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Bosonic pair creation and the Schiff-Snyder-Weinberg effect

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Interactions between different bound states in bosonic systems can lead to pair creation. We study this process in detail by solving the Klein-Gordon equation on space-time grids in the framework of time-dependent quantum field theory. By choosing specific external field configurations, two bound states can become pseudo degenerate, which is commonly referred to as Schiff-Snyder-Weinberg effect. These pseudo degenerate bound states, which have complex energy eigenvalues, are related to the pseudo hermiticity of the Klein-Gordon Hamiltonian. In this work, the influence of the Schiff-Snyder-Weinberg effect on pair production is studied. A generalized Schiff-Snyder-Weinberg effect, where several pairs of pseudo degenerate states appear, is found in combined electric and magnetic fields. The generalized Schiff-Snyder-Weinberg effect likewise triggers pair creation. The particle number in these situations obeys an exponential growth law in time enhancing the creation of bosons, which cannot be found in fermionic systems.

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I. INTRODUCTION

The possibility to create particle-antiparticle pairs from the vacuum in a strong external force field has sparked considerable interest among the theoretical [1–8] as well as the experimental communities [9–12]. The number of created particles during the interaction is associated directly with transitions from (initially occupied) states of negative energy to those of the positive energy. There are two mechanisms that can cause these upward transitions. The first one requires the external force to be time-dependent. Here the force field could be provided by one or several electromagnetic radiation pulses. In fact, several experimental efforts are presently planned worldwide to develop lasers with extremely high intensities that could breakdown the vacuum [13, 14].

The second mechanism of pair creation is based on static fields that can temporarily lead to effective spectral degeneracies among the energy states during the interaction. The most frequently studied case involves the degeneracy of two continua, as it is characteristic for the Schwinger mechanism, predicted by Sauter [15], Heisenberg [16], and Schwinger [17]. This scenario is also directly related to the so-called Klein paradox [18–23], where a sufficiently steep and large potential barrier can trigger the pair creation. In the early 1970s it was predicted [24–27] that the degeneracy between a potential's ground state and the lower continuum can also trigger pair creation.

It seems therefore natural to expect that the creation of particle pairs should also become possible due to a degeneracy of two discrete states. However, the generic behavior of two coupled discrete states is generally characterized by avoided crossings [28]. A recent work [29] examined the Dirac equation for a field configuration characterized by a combination of an attractive well and a repulsive well, which could support simultaneously bound states for particles as well as antiparticles. By

decreasing the spatial separation between the two neighboring wells it was possible to couple the electronic and positronic ground states with each other and to examine the effect of the corresponding avoided crossing (quasi degeneracy) on the change in the total particle number. Due to the unavoidable occurrence of avoided crossings it is not possible to construct a field configuration for a fermionic system for which an exact degeneracy can be found in the discrete spectrum of the Dirac equation.

For the Klein-Gordon equation there are field configurations, for which a pseudo degeneracy of discrete states can occur, in contrast to the Dirac equation. This was first pointed out by Schiff, Snyder, and Weinberg [30, 31] in 1940, who discovered (rather counter-intuitively) that a sufficiently strong and finite-range potential can support simultaneous bound states for the particle as well as the antiparticle. Later the restriction of the compact spatial support was generalized to potentials that only have to be short-range [32–34]. The Schiff-Snyder-Weinberg effect was related to pair creation by various authors [32, 34–36]. Here we are going to study bosonic pair creation under the Schiff-Snyder-Weinberg effect systematically in the framework of quantum field theory.

The article is organized as follows. In the Sec. II we discuss general properties of the Klein-Gordon equation in its Feshbach-Villars representation, in particular its spectrum and the properties of the eigenvectors, which leads us to the concept of pseudo degeneracy and the Schiff-Snyder-Weinberg effect, which is studied in Sec. III. Section IV examines the creation of bosonic particle-antiparticle pairs induced by the pseudo degeneracy of the Klein-Gordon equation and shows that the pair creation rate can be obtained from the imaginary parts of the Hamilton operator's eigenvalues. Pseudo degeneracies between discrete states do not necessarily require external binding fields. In fact, a suitable superposition of a non-binding electric field and a magnetic field can also lead to Schiff-Snyder-Weinberg-like states that eventually become pseudo degenerate, as shown in Sec. V. How a possible back reaction of the created particles

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on the electromagnetic environment modifies the pair creation dynamics is investigated in Sec. VI. Finally, we complete this article with an outlook and conclusions in Sec. VII.

II. THE KLEIN-GORDON EQUATION

Before considering pair creation within a quantum field theoretical framework let us firstly take a look at the spectrum of the Klein-Gordon equation for a single particle. A bosonic quantum particle of mass m and charge q interacting with the electromagnetic potentials $A(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ is described by the Klein-Gordon equation in the Feshbach-Villars representation [37–39]

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \hat{H}_{\text{KG}} \Psi(\mathbf{r}, t) = \left(\frac{\sigma_3 + i\sigma_2}{2m} (-i\hbar \nabla - qA(\mathbf{r}, t))^2 + q\phi(\mathbf{r}, t) + \sigma_3 mc^2 \right) \Psi(\mathbf{r}, t), \quad (1)$$

where the Pauli matrices σ_1 , σ_2 , and σ_3 are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

In contrast to the Schrödinger equation, the Hamilton operator $\hat{H}_{\text{KG}}$, which determines the time-evolution of the two-component wave function $\Psi(\mathbf{r}, t)$, is not Hermitian and the corresponding time-evolution operator is not unitary. The Klein-Gordon Hamilton operator is, however, σ_3 pseudo Hermitian and its time-evolution operator is σ_3 pseudo unitary [40, 41]. A linear operator $\hat{H}$ is called $\hat{\eta}$ pseudo Hermitian if there is a Hermitian operator $\hat{\eta}$ such that $\hat{H}$ equals its so-called $\hat{\eta}$ pseudo adjoint operator $\hat{H}^\# = \hat{\eta}^{-1} \hat{H}^\dagger \hat{\eta}$, i. e.,

$$\hat{\eta}^{-1} \hat{H}^\dagger \hat{\eta} = \hat{H}. \quad (3)$$

Hermitian Hamilton operators have real eigenvalues, their eigenfunctions form an orthonormal basis, and the vectors in the dual space which are associated to the eigenfunctions are given by the complex conjugated eigenfunctions. Pseudo Hermitian operators as the Klein-Gordon Hamilton operator, however, may have complex eigenvalues and the eigenfunctions and the dual functions are not just the complex conjugated of each other.

For simplicity let us assume that the Hilbert space has a finite dimension d . Then the pseudo Hermitian Hamilton operator can be represented by a finite matrix $\mathbf{H}$ with some set of right eigenvectors ψ_i (the so-called kets, represented by column vectors) and the corresponding left eigenvectors φ_i (the so-called bras or dual vectors, represented by row vectors). The left and right eigenvectors form a biorthogonal system, i. e., $\varphi_i \psi_j \neq 0$ if and only if $i = j$. Note that because left and right eigenvectors are defined only up to a multiplicative complex constant, one can always find left and right eigenvectors such that $\varphi_i \psi_i = 1$. Corresponding left and right eigenvectors φ_i and ψ_i have the same eigenvalue $\mathcal{E}_i$. If the number of left/right eigenvectors equals d then the eigenvectors span the whole Hilbert space and

any column vector Ψ can be expanded into the basis of right eigenvectors by multiplying with the left eigenvectors, i. e.,

$$\Psi = \sum_{i=1}^d \frac{\varphi_i \Psi}{\varphi_i \psi_i} \psi_i. \quad (4)$$

Thus, the biorthogonal system ψ_i and φ_i plays the same role as the orthogonal systems do in the Hermitian case. The notion of biorthogonal vectors and the results, which follow in the reminder of this section, can be generalized to biorthogonal function systems in case of infinite dimensional Hilbert spaces [42–44].

For each left eigenvector φ_i it holds

$$(\varphi_i \mathbf{H})^\dagger = (\mathcal{E}_i \varphi_i)^\dagger = \mathcal{E}_i^* \varphi_i^\dagger, \quad (5)$$

but due to pseudo hermiticity also

$$(\varphi_i \mathbf{H})^\dagger = \mathbf{H}^\dagger \varphi_i^\dagger = \boldsymbol{\eta} \mathbf{H} \boldsymbol{\eta}^{-1} \varphi_i^\dagger, \quad (6)$$

where $\boldsymbol{\eta}$ is in the case of the Klein-Gordon equation a blockdiagonal matrix with σ_3 replicated on the diagonal and $\boldsymbol{\eta} = \boldsymbol{\eta}^{-1} = \boldsymbol{\eta}^\dagger$. Consequently

$$\mathbf{H} \boldsymbol{\eta}^{-1} \varphi_i^\dagger = \mathcal{E}_i^* \boldsymbol{\eta}^{-1} \varphi_i^\dagger, \quad (7)$$

and therefore $\boldsymbol{\eta}^{-1} \varphi_i^\dagger$ is a right eigenvector of $\mathbf{H}$ with the eigenvalue $\mathcal{E}_i^*$. Thus, eigenvalues of $\mathbf{H}$ must be real or come in pairs of $\mathcal{E}_i$ and its complex conjugate $\mathcal{E}_i^*$, i. e., there is some $\mathcal{E}_j$ such that $\mathcal{E}_j = \mathcal{E}_i^*$. We call eigenstates with $\mathcal{E}_j = \mathcal{E}_i^*$ pseudo degenerate because the corresponding energy eigenvalues have equal real parts but different imaginary parts. Furthermore, it follows from (7) for a pair of pseudo degenerate states that the right eigenvector of state j is related to the left eigenvector of state i by $\psi_j = \boldsymbol{\eta}^{-1} \varphi_i^\dagger$ or equivalently $\varphi_i = \psi_j^\dagger \boldsymbol{\eta}$. If $\mathcal{E}_i$ is real, however, it follows from (7) that $\boldsymbol{\eta}^{-1} \varphi_i^\dagger$ is a right eigenvector corresponding to the left eigenvector φ_i , i. e., $\psi_i = \boldsymbol{\eta}^{-1} \varphi_i^\dagger$ or equivalently $\varphi_i = \psi_i^\dagger \boldsymbol{\eta}$.

Therefore, one may introduce the pseudo inner product $(\Psi_1, \Psi_2)_\eta$ between two quantum states Ψ_1 and Ψ_2 as

$$(\Psi_1, \Psi_2)_\eta = \Psi_1^\dagger \boldsymbol{\eta} \Psi_2, \quad (8)$$

which is commonly applied in the context of the Klein-Gordon equation [37–39], and the expansion (4) may be written as

$$\Psi = \sum_{i=1}^d \frac{\psi_i^\dagger \boldsymbol{\eta} \Psi}{\psi_i^\dagger \boldsymbol{\eta} \psi_i} \psi_i, \quad (9)$$

provided that the Hamilton matrix $\mathbf{H}$ has a real spectrum, e. g., in the free-particle case.

It is crucial, however, to realize that the pseudo inner product (8) is problematic, in particular if the spectrum contains complex eigenvalues. The first problem is that states cannot always be normalized to plus one with respect to the inner product (8). Secondly, the denominator in (9) becomes $\psi_i^\dagger \boldsymbol{\eta} \psi_i = 0$ if $\mathcal{E}_i$ is complex, which follows from $\psi_i^\dagger \boldsymbol{\eta} = \varphi_i$ and the orthogonality of φ_j and ψ_i . Therefore, the scalar product should be formed only between left and right eigenvectors as in Eq. (4).

The time evolution of an eigenstate ψ_i of H is

$$\psi_i(t) = e^{-i\mathcal{E}_i t/\hbar} \psi_i. \quad (10)$$

Thus, if $\mathcal{E}_i$ is complex then the components of $\psi_i(t)$ shrink or grow exponentially. However, the scalar product between $\psi_i(t)$ and its dual vector $\varphi_i(t)$ is constant, i. e.,

$$\varphi_i(t)\psi_i(t) = \varphi_i\psi_i = \text{const.} \quad (11)$$

This result follows from (10) and

$$\varphi_i(t) = \psi_j(t)^\dagger \eta = (e^{-i\mathcal{E}_j t/\hbar} \psi_j)^\dagger \eta = e^{i\mathcal{E}_j t/\hbar} \psi_j^\dagger \eta = e^{i\mathcal{E}_i t/\hbar} \varphi_i, \quad (12)$$

where ψ_i and ψ_j denote a pair of pseudo degenerate states. Obviously, $\varphi_i(t)\psi_i(t) = \varphi_i\psi_i$ holds for eigenstates with real energy eigenvalue, too. Furthermore, if the Hamilton matrix H has a real spectrum then each general column vector $\Psi(t)$, which evolves under H , satisfies

$$\Psi(t)^\dagger \eta \Psi(t) = \Psi(0)^\dagger \eta \Psi(0) = \text{const.} \quad (13)$$

III. THE SCHIFF-SNYDER-WEINBERG EFFECT

Back in 1940 Schiff, Snyder, and Weinberg investigated in their pioneering work [31] the unusual properties of the Klein-Gordon equation due to its non-Hermiticity. They determined the bound states of the equation for a deep square well potential for varying potential strengths. For a narrow but not too deep potential the Klein-Gordon equation features only bound states with energy close to but below the positive-continuum threshold at mc^2 . The deeper and therefore more attractive the potential is, the smaller the eigenenergies of these positive-continuum bound states are; see also Fig. 1. At a first critical potential strength $V_{\text{cr},1}$ a further bound state emerges above but close to the negative-continuum threshold at $-mc^2$. We call this state negative-continuum bound state. Contrary to other bound states its energy eigenvalue grows as the potential gets deeper. As the new state appears at the border to the negative continuum one may relate it to an antiparticle. This, however, leads to the paradox situation that the strong potential, which is repulsive for an antiparticle with charge $-q$, features a bound state. Thus, if the potential depth grows beyond $V_{\text{cr},1}$ the energy values move away from the continuum thresholds and approach each other and finally two energy eigenvalues merge at a second critical potential strength $V_{\text{cr},2}$. Thus, the states become degenerate at the point $V_{\text{cr},2}$, which is not possible in one-dimensional quantum systems with finite potentials within the nonrelativistic Schrödinger theory [45]. Note that the degenerate states at $V_{\text{cr},2}$ are linearly independent in contrast to one-dimensional complex PT-symmetric potentials, where linear independence is lost, when eigenvalue curves intersect [46]. Beyond the critical point $V_{\text{cr},2}$ the energy eigenvalues are complex and the states become pseudo degenerate. The real part of the energy eigenvalues decreases further with growing potential depth and enters finally the negative-energy continuum, thus there is a discrete bound state embedded in a continuum of states [47, 48].

In summary, one can distinguish four parameter regimes: In regime I the bound states move down towards the negative

FIG. 1. (Color online) The Schiff-Snyder-Weinberg effect for a one-dimensional smooth box potential $q\phi(x) = V_0/2 \times (\tanh((x+1/2)/w) - \tanh((x-1/2)/w))$ with the box width $l = 2.2 \lambda_C$ and $w = 0.2 \lambda_C$, where λ_C denotes the particle's Compton length. The solid lines show the real part and the imaginary part (if nonzero) of some bound-state energy eigenvalues as a function of the potential strength V_0 , the gray shaded area represents the negative-energy continuum. See main text for further details.

continuum. In regime II an antiparticle bound state is present. The two bound states become pseudo degenerate in regimes III and IV. They have complex energy eigenvalues with real part lying in gap between the positive-energy and negative-energy continua for the regime III, while the real part is in the negative-energy continuum for the regime IV.

This phenomenon of merging of two bound states into a pair of pseudo degenerate states with complex energy values is called the Schiff-Snyder-Weinberg effect. The asymptotic limit of the Schiff-Snyder-Weinberg effect, that is, for infinite walls, and its connection to the well-known Klein paradox has been studied by Fulling [36]. In 1970s, Popov came to the conclusion that the Schiff-Snyder-Weinberg effect is inherent to short-range interactions and should not be expected for long-range potentials [34]. Schroer and Swieca [49] have constructed a formal quantization of a charged Klein-Gordon field with strong stationary external interactions including the complex energy modes. An application of this quantization to a free scalar field with a tachyonic mass, i. e., $m^2 < 0$, was given by Schroer [50]. In all these studies, physicists argued that after the merging of the two bound states, the eigenstates with complex energy have zero norm, that there is no Fock-like representation and postulated a breakdown of the vacuum and a breakdown of the particle interpretation of quantum field theory. As we have shown in the previous section, however, zero norm does not occur if the scalar product of the correct left and right eigenvectors is utilized. Thus, there is no apparent reason to postulate a breakdown of the Klein-Gordon theory in the presence of complex energy values.

IV. PAIR CREATION PROCESS UNDER THE PSEUDO DEGENERACY

As discussed in Sec. III, pseudo degenerate bound states appear in the parameter regime III of strong localized potentials. This is a special effect in bosonic system as it can never be found in fermionic systems. This section will discuss bosonic pair creation via an external scalar potential within a quantum field theoretical framework. In particular, we will investigate the role of pseudo degenerate states and the occurrence complex energy levels in this process.

The quantum field operator of a many-particle system can be expressed as an integral or a sum (in case of a discretized Hamiltonian) over the simultaneous eigenstates of the free-particle Klein-Gordon Hamiltonian and the momentum operator:

$$\hat{\Psi}(\mathbf{r}, t) = \sum_{\mathbf{p}} \hat{b}_{\mathbf{p}}^+(t) \varphi_{\mathbf{p}}^+(\mathbf{r}) + \sum_{\mathbf{p}} \hat{b}_{\mathbf{p}}^{+\dagger}(t) \varphi_{\mathbf{p}}^-(\mathbf{r}). \quad (14)$$

Here, $\varphi_{\mathbf{p}}^+(\mathbf{r})$ denotes a free-particle state with positive energy and momentum eigenvalue $\mathbf{p}$ and correspondingly $\varphi_{\mathbf{p}}^-(\mathbf{r})$ denotes a free-particle state with negative energy. The operators $\hat{b}_{\mathbf{p}}^+(t)$ and $\hat{b}_{\mathbf{p}}^{+\dagger}(t)$ denote the annihilation operators for the particle and antiparticle, respectively, with momentum $\mathbf{p}$. Together with the respective creation operators $\hat{b}_{\mathbf{p}}^{+\dagger}(t)$ and $\hat{b}_{\mathbf{p}}^+(t)$ they satisfy the commutator relations of bosonic annihilation and creation operators

$$[\hat{b}_{\mathbf{p}}^{\pm}(t), \hat{b}_{\mathbf{p}'}^{\pm\dagger}(t)] = \delta_{\mathbf{p}, \mathbf{p}'}, \quad (15)$$

where $\delta_{\mathbf{p}, \mathbf{p}'}$ denotes a Kronecker delta. The particle and antiparticle annihilation operators $\hat{b}_{\mathbf{p}}^+(t)$ and $\hat{b}_{\mathbf{p}}^{+\dagger}(t)$ both satisfy the Heisenberg equation

$$i\hbar \frac{\partial \hat{b}_{\mathbf{p}}^{\pm}(t)}{\partial t} = [\hat{b}_{\mathbf{p}}^{\pm}(t), \hat{H}], \quad (16)$$

where $\hat{H}$ denotes the quantum field theoretical Hamiltonian of a particle coupled to an external field. This can be expressed in terms of the first-quantization Klein-Gordon Hamiltonian $\hat{H}_{\text{KG}}$

$$\hat{H} = \int \hat{\Psi}^{\dagger}(\mathbf{r}, t) \sigma_3 \hat{H}_{\text{KG}} \hat{\Psi}(\mathbf{r}, t) d^3r. \quad (17)$$

It should be noted that, while this Hamiltonian fully accounts for the interaction of the particle due to the external field through the minimum coupling principle, which is implemented in $\hat{H}_{\text{KG}}$, it neglects all internal forces between particles and antiparticles.

The field operator $\hat{\Psi}(\mathbf{r}, t)$ defined in Eq. (14) satisfies the Schrödinger-like equation

$$i\hbar \frac{\partial \hat{\Psi}(\mathbf{r}, t)}{\partial t} = \hat{H}_{\text{KG}} \hat{\Psi}(\mathbf{r}, t). \quad (18)$$

Consequently, the time-dependent field operator (14) may be equivalently expressed as

$$\hat{\Psi}(\mathbf{r}, t) = \sum_{\mathbf{p}} \hat{b}_{\mathbf{p}}^+ \varphi_{\mathbf{p}}^+(\mathbf{r}, t) + \sum_{\mathbf{p}} \hat{b}_{\mathbf{p}}^{+\dagger} \varphi_{\mathbf{p}}^-(\mathbf{r}, t), \quad (19)$$

where $\hat{b}_{\mathbf{p}}^+ = \hat{b}_{\mathbf{p}}^+(0)$ and $\hat{b}_{\mathbf{p}}^{+\dagger} = \hat{b}_{\mathbf{p}}^{+\dagger}(0)$, and the functions $\varphi_{\mathbf{p}}^+(\mathbf{r}, t)$ and $\varphi_{\mathbf{p}}^-(\mathbf{r}, t)$ denote the solutions of the time-dependent Klein-Gordon equation (1) with $\varphi_{\mathbf{p}}^+(\mathbf{r})$ and $\varphi_{\mathbf{p}}^-(\mathbf{r})$, respectively, as

initial conditions at time $t = 0$. By equating (14) and (19) we find

$$\hat{b}_{\mathbf{p}}^+(t) = \sum_{\mathbf{p}'} \hat{b}_{\mathbf{p}'}^+ \langle \varphi_{\mathbf{p}}^+(\mathbf{r}) | \varphi_{\mathbf{p}'}^+(\mathbf{r}, t) \rangle + \hat{b}_{\mathbf{p}'}^{+\dagger} \langle \varphi_{\mathbf{p}}^+(\mathbf{r}) | \varphi_{\mathbf{p}'}^-(\mathbf{r}, t) \rangle \quad (20)$$

and

$$\hat{b}_{\mathbf{p}}^{+\dagger}(t) = \sum_{\mathbf{p}'} \hat{b}_{\mathbf{p}'}^{+\dagger} \langle \varphi_{\mathbf{p}}^-(\mathbf{r}) | \varphi_{\mathbf{p}'}^-(\mathbf{r}, t) \rangle + \hat{b}_{\mathbf{p}'}^+ \langle \varphi_{\mathbf{p}}^-(\mathbf{r}) | \varphi_{\mathbf{p}'}^+(\mathbf{r}, t) \rangle, \quad (21)$$

where we have employed the pseudo scalar product

$$\langle \varphi_{\mathbf{p}}^{\pm}(\mathbf{r}) | \Psi(\mathbf{r}) \rangle = \int \varphi_{\mathbf{p}}^{\pm\dagger}(\mathbf{r}) \sigma_3 \Psi(\mathbf{r}) d^3r. \quad (22)$$

We can calculate the particles' spatial density $\varrho(\mathbf{r}, t)$ from the particle portion of the particle-antiparticle field operator, which is defined as

$$\hat{\Psi}^+(\mathbf{r}, t) = \sum_{\mathbf{p}} \hat{b}_{\mathbf{p}}^+(t) \varphi_{\mathbf{p}}^+(\mathbf{r}), \quad (23)$$

via the expectation value of the corresponding density operator with respect to the vacuum state $|0\rangle$, i. e.,

$$\varrho(\mathbf{r}, t) = \langle 0 | \hat{\Psi}^{+\dagger}(\mathbf{r}, t) \hat{\Psi}^+(\mathbf{r}, t) | 0 \rangle. \quad (24)$$

After some operator algebra and employing (15) we find that $\varrho(\mathbf{r}, t)$ can be expressed in terms of the time-dependent solutions of the single-particle Klein-Gordon equation (1), viz.

$$\varrho(\mathbf{r}, t) = \sum_{\mathbf{p}'} \left| \sum_{\mathbf{p}} \langle \varphi_{\mathbf{p}}^+(\mathbf{r}) | \varphi_{\mathbf{p}'}^-(\mathbf{r}, t) \rangle \varphi_{\mathbf{p}}^+(\mathbf{r}) \right|^2. \quad (25)$$

By integrating (25) over the whole space, one obtains the total number of the created pairs as

$$N(t) = \int \varrho(\mathbf{r}, t) d^3r = \sum_{\mathbf{p}'} \sum_{\mathbf{p}} \left| \langle \varphi_{\mathbf{p}}^+(\mathbf{r}) | \varphi_{\mathbf{p}'}^-(\mathbf{r}, t) \rangle \right|^2. \quad (26)$$

These expressions permit us to study the details of the pair creation process for various parameters (such as the potential height and width) by investigating the total number of created particles $N(t)$ and the spatial and the momentum probability distributions of the created pairs. For an alternative approach based on in- and out-states see, e. g., Refs. [51, 52]. While this approach leads to the same result as the asymptotic in- and out-state-based S-matrix formalism after the external field is turned off, our approach permits us to follow the dynamics with space-time resolution [53].

Figure 1 indicates that the one-dimensional Klein-Gordon equation may feature complex eigenvalues depending on the parameters and one may distinguish four parameter regimes with different characteristics regarding the energy eigenvalues. The time-dependent process of pair-creation from the initial vacuum and how the Schiff-Snyder-Weinberg effect affects the pair-creation dynamics may be studied by utilizing the potential energy

$$q\phi(x, t) = \begin{cases} 0 & \text{for } t \leq 0, \\ \frac{V_0}{2} \left(\tanh \frac{x + l/2}{w} - \tanh \frac{x - l/2}{w} \right) & \text{else.} \end{cases} \quad (27)$$

FIG. 2. The mean value of the number of created particles $N(t)$ as a function of time t for different parameter regimes as indicated in Fig. 1 with the same potential and $V_0 = -2.17 mc^2$, $V_0 = -2.195 mc^2$, $V_0 = -2.22 mc^2$, and $V_0 = -2.25 mc^2$ for the regimes I–IV. In the parameter regimes I and II, where all eigenenergies are real-valued, the number of created particles remains bounded. In the parameter regimes III and IV, however, the number of created particles grows exponentially and the growth rate is given by twice the absolute value of the imaginary part of the pseudo degenerate complex-valued eigenenergies, which appear in these regimes. This exponential growth law is indicated by the solid gray lines. Atomic units (indicated as a. u.) are employed.

In Fig. 2 the mean value of the number of created particles $N(t)$ is presented as a function of time t for a parameter set from each of the four parameter regimes.

Parameter regimes I and II show qualitatively a completely different behavior compared to regimes III and IV, where pseudo degeneracy is present. In the parameter regimes I and II the number of created particles initially grows and then oscillates, i. e., it remains bounded, whereas the number of created particles grows exponentially in the parameter regimes III and IV. The oscillation of the particle number in parameter regimes I and II is caused by the sudden turn-on of the external field. The oscillation frequency times $\hbar$ equals approximately the energy difference between the lowest positive-continuum bound state and the negative continuum (regime I) or the lowest positive-continuum bound state and the negative-continuum bound state (regime II). Thus, the system behaves like an effective two-level system.

The lowest positive-continuum bound state and the negative-continuum bound state are degenerate with respect to the real part of their energies in the parameter regimes III and IV. In this way, the positive-energy continuum gets coupled to the negative-energy continuum, which induces pair creation and an exponential growth of the mean value of the number of created particles as a function of time. To calculate the number of created particles we write (26) as

$$\begin{aligned} N(t) &= \sum_{p'} \sum_p \left| \langle \varphi_p^+(\mathbf{r}) | \varphi_{p'}^-(\mathbf{r}, t) \rangle \right|^2 \\ &= \sum_{p'} \sum_p \left| \langle \varphi_p^+(\mathbf{r}) | \exp(-i\hat{H}_{\text{KG}}t/\hbar) | \varphi_{p'}^-(\mathbf{r}) \rangle \right|^2 \\ &= \sum_{p'} \sum_p \left| \langle \varphi_p^+(\mathbf{r}) | \exp(-i\hat{H}_{\text{KG}}t/\hbar) \sum_{\mathcal{E}} |\mathcal{E}\rangle \langle \mathcal{E}| \varphi_{p'}^-(\mathbf{r}) \rangle \right|^2 \end{aligned}$$

$$= \sum_{p'} \sum_p \left| \langle \varphi_p^+(\mathbf{r}) | \sum_{\mathcal{E}} \exp(-i\mathcal{E}t/\hbar) |\mathcal{E}\rangle \langle \mathcal{E}| \varphi_{p'}^-(\mathbf{r}) \rangle \right|^2, \quad (28)$$

where $|\mathcal{E}\rangle$ and $\langle \mathcal{E}|$ denote all the left and right eigenstates of the Klein-Gordon Hamiltonian with energy $\mathcal{E}$. Supposing that the pseudo degenerate states with energies $\mathcal{E}$ and $\mathcal{E}^*$ (where without loss of generality $\text{Im } \mathcal{E} > 0$) mainly contribute to the pair creation as the continua are far away from each other we can approximate (28) by

$$\begin{aligned} N(t) &\approx \sum_{p'} \sum_p \left| \langle \varphi_p^+(\mathbf{r}) | \left(\exp(-i\mathcal{E}t/\hbar) |\mathcal{E}\rangle \langle \mathcal{E}| \right. \right. \\ &\quad \left. \left. + \exp(-i\mathcal{E}^*t/\hbar) |\mathcal{E}^*\rangle \langle \mathcal{E}^*| \right) \varphi_{p'}^-(\mathbf{r}) \right|^2 \\ &\approx \exp(2 \text{Im } \mathcal{E} t) \sum_{p'} \sum_p \left| \langle \varphi_p^+(\mathbf{r}) | \mathcal{E} \rangle \langle \mathcal{E}| \varphi_{p'}^-(\mathbf{r}) \rangle \right|^2. \quad (29) \end{aligned}$$

Thus, the growth rate is given by twice the absolute value of the imaginary part of the pseudo degenerate eigenenergies, which agrees very well with our numerical calculations as indicated Fig. 2. This self-amplified creation process in bosonic systems can be understood as the bosons obey the so-called anti-Pauli blocking principle. As long as the created particles are localized in the interaction regime, the sequential creation can be amplified and the particle number shows exponential increase in time.

V. GENERALIZED SCHIFF-SYNDER-WEINBERG EFFECT AND ITS INFLUENCE ON PAIR CREATION

The addition of a magnetic field to an electric field can lead to new phenomena in the Schiff-Snyder-Weinberg effect. In the following, we will consider pair creation at the step potential

$$q\phi(x, t) = \frac{V_0}{2} \left(\tanh \frac{x}{w_E} + 1 \right), \quad (30)$$

which corresponds to a strong localized electric field. The electric field of the potential (30) is not able to support bound states and, therefore, the Schiff-Snyder-Weinberg effect cannot occur. It may, however, feature bound states if a sufficiently strong localized magnetic field is superimposed perpendicularly to the localized electric field [54]. In the magnetic field the charged particle can undergo bound cyclotron motion. Such a magnetic field may be given by the vector potential

$$\mathbf{A}(x, t) = \begin{pmatrix} 0 \\ A_y(x, t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{A_0}{2} \left(\tanh \frac{x}{w_B} + 1 \right) \\ 0 \end{pmatrix}. \quad (31)$$

As the scalar potential (30) and the vector potential (31) depend on the x coordinate only, the system is quasi one-dimensional and the canonical momenta in the y and z directions are conserved. Consequently, the three-dimensional system can be simplified to a set of non-coupled one-dimensional systems with different canonical momenta p_y and p_z . In the following,

FIG. 3. (Color online) Spectrum of the one-dimensional Klein-Gordon Hamiltonian (32) with $p_y = qA_0/2$ and $p_z = 0$, $A_0 = 1.2 w_B mc / (\lambda_C q)$, $w_E = 0.3 \lambda_C$, and $w_B = 2.2 \lambda_C$, where λ_C denotes the Compton wave length. The solid lines show the real part and the imaginary part (if nonzero) of some bound-state energy eigenvalues as a function of the potential strengths V_0 , the shaded areas represent the negative-energy and the positive-energy continua. States with complex energy eigenvalues are marked in blue.

we choose $p_y = qA_0/2$ and $p_z = 0$, which lets the kinematic momentum components along the y and z direction vanish at $x \approx 0$. These particular parameters are motivated by the fact that particles with low kinetic energy are favored in pair creation in strong electromagnetic fields.

Figure 3 shows the spectrum of the Hamiltonian of the one-dimensional Klein-Gordon equation

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = \left(\frac{\sigma_3 + i\sigma_2}{2m} \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + p_y^2 + p_z^2 \right) - 2qp_y A_y(x, t) + q^2 A_y(x, t)^2 \right) + q\phi(x, t) + \sigma_3 mc^2 \Psi(x, t). \quad (32)$$

Its spectrum features a lower continuum, an upper continuum of states, and energetically isolated bound states in between. The lower limit of the upper continuum is given by

$$\mathcal{E} = \sqrt{c^2(p_y^2 + p_z^2) + m^2 c^4}, \quad (33)$$

while the upper limit of the lower continuum is given by

$$\mathcal{E} = -\sqrt{c^2((p_y - qA_0)^2 + p_z^2) + m^2 c^4} + V_0. \quad (34)$$

If the scalar potential is strong enough, the two continua intersect, which happens at

$$V_0 = \sqrt{c^2(p_y^2 + p_z^2) + m^2 c^4} + \sqrt{c^2((p_y - qA_0)^2 + p_z^2) + m^2 c^4}. \quad (35)$$

Analog to the electric-field-only Schiff-Snyder-Weinberg effect, we find for not too strong electric potentials bound states with real-valued energy, indicated as regime I in Fig. 3. At some critical potential strength these states become degenerate and beyond this critical point the states are pseudo degenerate with complex conjugated energy values (regime II). In contrast to the standard Schiff-Snyder-Weinberg effect the pseudo degenerate states do not dive into the lower continuum if the strength of the scalar potential is further increased. Instead the bound states become degenerate again at some further critical point and the energies become real and separate again (regime III). The increase of the potential strength also induces two new bound states between the energy continua and finally triggers a second occurrence of the Schiff-Snyder-Weinberg effect. This means, beyond some critical point two pairs of states become simultaneously pseudo degenerate (regime IV). Remarkably, the imaginary parts of all four states have the same absolute value. These states finally dive into the upper and lower continua. At the potential strength given by (35) the two continua overlap. In this regime, indicated as regime V in Fig. 3, a whole continuum of states with complex energy exists.

The observed coalescing followed by anticoalescing of two levels when increasing the potential strength does not occur in the standard Schiff-Snyder-Weinberg effect as described in Sec. III. We call this new phenomenon therefore generalized Schiff-Snyder-Weinberg effect. Note that the Schiff-Snyder-Weinberg effect has been studied recently for a one-dimensional system including both a scalar as well as a vector potential, which corresponds to a vanishing magnetic field in this case [55]. For this system the generalized Schiff-Snyder-Weinberg effect could not be observed. Thus, one might argue that the presence of a magnetic field is pivotal for the generalized Schiff-Snyder-Weinberg effect.

It is instructive to analyze the wave functions of the bound states of the system in more detail. For this purpose the left and right eigenstates $\varphi(x)$ and $\psi(x)$ (of the discretized version) of the Hamilton operator in Eq. (32) are calculated, where the states are two-component functions. More precisely, $\varphi(x)$ and $\psi(x)$ assign to each space point a two-component complex row vector and a two-component complex column vector, respectively. This allows us to define the density

$$\rho(x) = \varphi(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \psi(x) \quad (36)$$

and the eigenstates shall be normalized such that

$$\int \rho(x) dx = 1. \quad (37)$$

Note that despite this normalization the density (36) may be negative or even complex. Figure 4 shows the real and imaginary parts of two bound states' densities of the Hamilton operator in Eq. (32) for parameters in the regimes I, II, and IV of Fig. 3. In the regime I, that is where the eigenenergies are real-valued, the densities are real-valued too. The bound state that is associated with the upper continuum is located left from the scalar potential step (solid black line in Fig. 4), while the bound state that is associated with the lower continuum is located right from the scalar potential step (dashed black line in Fig. 4). This changes

FIG. 4. (Color online) Real and imaginary parts of two bound states' densities of the Hamilton operator in Eq. (32) denoted by $\rho_1(x)$ and $\rho_2(x)$ for $V_0 = 2.6 mc^2$ (black), $V_0 = 2.9 mc^2$ (blue), and $V_0 = 3.2 mc^2$ (red). Other parameters as for Fig. 3. For $V_0 = 3.2 mc^2$ there are two pairs of pseudo degenerate bound states. Only the densities of the pair with larger real part of the energy eigenvalue are shown. The densities of the other pair of states can be obtained by mirroring the shown densities at $x = 0$.

if some energy values become complex as in regime II. In this case it is no longer possible to associate the bound states with one of the two energy continua. The densities are complex-valued and the bound states are each located on both sides of the scalar potential step (solid and dashed blue lines in Fig. 4). They occupy the spatial region approximately from $-w_B$ to w_B . In this way these states bridge states related to the upper continuum, which are located on the left to the potential step at $x = 0$, and states related to the lower continuum, which are located on the right to the potential step at $x = 0$. In regime III the energy eigenvalues become real again. In total there are four bound states in regime III. Two of them emerged from the upper and the lower continuum, respectively, and can be interpreted as particle and antiparticle states. The other two states, which lie energetically between them, emerged from the pseudo degenerate pair of states in the regime II and therefore one cannot associate them unambiguously to particle or antiparticle states. As a consequence of the generalized Schiff-Snyder-Weinberg effect two pairs of pseudo degenerate states appear in regime IV. The densities of one of these pairs are indicated in Fig. 4 by red lines. Similarly to regime II the densities are complex-valued but they are no longer symmetric around the potential step at $x = 0$. The densities of the second pair of states (not shown in Fig. 4) can be obtained by mirroring the densities of the first pair at $x = 0$.

The generalized Schiff-Snyder-Weinberg effect controls the pair creation process as illustrated in Fig. 5. Depending on the system parameters and the resulting energy eigenvalues the number of created articles may grow exponentially or oscillate in time. In the regimes I and III all eigenenergies are real-valued and consequently the number of created particles grows only

FIG. 5. The mean value of the number of created particles $N(t)$ as a function of time t for different parameter regimes as indicated in Fig. 3 with the same potential and $V_0 = 2.6 mc^2$, $V_0 = 2.9 mc^2$, $V_0 = 3.07 mc^2$, $V_0 = 3.2 mc^2$, and $V_0 = 3.4 mc^2$ for the regimes I–V. In the parameter regimes I and III, where all eigenenergies are real valued, the number of created particles remains bounded. In the parameter regimes II, IV, and V, however, the number of created particles grows exponentially and the growth rate is given by twice the absolute value of the imaginary part of the pseudo degenerate complex valued eigenenergies, which appear in these regimes.

initially due to the sudden turn-on of the potentials but then evolves into an oscillatory behavior with a frequency, which is approximately given by the energy difference of the two bound states divided by $\hbar$. This corresponds to regime II of the system as described in Sec. IV. In regime II of Fig. 3, a single pseudo degenerate pair of bound states is present and the mean number of created particles grows exponentially with a rate that is approximately given by twice the absolute value of the imaginary part of the energy eigenvalues. As a consequence of the generalized Schiff-Snyder-Weinberg effect two pseudo degenerate pairs of bound states are present in the regimes IV and V. Also in these regimes, the mean number of created particles grows exponentially with a rate that equals twice the absolute value of the imaginary part of the energy eigenvalues. Note that the growth rate can be larger in regime II than in regime IV although the scalar potential is always larger in regime IV, as it can be inferred from the imaginary part of energy eigenvalues shown in Fig. 3. In addition to the exponential growth of the particle number there is a small oscillation superimposed to the exponential growth law. The oscillation frequency times $\hbar$ equals the difference of the real parts of the energies of the two pairs of pseudo degenerate bound states. Also here, the system behaves like an effective two-level system. This holds true even in regime V, where the two continua of the energy spectrum overlap and the system has a continuum of pseudo degenerate states. These scattering states, however, are not localized around the potential step at $x = 0$ and cause a linear growth only of the mean number of created particles, which is covered by the exponential growth law [56].

VI. BACK REACTION FROM THE CREATED PARTICLES

In the previous sections pair creation was studied in a model, where the electromagnetic field which triggers the pair-creation dynamics, is incorporated via given potentials. Thus, the electromagnetic field is not a dynamical variable because a possible back reaction of the created particles on the external field has been neglected. Such a back reaction may eventually stop the seemingly unlimited growth of the number of created bosons. In order to account for how the created particles modify the external field, we introduce a purely phenomenological model here by considering the energy transfer between the particles and the field.

The energy of external electric and magnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ is with the permittivity of the vacuum ϵ_0 given by

$$\mathcal{E} = \frac{\epsilon_0}{2} \int (\mathbf{E}(\mathbf{r})^2 + c^2 \mathbf{B}(\mathbf{r})^2) d^3r. \quad (38)$$

The created pairs have the rest mass energy $2mc^2N(t)$, where $N(t)$ is the number of pairs. The created antiparticles are able to escape from the potential, the created particles, however, remain in the interaction zone. As these particles carry charges, they induce an additional electric field $\mathbf{E}_{\text{in}}(\mathbf{r}, t)$, which reduces the total electric field and can be obtained by solving the equation

$$\nabla \mathbf{E}_{\text{in}}(\mathbf{r}, t) = 4\pi \varrho(\mathbf{r}, t). \quad (39)$$

Here $\varrho(\mathbf{r}, t)$ denotes the spatial distribution of the created bosons (25). Therefore, the total energy needed to create the particles is

$$\mathcal{E}_{\text{in}}(t) = 2mc^2N(t) + \frac{\epsilon_0}{2} \int \mathbf{E}_{\text{in}}(\mathbf{r}, t)^2 d^3r. \quad (40)$$

The kinetic energy and the magnetic field, which is triggered by the created bosons, are neglected as the particles are created at rest. Thus, the energy that is left in the external field after the pair creation is

$$\mathcal{E}_{\text{ex}}(t) = \mathcal{E}_0 - \mathcal{E}_{\text{in}}(t), \quad (41)$$

where $\mathcal{E}_0$ denotes the energy of the initial field configuration.

Incorporating this change of the energy of the external field affects significantly the pair creation process as show in Fig. 6. Here we have fixed the spatial distribution of the external field and the change of the energy is only reflected by a time-dependent potential strength $V_0(t)$, which is dynamically adjusted such that the energy of the scalar potential equals (41). There is no sustained pair creation if back reaction is included in our model. Initially, the number of particles grows but it turns soon into an oscillatory behavior. Due to the created particles and back reaction, the potential can become subcritical. In this case, the anti-bosons do not have enough kinetic energy to escape from the interaction region and will therefore annihilate with bosons and, consequently, the external field will become supercritical again and create another pair. This process lasts forever as the external field oscillates between supercritical and subcritical.

As an alternative to our purely phenomenological approach, the inclusion of the back reaction could be implemented on a

FIG. 6. The created particle number $N(t)$ and the depth of the potential $V_0(t)$ with and without back reaction as functions of time t . The parameters used here are the same as in regime IV of Fig. 2

more fundamental level, which would extend the theoretical description significantly. In the case of the related fermionic pair creation, some progress in this direction has been reported recently [12, 57, 58], where the back reaction of the created electron-positron pairs on the electromagnetic field has been taken into account by coupling the Dirac equation with the Maxwell equation under the assumption that the force field is classical. However, such an approach lies beyond the scope of this article and, furthermore, it is presently not clear how this procedure could be applied to bosonic systems.

VII. CONCLUSIONS

In this paper, we studied the role of pseudo degenerate bound states on pair creation in bosonic systems. The coalescing of particle and antiparticle bound states when increasing the potential strength is known as the Schiff-Snyder-Weinberg effect. These coalesced states have complex energy eigenvalues due to the pseudo hermiticity of the Klein-Gordon Hamiltonian. By employing the biorthogonal left and right eigenvectors of the Hamiltonian, these pseudo degenerate states can be used as basis sets to quantize the bosonic matter field. The problem that states with complex energy have zero norm does not coocur when one discriminates thoroughly between left and right eigenvectors. When distinct bound particle and antiparticle states merge into pseudo degenerate states, they trigger pair creation from the vacuum. The mean particle number obeys an exponential growth law in time. The characteristic parameter of this exponential behavior equals to twice the absolute value of the imaginary part of the energy of the pseudo degenerate states.

In addition to the simple scalar potential well, we also studied a field configuration that consists of a superposition of an electric and a magnetic field. Our findings are contrary to the common belief that the Schiff-Snyder-Weinberg effect can only occur for short range potentials. The scalar and the vector potential used here are both long-range, nevertheless

the coalescing of bound states is found here, too. The magnetic field given by the vector potential is essential for this generalized Schiff-Snyder-Weinberg effect. The generalized Schiff-Snyder-Weinberg effect induces pair creation from the vacuum, which leads to an exponential increase of the particle number in time. The exponential creation rate is related to the imaginary part of the pseudo degenerate energies as for the standard Schiff-Snyder-Weinberg effect.

Exceeding some critical potential strength, not only the bound states are degenerate with respect to another bound state or states in the continuum but also the lower and upper continuum bands become degenerate to each other as the potentials are long-range. The resulting continuum-continuum overlap induces a different mechanism for pair creation that (by itself) would lead to a permanent linear growth of the particle yield. As a result of both mechanisms, this continuum-continuum transition plays a minor role for the overall long-time behavior, which shows exponential growth.

The characteristic of the generalized Schiff-Snyder-Weinberg effect is that pseudo degenerate states with complex energy emerge and dissolve again a function of some potential parameter. Consequently, a remarkable feature of the generalized Schiff-Snyder-Weinberg effect is that the growth rate of the number of created particles is not a monotonous function of the potential strength. This means, a strong field may create fewer particles than some weaker field.

Contrary, a Schiff-Snyder-Weinberg-like effect has never been found in the fermionic systems and the Dirac Hamiltonian usually features the avoided crossing mechanism, when two states approach each other. The comparison of the pseudo degeneracy and the avoided crossing may give us some insights about the physical meaning of the spatial density of the states. More systematic studies of the pseudo degeneracy in the frame-

work of quantum field theory are needed.

Pair creation via strong static fields has not been experimentally verified for either fermions or bosons. While we expect that due to their smaller rest masses electron-positron pairs might be observed in the near future, the lowest massive charged bosons (such as π mesons) are heavier and require therefore even stronger external fields for their production. However, the theoretical study of bosonic systems in extreme parameter regimes allows us to explore the limits of quantum theories. To stress this we would like to finish this article by drawing the reader's attention to the following fundamental issue. The Hamiltonian of the Dirac equation for an electron in a Coulomb potential loses its mathematical property of being self-adjoint [59] if the nuclear charge exceeds 137. Therefore, it is often conjectured that in this particular regime this theoretical framework has reached the principal limits of its applicability and a collapse of the vacuum is postulated [38, chapter 9]. As we have discussed in the present work there is also a parameter regime where a discrete subset of the spectrum of the Klein-Gordon Hamiltonian becomes complex. We should point out that it is possible that the Klein-Gordon theory becomes physically meaningless in this regime, similar to the conjectures regarding the Dirac equation, or requires a modified interpretation. However, a definite answer to this extremely fundamental question cannot be obtained within any theoretical framework alone and would clearly require experimental data.

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- [1] Ralf Schützhold, Holger Gies, and Gerald Dunne, “Dynamically assisted Schwinger mechanism,” *Phys. Rev. Lett.* **101**, 130404 (2008).
 - [2] Matthias Ruf, Guido R. Mocken, Carsten Müller, Karen Z. Hatsagortsyan, and Christoph H. Keitel, “Pair production in laser fields oscillating in space and time,” *Phys. Rev. Lett.* **102**, 080402 (2009).
 - [3] S. S. Bulanov, V. D. Mur, N. B. Narozhny, J. Nees, and V. S. Popov, “Multiple colliding electromagnetic pulses: A way to lower the threshold of e^+e^- pair production from vacuum,” *Phys. Rev. Lett.* **104**, 220404 (2010).
 - [4] T. Cheng, Q. Su, and R. Grobe, “Creation of multiple electron-positron pairs in arbitrary fields,” *Phys. Rev. A* **80**, 013410 (2009).
 - [5] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, “Extremely high-intensity laser interactions with fundamental quantum systems,” *Rev. Mod. Phys.* **84**, 1177–1228 (2012).
 - [6] Christian Kohlfürst, Holger Gies, and Reinhard Alkofer, “Effective mass signatures in multiphoton pair production,” *Phys. Rev. Lett.* **112**, 050402 (2014).
 - [7] O. J. Pike, F. Mackenroth, E. G. Hill, and S. J. Rose, “A photon-photon collider in a vacuum hohlraum,” *Nature Photon.* **4**, 434–436 (2014).
 - [8] Anton Wöllert, Heiko Bauke, and Christoph H. Keitel, “Spin polarized electron-positron pair production via elliptical polarized laser fields,” *Phys. Rev. D* **91**, 125026 (2015).
 - [9] D. L. Burke, R. C. Field, G. Horton-Smith, J. E. Spencer, D. Walz, S. C. Berridge, W. M. Bugg, K. Shmakov, A. W. Weidemann, C. Bula, and et al., “Positron production in multiphoton light-by-light scattering,” *Phys. Rev. Lett.* **79**, 1626–1629 (1997).
 - [10] I. Ahmad, Sam M. Austin, B. B. Back, R. R. Betts, F. P. Calaprice, K. C. Chan, A. Chishti, C. Conner, R. W. Dunford, J. D. Fox, and et al., “Search for monoenergetic positron emission from heavy-ion collisions at Coulomb-barrier energies,” *Phys. Rev. Lett.* **78**, 618–621 (1997).
 - [11] C. Bamber, S. J. Boege, T. Koffas, T. Kotseroglou, A. C. Melissinos, D. D. Meyerhofer, D. A. Reis, W. Ragg, C. Bula, K. T. McDonald, and et al., “Studies of nonlinear QED in collisions of 46.6 GeV electrons with intense laser pulses,” *Phys. Rev. D* **60**, 092004 (1999).
 - [12] V. Kasper, F. Hebenstreit, M. Oberthaler, and J. Berges, “Schwinger pair production with ultracold atoms,” (2015), arXiv:1506.01238.
 - [13] Brian W. J. McNeil and Neil R. Thompson, “X-ray free-electron lasers,” *Nature Photon.* **4**, 814–821 (2010).

- [14] G.A. Mourou, N.J. Fisch, V.M. Malkin, Z. Toroker, E.A. Khazanov, A.M. Sergeev, T. Tajima, and B. Le Garrec, "Exawatt-zettawatt pulse generation and applications," *Optics Communications* **285**, 720–724 (2012).
- [15] Fritz Sauter, "Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs," *Z. Physik* **69**, 742–764 (1931).
- [16] W. Heisenberg and H. Euler, "Folgerungen aus der Diracschen Theorie des Positrons," *Z. Physik* **98**, 714–732 (1936).
- [17] Julian Schwinger, "On gauge invariance and vacuum polarization," *Phys. Rev.* **82**, 664–679 (1951).
- [18] O. Klein, "Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac," *Z. Physik* **53**, 157–165 (1929).
- [19] O. Klein, "Elektrodynamik und Wellenmechanik vom Standpunkt des Korrespondenzprinzips," *Z. Physik* **41**, 407–442 (1927).
- [20] Alex Hansen and Finn Ravndal, "Klein's paradox and its resolution," *Physica Scripta* **23**, 1036–1042 (1981).
- [21] N. Dombey, "Seventy years of the Klein paradox," *Phys. Rep.* **315**, 41–58 (1999).
- [22] Barry R. Holstein, "Strong field pair production," *Am. J. Phys.* **67**, 499 (1999).
- [23] P. Krekora, Q. Su, and R. Grobe, "Klein paradox in spatial and temporal resolution," *Phys. Rev. Lett.* **92**, 040406 (2004).
- [24] Ya B. Zeldovich and Valentin S. Popov, "Electronic structure of superheavy atoms," *Sov. Phys. Usp.* **14**, 673–694 (1972).
- [25] Berndt Müller, "Quantum electrodynamics of strong fields," in *Quantum Electrodynamics of Strong Fields*, NATO Advanced Study Institutes Series Volume 80, edited by W. Greiner (Springer, Heidelberg, 1983) pp. 41–75.
- [26] Berndt Müller, Heinrich Peitz, Johann Rafelski, and Walter Greiner, "Solution of the Dirac equation for strong external fields," *Phys. Rev. Lett.* **28**, 1235–1238 (1972).
- [27] Berndt Müller, Johann Rafelski, and Walter Greiner, "Electron shells in over-critical external fields," *Z. Physik* **257**, 62–77 (1972).
- [28] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Butterworth-Heinemann, Oxford, 1981).
- [29] Y. Liu, Q. Z. Lv, Y. T. Li, R. Grobe, and Q. Su, "Pair creation induced by transitions between electronic and positronic bound states," *Phys. Rev. A* **91**, 052123 (2015).
- [30] H. Snyder and J. Weinberg, "Stationary states of scalar and vector fields," *Phys. Rev.* **57**, 307–314 (1940).
- [31] L. I. Schiff, H. Snyder, and J. Weinberg, "On the existence of stationary states of the mesotron field," *Phys. Rev.* **57**, 315–318 (1940).
- [32] Víctor M. Villalba and Clara Rojas, "Bound states of the Klein-Gordon equation in the presence of short range potentials," *Int. J. Mod. Phys. A* **21**, 313–325 (2006).
- [33] Johann Rafelski, Lewis P. Fulcher, and Abraham Klein, "Fermions and bosons interacting with arbitrarily strong external fields," *Phys. Rep.* **38**, 227–361 (1978).
- [34] V. S. Popov, "Position production in a Coulomb field with $Z > 137$," *Soviet Physics JETP* **32**, 526–535 (1971).
- [35] B. A. Arbuzov and V. E. Rochev, "On the production of boson pairs by a static field," *Theor. Math. Phys.* **12**, 761–764 (1972).
- [36] S. A. Fulling, "Varieties of instability of a boson field in an external potential," *Phys. Rev. D* **14**, 1939–1943 (1976).
- [37] Herman Feshbach and Felix Villars, "Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles," *Rev. Mod. Phys.* **30**, 24–45 (1958).
- [38] W. Greiner, *Relativistic Quantum Mechanics* (Springer, Heidelberg, 1997).
- [39] Armin Wachter, *Relativistic Quantum Mechanics* (Springer, Heidelberg, 2011).
- [40] Ali Mostafazadeh, "Pseudounitary operators and pseudounitary quantum dynamics," *J. Math. Phys.* **45**, 932 (2004).
- [41] Ali Mostafazadeh, "Pseudo-hermitian representation of quantum mechanics," *Int. J. Geom. Methods Mod. Phys.* **07**, 1191–1306 (2010).
- [42] Anna Johnson Pell, "Biorthogonal systems of functions," *Trans. Amer. Math. Soc.* **12**, 135–135 (1911).
- [43] Thomas Curtright and Luca Mezincescu, "Biorthogonal quantum systems," *J. Math. Phys.* **48**, 092106 (2007).
- [44] Dorje C Brody, "Biorthogonal quantum mechanics," *J. Phys. A: Math. Theor.* **47**, 035305 (2013).
- [45] Sayan kar and Rajesh R. Parwani, "Can degenerate bound states occur in one-dimensional quantum mechanics?" *Europhys. Lett.* **80**, 30004 (2007).
- [46] Zafar Ahmed, Dona Ghosh, Joseph Amal Nathan, and Gaurang Parkar, "Accidental crossings of eigenvalues in the one-dimensional complex PT-symmetric Scarf-II potential," *Phys. Lett. A* **379**, 2424–2429 (2015).
- [47] Frank H. Stillinger and David R. Herrick, "Bound states in the continuum," *Phys. Rev. A* **11**, 446–454 (1975).
- [48] Q. Z. Lv, Y. Liu, Y. J. Li, R. Grobe, and Q. Su, "Noncompeting channel approach to pair creation in supercritical fields," *Phys. Rev. Lett.* **111**, 183204 (2013).
- [49] Bert Schroer and Jorge Andre Swieca, "Indefinite metric and stationary external interactions of quantized fields," *Phys. Rev. D* **2**, 2938–2944 (1970).
- [50] Bert Schroer, "Quantization of $m^2 < 0$ field equations," *Phys. Rev. D* **3**, 1764–1770 (1971).
- [51] E. S. Fradkin, Dmitri M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum*, Springer Series in Nuclear and Particle Physics (Springer, Heidelberg, 1991).
- [52] Walter Greiner, Berndt Müller, and Johann Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer, Heidelberg, 1985).
- [53] T. Cheng, Q. Su, and R. Grobe, "Introductory review on quantum field theory with space-time resolution," *Contemp. Phys.* **51**, 315–330 (2010).
- [54] Q. Z. Lv, A. C. Su, M. Jiang, Y. J. Li, R. Grobe, and Q. Su, "Pair creation for bosons in electric and magnetic fields," *Phys. Rev. A* **87**, 023416 (2013).
- [55] Hassan Hassanabadi and Antonio Soares de Castro, "Bound states of spinless particles in a short-range potential," *Zeitschrift für Naturforschung A* **70**, 245–249 (2015).
- [56] Q. Z. Lv, Y. Liu, Y. J. Li, R. Grobe, and Q. Su, "Degeneracies of discrete and continuum states with the Dirac sea in the pair-creation process," *Phys. Rev. A* **90**, 013405 (2014).
- [57] F. Hebenstreit, J. Berges, and D. Gelfand, "Real-time dynamics of string breaking," *Phys. Rev. Lett.* **111**, 201601 (2013).
- [58] A. T. Steinacher, R. E. Wagner, Q. Su, and R. Grobe, "Impact of interfermionic forces on the pair-creation process," *Phys. Rev. A* **89**, 032119 (2014).
- [59] M. Ruf, C. Müller, and R. Grobe, "Numerical signatures of non-self-adjointness in quantum Hamiltonians," *J. Phys. A: Math. Theor.* **44**, 345205 (2011).