and take into account Eq. (2). Condition (1) imposes certain relations between the matrix elements of Eq. (30). To this end, we use the fact that the eigenvalues of Eq. (30) are either real or grouped into complex-conjugated pairs, and use the method described in Appendix B for the two-level case, cf. Eqs. (B1a) and (B1b). One of the conditions is

$$\Gamma_1 + \Gamma_2 + \Gamma_3 = 0. \quad (31)$$

The most general case is rather cumbersome to be displayed here. Thus we confine ourselves to the quantum case, where due to the restrictions $\Gamma_k \leqq 0$, we must have $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. Continuing with the other conditions, derived from the characteristic equation for $\mathbf{H}$ we also deduce that

$$\psi_1 = \psi_2 = 0. \quad (32)$$

Therefore the Hamiltonian becomes

$$\mathbf{H} = \begin{bmatrix} \Delta_1 & \Omega_1 e^{\mathrm{i}\,\phi_1} & 0 \\ \Omega_2 e^{-\mathrm{i}\,\phi_1} & \Delta_2 & \Omega_3 e^{\mathrm{i}\,\phi_2} \\ 0 & \Omega_4 e^{-\mathrm{i}\,\phi_2} & \Delta_3 \end{bmatrix}. \quad (33)$$

For this Hamiltonian, only one matrix $\boldsymbol{\eta}$ exists (up to an arbitrary real factor $f(t)$), which obeys condition (1), viz.

$$\boldsymbol{\eta} = \begin{bmatrix} \Omega_2/\Omega_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Omega_3/\Omega_4 \end{bmatrix}. \quad (34)$$

Now it is straightforward to derive the conservation laws (3). The first two of them read

$$\mathrm{Tr}(\boldsymbol{\eta\rho}) = \frac{\Omega_2}{\Omega_1}\rho_{11} + \rho_{22} + \frac{\Omega_3}{\Omega_4}\rho_{33}, \quad (35)$$

$$\mathrm{Tr}(\boldsymbol{\eta\rho})^2 = \frac{\Omega_2^2}{\Omega_1^2}\rho_{11}^2 + \rho_{22}^2 + \frac{\Omega_3^2}{\Omega_4^2}\rho_{33}^2 + 2\frac{\Omega_2}{\Omega_1}|\rho_{12}|^2 + 2\frac{\Omega_2\Omega_3}{\Omega_1\Omega_4}|\rho_{13}|^2 + 2\frac{\Omega_3}{\Omega_4}|\rho_{32}|^2. \quad (36)$$

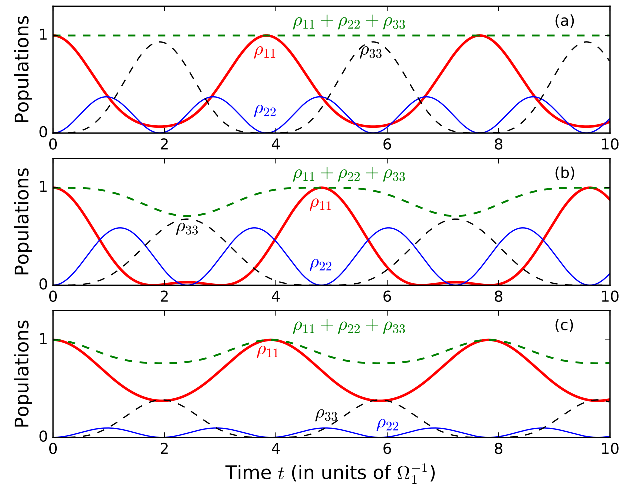

FIG. 2: (Color online) Evolution of populations for pseudo-hermitian three-level system. (a) $\Omega_1 = \Omega_2$ and $\Omega_3 = \Omega_4 = 1.3\Omega_1$. (b) $\Omega_2 = \Omega_1$, $\Omega_3 = \Omega_1$ and $\Omega_4 = 0.7\Omega_1$. (c) All couplings are different, $j = 1, 2, 3, 4$: $\Omega_2 = 0.5\Omega_1$, $\Omega_3 = 1.3\Omega_1$ and $\Omega_4 = 1.6\Omega_1$.

The third one is $\mathrm{Tr}(\boldsymbol{\eta\rho})^3$. By using the Cayley-Hamilton theorem it can be calculated by using the characteristic equation of $\boldsymbol{\eta\rho}$. For example, on two-photon resonance ($\Delta_1 = \Delta_3 = 0$) we find

$$\mathrm{Tr}(\boldsymbol{\eta\rho})^3 = \mathrm{Tr}(\boldsymbol{\eta\rho})^2\mathrm{Tr}(\boldsymbol{\eta\rho}) + (\Omega_2^2 + \Omega_3^2)\mathrm{Tr}(\boldsymbol{\eta\rho}) + 3\det(\boldsymbol{\eta\rho}) \quad (37)$$

The above conservation laws are valid when $\boldsymbol{\eta} =$ const, i.e., Ω_1 and Ω_2, as well as Ω_3 and Ω_4, must have the same time-dependence: $\Omega_1(t) = \alpha_1 f(t)$, $\Omega_2(t) = \alpha_2 f(t)$, $\Omega_3(t) = \alpha_3 g(t)$, and $\Omega_4(t) = \alpha_4 g(t)$.

In the extreme case when $\Omega_1 = -\Omega_2$ and $\Omega_3 = -\Omega_4$, which is of interest in optical waveguides, we have

$$\mathrm{Tr}(\boldsymbol{\eta\rho}) = -\rho_{11} + \rho_{22} - \rho_{33}, \quad (38a)$$

$$\mathrm{Tr}(\boldsymbol{\eta\rho})^2 = \rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 - 2|\rho_{12}|^2 + 2|\rho_{13}|^2 - 2|\rho_{32}|^2. \quad (38b)$$

B. Resonant solution

We provide the exact solution of Eq. (2) on resonance, i.e., when $\Delta_1 = \Delta_2 = \Delta_3 = 0$. We have

$$\boldsymbol{\rho}(t) = e^{-\mathrm{i}\,\mathbf{H}t}\boldsymbol{\rho}(0)e^{\mathrm{i}\,\mathbf{H}^\dagger t}. \quad (39)$$

The above equation yields

$$\rho_{11} = \frac{[\Omega_1\Omega_2\cos(\omega t) + \Omega_3\Omega_4]^2}{\omega^4}, \tag{40}$$

$$\rho_{22} = \frac{\Omega_2^2\sin^2(\omega t)}{\omega^2}, \tag{41}$$

$$\rho_{33} = \frac{4\Omega_2^2\Omega_4^2\sin^4(\omega t/2)}{\omega^4}. \tag{42}$$

with $\omega = \sqrt{\Omega_1\Omega_2 + \Omega_3\Omega_4}$.

Figure 2 shows examples of this solution. The total probability $\rho_{11} + \rho_{22} + \rho_{33}$ is conserved only in the hermitian limit, frame (a). Instead, it is easily verified by inspection that this solution satisfies the dynamical invariants (38).

V. CONCLUSION

We have investigated pseudo-hermitian quantum systems and derived conservation laws, when the matrix elements of $\boldsymbol{\eta}$ are time-independent. The latter condition, as it turns out, is still rather general for it only implies the same time dependence of pairs of non-diagonal elements of the Hamiltonian. We have applied the conservation laws to the specific cases of two- and three-state quantum systems, and we have exemplified them with the explicit analytic solutions for constant couplings . For the non-hermitian two-state system, we have derived the analogue of the Bloch vector and the Bloch equation. The dynamical invariants can be obtained for even larger pseudo-hermitian matrices, under rather general physical assumptions.

Appendix A: Derivation of $\boldsymbol{\eta}$ for Two-Level System

Let us substitute

$$\mathcal{H} = \boldsymbol{\eta}\mathbf{H}\boldsymbol{\eta}^{-1}. \tag{A1}$$

Thus, Eq. (1) is equivalent to $\mathcal{H} = \mathbf{H}^\dagger$. From $\mathcal{H}_{11} = H_{11}^\dagger$, we have

$$\eta_{11} = \eta_{12}\frac{|\eta_{12}|(\Delta_1 - \Delta_2 - \mathrm{i}\,(\Gamma_1 + \Gamma_2)) + \eta_{22}\Omega_2 e^{\mathrm{i}\,(\theta-\phi)}}{\Omega_1|\eta_{12}|e^{\mathrm{i}\,(\phi+\psi)} - 2\mathrm{i}\,\Gamma_1\eta_{22}e^{\mathrm{i}\,\theta}}, \tag{A2}$$

where $\arg(\eta_{12}) = -\arg(\eta_{21}) = \theta$, since $\boldsymbol{\eta}$ is hermitian operator. Upon substitution of the above equation into $\mathcal{H}$, we have

$$\mathcal{H}_{22} = \Delta_2 + \mathrm{i}\,(2\Gamma_1 + \Gamma_2). \tag{A3}$$

Using $\mathcal{H}_{22} = H_{22}^\dagger$ we derive

$$\Gamma_1 + \Gamma_2 = 0. \tag{A4}$$

Next, we use $\mathcal{H}_{21} = H_{21}^\dagger$, which yields

$$|\eta_{21}| = \frac{2\mathrm{i}\,\eta_{22}\Gamma e^{\mathrm{i}\,(\theta+\phi+\psi)}}{\Omega_1(e^{2\mathrm{i}\,(\phi+\psi)} - e^{2\mathrm{i}\,\theta})}. \tag{A5}$$

We substitute the above equation into $\mathcal{H}$. After we apply the condition $\mathcal{H}_{12} = H_{12}^\dagger$ we derive

$$\sin\psi = \frac{(\Delta_2 - \Delta_1)\Gamma}{\Omega_1\Omega_2}. \tag{A6}$$

Finally, we use Eqs. (A2), (A4), (A5), (A6) as well as the fact that $\boldsymbol{\eta}$ is defined up to arbitrary real factor. In this way, after some algebra, we derive the matrix $\boldsymbol{\eta}$ in Eq. (11).

Appendix B: Derivation of conditions for pseudo-Hermiticity

We provide an additional method for deriving conditions (8), which is applicable for more than two levels. Following Ref. [4], we take into account that any diagonalizable pseudo-hermitian matrix has eigenvalues, which are either real or grouped into complex-conjugated pairs. In other words, if λ is an eigenvalue of the Hamiltonian (7), so is λ^*. Then the secular equations

$$\lambda^2 - \lambda(\Delta_1 + \Delta_2 + \mathrm{i}\,\Gamma_1 + \mathrm{i}\,\Gamma_2) + (\Delta_1 + \mathrm{i}\,\Gamma_1)(\Delta_2 + \mathrm{i}\,\Gamma_2) - \Omega_1\Omega_2 e^{\mathrm{i}\,\psi} = 0, \tag{B1a}$$

and

$$\lambda^2 - \lambda(\Delta_1 + \Delta_2 - \mathrm{i}\,\Gamma_1 - \mathrm{i}\,\Gamma_2) + (\Delta_1 - \mathrm{i}\,\Gamma_1)(\Delta_2 - \mathrm{i}\,\Gamma_2) - \Omega_1\Omega_2 e^{-\mathrm{i}\,\psi} = 0, \tag{B1b}$$

should yield the same roots. Comparing the coefficients of the above two equations we get exactly conditions (8). This method is quite simple and therefore it is useful for matrices of higher dimensions.

Appendix C: Proof that $\det\boldsymbol{\eta\rho}$ is a dynamical invariant

We can easily prove that $\det\boldsymbol{\eta\rho}$ is a dynamical invariant when $\mathbf{H}$ is time-independent. First, it is obvious that

$$\boldsymbol{\rho}(t) = e^{-\mathrm{i}\mathbf{H}t}\rho(0)e^{\mathrm{i}\mathbf{H}^\dagger t}. \tag{C1}$$

The above equation is immediately seen upon differentiation and comparison with Eq. (2).

Below, we shall make use of the formula

$$\det\mathbf{AB} = \det\mathbf{A}\det\mathbf{B}, \tag{C2}$$

as well as its elementary consequence

$$\det\mathbf{A}^{-1} = 1/\det\mathbf{A}. \tag{C3}$$

In order to prove that $\det\boldsymbol{\eta\rho}$ is constant in time, we shall need one last equation

$$\boldsymbol{\eta}f(\mathbf{H}) = f(\mathbf{H}^\dagger)\boldsymbol{\eta}, \tag{C4}$$

where $f(x)$ is any function, which can be Taylor expanded. The above equation is easily proved, if we take into account Eq. (1). Indeed, from Eq. (1) it follows by induction that $\boldsymbol{\eta}\mathbf{H}^n = \mathbf{H}^{\dagger n}\boldsymbol{\eta}$ for any positive integer n. Using this result and Taylor expanding $f(\mathbf{H})$, Eq. (C4) is proved.

With this preparation in hand, we have as follows

$$\det\left[\boldsymbol{\eta}\boldsymbol{\rho}(t)\right] = \det\left[\boldsymbol{\eta}e^{-\mathrm{i}\mathbf{H}t}\rho(0)e^{\mathrm{i}\mathbf{H}^\dagger t}\right] = \det\left[e^{-\mathrm{i}\mathbf{H}^\dagger t}\boldsymbol{\eta}\rho(0)e^{\mathrm{i}\mathbf{H}^\dagger t}\right]. \tag{C5}$$

In the last line, we have used (C4). Now, taking into account Eqs. (C2) and (C3), we finally see that

$$\det\left[\boldsymbol{\eta}\boldsymbol{\rho}(t)\right] = \det\left[\boldsymbol{\eta}\boldsymbol{\rho}(0)\right], \tag{C6}$$

i.e., $\det\left[\boldsymbol{\eta}\boldsymbol{\rho}(t)\right]$ is a dynamical invariant.

When $\mathbf{H}$ is time-dependent, the proof is slightly more cumbersome. From Eq. (2) we have

$$\begin{aligned}\boldsymbol{\rho}(t+\delta t) &= \boldsymbol{\rho}(t) - \mathrm{i}\delta t\mathbf{H}(t)\boldsymbol{\rho}(t) + \mathrm{i}\delta t\boldsymbol{\rho}(t)\mathbf{H}^\dagger(t)\\ &= e^{-\mathrm{i}\mathbf{H}(t)\delta t}\boldsymbol{\rho}(t)e^{\mathrm{i}\mathbf{H}^\dagger\delta t} + O(\delta t^2),\end{aligned} \tag{C7}$$

where δt is infinitesimally small time interval. By induction, we can find $\boldsymbol{\rho}(t+2\delta t)$, $\boldsymbol{\rho}(t+3\delta t)$,..., until we reach the final moment t_{f}. Thus we obtain

$$\boldsymbol{\rho}(t_{\mathrm{f}}) = \lim_{\delta t\to 0}\prod_\alpha e^{-\mathrm{i}\mathbf{H}(t_\alpha)\delta t}\boldsymbol{\rho}(0)\lim_{\delta t\to 0}\prod_\alpha e^{\mathrm{i}\mathbf{H}^\dagger(t_\alpha)\delta t}. \tag{C8}$$

Generally speaking $\mathbf{H}(t_\alpha)$ does not commute with $\mathbf{H}(t_\beta)$, for $\alpha \neq \beta$. If they commute, the above product is a simple exponent. However, since the commutator is not zero, we define as is customary the so called T-exponent,

$$\mathrm{T}\exp\left(-\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}(t')dt'\right) \equiv \lim_{\delta t\to 0}\prod_\alpha e^{-\mathrm{i}\mathbf{H}(t_\alpha)\delta t} \tag{C9}$$

Using the above definition of T exponent as well as Eq. (C4), we have that

$$\boldsymbol{\eta}\mathrm{T}\exp\left(-\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}(t')dt'\right) = \mathrm{T}\exp\left(-\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}^\dagger(t')dt'\right)\boldsymbol{\eta}. \tag{C10}$$

Note $\mathbf{H}^\dagger$ on the right. We can now prove that $\det\boldsymbol{\eta}\rho$ is a dynamical invariant even for time-dependent Hamiltonian.

$$\begin{aligned}&\det\boldsymbol{\eta}\boldsymbol{\rho}(t_{\mathrm{f}}) =\\ &\det\left[\boldsymbol{\eta}\mathrm{T}\exp\left(-\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}(t')dt'\right)\boldsymbol{\rho}(0)\mathrm{T}\exp\left(\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}^\dagger(t')dt'\right)\right]\\ &\det\left[\mathrm{T}\exp\left(-\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}^\dagger(t')dt'\right)\boldsymbol{\eta}\boldsymbol{\rho}(0)\mathrm{T}\exp\left(\mathrm{i}\int_0^{t_{\mathrm{f}}}\mathbf{H}^\dagger(t')dt'\right)\right].\end{aligned} \tag{C11}$$

Finally, using Eqs. (C2) and (C3) we obtain the $\det\boldsymbol{\eta}\boldsymbol{\rho}$ is a dynamical invariant.

Appendix D: Bloch Vector for General N-Level Pseudo-hermitian System

We define the operators [23]

$$\mathbf{w}_l = -\sqrt{\frac{2}{l(l+1)}}\left[|1\rangle\langle 1| + \cdots + |l\rangle\langle l| - l|l+1\rangle\langle l+1|\right], \tag{D1a}$$

$$\mathbf{u}_{kj} = |k\rangle\langle j| + |j\rangle\langle k|, \quad (1 \leqq k \leqq j \leqq N), \tag{D1b}$$

$$\mathbf{v}_{kj} = \mathrm{i}\left(|j\rangle\langle k| - |k\rangle\langle j|\right), \quad (1 \leqq k \leqq j \leqq N), \tag{D1c}$$

where $l = 1, 2, ..., N-1$. Moreover, we define the vector-operator

$$\mathbf{s} = (\mathbf{u}_{12}, ..., \mathbf{v}_{12}, ..., \mathbf{w}_1, ..., \mathbf{w}_{N-1}). \tag{D2}$$

It is well-known that [23]

$$[\mathbf{s}_i, \mathbf{s}_j] = 2\mathrm{i}\sum_k f_{ijk}\mathbf{s}_k, \tag{D3}$$

as well as

$$\mathrm{Tr}(\mathbf{s}_i\mathbf{s}_j) = 2\delta_{ij}, \tag{D4}$$

where f_{ijk} are the antisymmetric structure constants of the group $\mathrm{SU}(N)$, and δ_{ij} is Kronecker's delta. Equations (D1a), (D1b) and (D1c) make it evident that

$$\mathrm{Tr}(\mathbf{s}_i) = 0. \tag{D5}$$

For $N = 2$, $\mathbf{s}_i$ become the Pauli matrices and $f_{ijk} = \epsilon_{ijk}$, the Levi-Civita symbol.

Any matrix can be represented as a linear combination of the matrices $\mathbf{s}_i$ and $\mathbf{I}$. Thus we represent

$$\boldsymbol{\eta}\boldsymbol{\rho}(t) = K(t)\mathbf{I} + \frac{1}{2}\sum_{j=1}^{N^2-1}S_j(t)\mathbf{s}_j, \tag{D6}$$

and

$$\mathbf{H}^\dagger = \frac{1}{2}\sum_m \Upsilon_m(t)\mathbf{s}_m. \tag{D7}$$

Here $K(t), S_j(t)$ and $\Upsilon_m(t)$ are complex functions. Obviously,

$$S_j = \mathrm{Tr}\left(\boldsymbol{\eta}\boldsymbol{\rho}\mathbf{s}_j\right). \tag{D8}$$

Indeed, in order to obtain the above equation we multiply Eq. (D6) on the right with $\mathbf{s}_k$, take trace and use Eqs. (D4) and (D5). Next we differentiate Eq. (D8) and use $\boldsymbol{\eta} = \mathrm{const}$. Thus,

$$\begin{aligned}\dot{S}_k &= \mathrm{Tr}(\boldsymbol{\eta}\dot{\boldsymbol{\rho}}\mathbf{s}_k)\\ &= -\mathrm{i}\mathrm{Tr}\left(\boldsymbol{\eta}(\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H}^\dagger)\mathbf{s}_k\right)\\ &= -\mathrm{i}\mathrm{Tr}\left(\mathbf{H}^\dagger(\boldsymbol{\eta}\boldsymbol{\rho})\mathbf{s}_k - (\boldsymbol{\eta}\boldsymbol{\rho})\mathbf{H}^\dagger\mathbf{s}_k\right).\end{aligned} \tag{D9}$$

It is crucial that we have used Eq. (1) in the last line. Now, we can apply Eqs. (D6) and (D7). Upon substitution of these into the above equation we obtain

$$\dot{S}_k = \sum_{mj} f_{mjk} \Upsilon_m S_j, \tag{D10}$$

Now we are ready to prove that the magnitude of the vector $\mathbf{S} = [S_1(t), ..., S_{N^2-1}(t)]$ is time-independent. Indeed,

$$\frac{d}{dt} \sum_k S_k^2 = 2 \sum_{k,m,j} f_{mkj} \Upsilon_m S_j S_k = 0, \tag{D11}$$

due to the antisymmetry of the structure constants f_{ijk}. Therefore we can consider the vector $\mathbf{S}$ as the pseudo-hermitian analogue of the Bloch vector.

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