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# Scattering in $\mathcal{PT}$ and $\mathcal{RT}$ Symmetric Multimode Waveguides: Generalized Conservation Laws and Spontaneous Symmetry Breaking beyond One Dimension

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We extend the generalized conservation law of light propagating in a one-dimensional  $\mathcal{PT}$ -symmetric system, i.e.,  $|T - 1| = \sqrt{R_L R_R}$  for the transmittance  $T$  and the reflectance  $R_{L,R}$  from the left and right, to a multimode waveguide with either  $\mathcal{PT}$  or  $\mathcal{RT}$  symmetry, in which higher dimensional investigations are necessary. These conservation laws exist not only in a matrix form for the transmission and reflection matrices; they also exist in a scalar form for real-valued quantities by defining generalized transmittance and reflectance. We then discuss, for the first time, how a multimode  $\mathcal{PT}$ -symmetric waveguide can be used to observe spontaneous symmetry breaking of the scattering matrix, which typically requires tuning the non-hermiticity of the system (i.e. the strength of gain and loss). Here the advantage of using a multimode waveguide is the elimination of tuning any system parameters: the transverse mode order  $m$  plays the role of the symmetry breaking parameter, and one observes the symmetry breaking by simply performing scattering experiment in each waveguide channel at a single frequency and fixed strength of gain and loss.

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## I. INTRODUCTION

Parity-Time ( $\mathcal{PT}$ ) symmetric optical systems have attracted growing interest in the past several years [1–21]. These systems are non-hermitian due to the presence of gain and loss, which are delicately balanced to make the refractive index satisfying  $n(x) = n^*(-x)$  with respect to a chosen symmetry plane at  $x = 0$ . The plethora of findings in such systems are tied to the spontaneous symmetry breaking at an exceptional point (EP) [22–30]. This spontaneous symmetry breaking was first suggested in non-hermitian quantum mechanism [31–33] and later found in the evolution of waves in the paraxial regime [1–4], which takes the system from a regime of real energy eigenvalues to complex conjugate pairs of eigenvalues.

Recently it was found that the scattering eigenstates of a  $\mathcal{PT}$ -symmetric system also display a spontaneous symmetry breaking [9], independent of its shape and dimension: the eigenvalues of the scattering ( $S$ ) matrix can remain on the unit circle in the complex plane, conserving optical flux despite the non-hermiticity; the symmetry breaking results in pairs of scattering eigenvalues with inverse moduli [9, 11, 12]. Using the symmetry property

of the  $S$  matrix [9] or equivalently that of the transfer matrix [10], one can derive a generalized conservation law in one dimension (1D) [11], i.e.,

$$|T - 1| = \sqrt{R_L R_R}, \quad (1)$$

in contrast to the usual conservation relation  $T + R_{L(R)} = 1$  when the system is hermitian. Here  $T$  is the transmittance from either the left or right side (they are identical due to optical reciprocity [34–36]) and  $R_{L,R}$  are the reflectance from the two sides, respectively. At an accidental reflection degeneracy where  $R_L = R_R \equiv R$ , the generalized conservation law above indicates two possible scenarios, where either  $T + R = 1$  or  $T - R = 1$ . The former is identical to the hermitian conservation law even though the system is non-hermitian, and the transmittance in the latter is clearly super-unitary.

In this report we first extend the generalized conservation law above in 1D to higher dimensions, i.e., in a multimode waveguide with either  $\mathcal{PT}$  or Rotation-Time ( $\mathcal{RT}$ ) symmetry [13] (see Fig. 1), where  $\mathcal{R}$  is rotation by  $\pi$  about a given axis. Using the matrix representation of  $\mathcal{P}$  and  $\mathcal{R}$  in block form, we show that a similar expression exists for the transmission and reflection matrices, which holds independent of the details of the channels, i.e., whether they are sinusoidal waves in a ridge waveguide or cylindrical waves in an optical fiber. By further defining the generalized transmittance and reflectance

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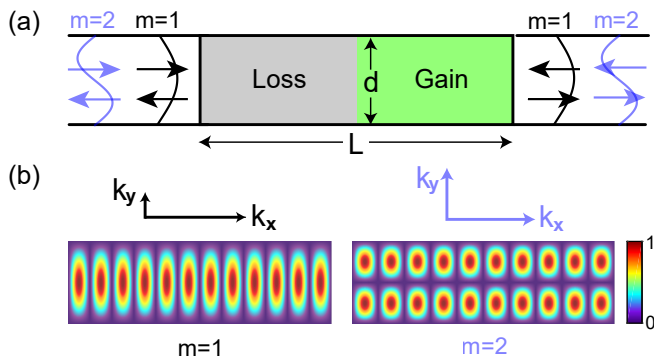


FIG. 1. (Color online) (a) Schematic of a  $\mathcal{PT}$ -symmetric multimode waveguide of length  $L$  and width  $d$ . Incoming and outgoing channels of transverse order  $m = 1, 2$  are illustrated in both the left and right “leads” of the waveguide. (b) Standing-wave intensity patterns with transverse order  $m = 1, 2$  in the leads. The longitudinal and transverse wavevectors corresponding to their right-traveling components are contrasted schematically.

suitable for  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric systems, we also reduce these conservation laws to their scalar forms with only real-valued quantities.

We then discuss how a multimode  $\mathcal{PT}$ -symmetric waveguide can be used to observe spontaneous symmetry breaking of the  $S$  matrix. It was suggested in Ref. [9] that this symmetry breaking can be observed in 1D (i.e., in a single-mode waveguide) by tuning either the gain and loss strength  $\tau$  of the system or the product of the waveguide length  $L$  and the frequency  $\omega$  of the incident light. Such an approach was successfully implemented in phononics, where clear regimes of  $\mathcal{PT}$ -symmetric and  $\mathcal{PT}$ -broken phases were observed [37]. In the optical regime however, this approach was not well received due to the stringent requirement on maintaining  $\mathcal{PT}$  symmetry while tuning across the symmetry breaking point. In our current proposal of using a multimode waveguide, the advantage is the elimination of tuning any system parameters; the transverse mode order  $m = 1, 2, \dots, N$  plays the role of the symmetry breaking parameter, as we will show below. Hence one can observe the symmetry breaking by simply performing scattering experiment in each waveguide channel, at a *single* frequency and *fixed* strength of gain and loss.

## II. GENERALIZED CONSERVATION LAWS

There are two complexities when extending the generalized conservation law given by Eq. (1) to a multimode waveguide. The first one is that Eq. (1) only includes three quantities, which are the reflectance from both left and right sides and the transmittance. For a multimode waveguide with  $N$  channels however, each of these quantities become a  $N \times N$  matrix (unless there is no coupling

between the channels in the system), and the increased degrees of freedom prompt us to look for a conservation law in a *matrix* form first. The second complexity is that the three quantities in Eq. (1) are non-negative real numbers, with the phases of the transmission coefficient  $t$  and reflection coefficients  $r_{L,R}$  eliminated through the definition  $T = |t|^2$  and  $R_{L,R} = |r_{L,R}|^2$ . In a multimode waveguide however, it seems difficult to eliminate the phases of all the transmission and reflection coefficients. Therefore, the generalized conservation law is hence more likely a matrix identity for the complex-valued transmission matrix  $\mathbf{t}$  and reflection matrices  $\mathbf{r}_{L,R}$ , instead of for the real-valued transmittance matrix and reflectance matrices. Nevertheless, we indeed find a *scalar* and *real-valued* form of the generalized conservation law for both  $\mathcal{PT}$ -symmetric and  $\mathcal{RT}$ -symmetric systems, using generalized definitions of transmittance and reflectance as we show below.

By denoting the amplitudes of the  $N$  incoming (outgoing) channels in the left lead by  $\varphi_{\text{in}}^L$  ( $\varphi_{\text{out}}^L$ ) and those in the right lead by  $\varphi_{\text{in}}^R$  ( $\varphi_{\text{out}}^R$ ),  $\mathbf{t}$  and  $\mathbf{r}_{L,R}$  are defined by

$$\varphi_{\text{out}}^L = \mathbf{t}\varphi_{\text{in}}^R, \quad \varphi_{\text{out}}^R = \mathbf{r}_R\varphi_{\text{in}}^R \quad (2)$$

when there are only incident waves in the right lead, and

$$\varphi_{\text{out}}^R = \mathbf{t}^T\varphi_{\text{in}}^L, \quad \varphi_{\text{out}}^L = \mathbf{r}_L\varphi_{\text{in}}^L \quad (3)$$

when there are only incident waves in the left lead. The superscript “ $T$ ” denotes matrix transpose in Eq. (3), and it appears due to optical reciprocity [34–36]: the transmission coefficient from channel  $m$  on the left to channel  $m'$  on the right is the same as the reversed process. This is also the case in 1D, where  $\mathbf{t}$  reduces to a complex number and the matrix transpose can be omitted. Here we do not consider systems that break optical reciprocity, including but not limited to magneto-optical systems. The  $S$  matrix is then defined by

$$\begin{pmatrix} \varphi_{\text{out}}^L \\ \varphi_{\text{out}}^R \end{pmatrix} = \begin{pmatrix} \mathbf{r}_L & \mathbf{t} \\ \mathbf{t}^T & \mathbf{r}_R \end{pmatrix} \begin{pmatrix} \varphi_{\text{in}}^L \\ \varphi_{\text{in}}^R \end{pmatrix} \equiv \mathbf{S} \begin{pmatrix} \varphi_{\text{in}}^L \\ \varphi_{\text{in}}^R \end{pmatrix}, \quad (4)$$

which is a  $2N \times 2N$  matrix. Note that we do not include the channels of evanescent waves, which decays exponentially away from the waveguide in both leads.

The derivation of Eq. (1) in Ref. [11] is based on the symmetry relation of the transfer matrix [10]. Here we use the symmetry relation of the  $S$  matrix instead [9], i.e.,

$$\mathcal{PTSP} = \mathbf{S}^{-1} \quad (5)$$

which is equivalent but more convenient in the case of a  $\mathcal{PT}$ -symmetric multimode waveguide. Here the time-reversal operator  $\mathcal{T}$  is simply complex conjugate and denoted by a superscript “ $*$ ”, and the parity operator  $\mathcal{P}$  along the longitudinal direction can be represented by a matrix permutation  $\mathbf{P}$  satisfying  $\mathbf{P}^2 = \mathbf{1}_{2N}$  [9], where  $\mathbf{1}_{2N}$  is the identity matrix of rank  $2N$ . Therefore, we can rewrite Eq. (5) as

$$\mathbf{PS}^*\mathbf{P} = \mathbf{S}^{-1}. \quad (6)$$

It is important to note that in a multimode waveguide, the incoming/outgoing channel  $m$  on the left side becomes the same incoming/outgoing channel  $m$  on the right side after the parity operation, since the parity operation is only about the longitudinal coordinate and leaves the transverse coordinate(s) unchanged. (This is not the case in a  $\mathcal{RT}$ -symmetric waveguide as we shall see below.) Therefore, the parity operator here does not change the order of the left channels or the right channels, and its matrix representation  $\mathbf{P}$  takes the following block form

$$\mathbf{P} = \begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix}. \quad (7)$$

This form of  $\mathbf{P}$  greatly simplifies Eq. (6), the left hand side of which becomes

$$\mathbf{P}\mathbf{S}^*\mathbf{P} = \begin{pmatrix} \mathbf{r}_R & \mathbf{t}^T \\ \mathbf{t} & \mathbf{r}_L \end{pmatrix}^*. \quad (8)$$

Using matrix inversion in block form, we find the following four relations

$$(\mathbf{r}_R^*)^{-1} = \mathbf{r}_L - \mathbf{t}(\mathbf{r}_R)^{-1}\mathbf{t}^T, \quad (9)$$

$$(\mathbf{r}_L^*)^{-1} = \mathbf{r}_R - \mathbf{t}^T(\mathbf{r}_L)^{-1}\mathbf{t}, \quad (10)$$

$$\mathbf{r}_R\mathbf{t}^* + \mathbf{t}^T\mathbf{r}_R^* = 0, \quad (11)$$

$$\mathbf{r}_L^*\mathbf{t}^T + \mathbf{t}^*\mathbf{r}_L = 0, \quad (12)$$

by equating the four  $N \times N$  blocks on the left and right sides of Eq. (6). Next we combine Eqs. (9),(11) and Eqs. (10),(12) respectively, which lead to

$$\mathbf{t}\mathbf{t}^* - \mathbf{1}_N = -\mathbf{r}_L\mathbf{r}_R^*, \quad (13)$$

$$\mathbf{t}^\dagger\mathbf{t}^T - \mathbf{1}_N = -\mathbf{r}_R^*\mathbf{r}_L, \quad (14)$$

where the superscript “ $\dagger$ ” denotes hermitian conjugate as usual.

Eq. (14) is in fact identical to Eq. (13), once we take its transpose and use the property that  $\mathbf{r}_{L,R}$  are symmetric, which comes from optical reciprocity [34–36], i.e., the scattering coefficient from incoming channel  $m$  to outgoing channel  $m'$  on the same side is the same as that of the reversed process, for both left- and right-side incidence. In addition, we note that using the properties of matrix trace, i.e.,  $Tr(\mathbf{A}^*) = (Tr\mathbf{A})^*$  and  $Tr(\mathbf{A}\mathbf{B}) = Tr(\mathbf{B}\mathbf{A})$  for two arbitrary square matrices  $\mathbf{A}$  and  $\mathbf{B}$ , we find that  $Tr(\mathbf{t}\mathbf{t}^*) \equiv \bar{T}$  is real (see further discussion in the Appendix), which implies that  $Tr(\mathbf{r}_L\mathbf{r}_R^*) \equiv \bar{R}$  is also real and

$$\bar{T} + \bar{R} = N \quad (15)$$

by considering Eq. (13). We will refer to  $\bar{T}$  and  $\bar{R}$  as the generalized transmittance and reflectance in  $\mathcal{PT}$ -symmetric multimode waveguides.

The corresponding relations to Eqs. (13-15) in a hermitian system are given by

$$\mathbf{t}\mathbf{t}^\dagger - \mathbf{1}_N = -\mathbf{r}_L\mathbf{r}_L^\dagger, \quad (16)$$

$$\mathbf{t}^\dagger\mathbf{t} - \mathbf{1}_N = -\mathbf{r}_R^\dagger\mathbf{r}_R, \quad (17)$$

$$T + R_L = T + R_R = N. \quad (18)$$

The first two are derived from  $\mathbf{S}\mathbf{S}^\dagger = 1$ , and their traces lead to Eq. (18), where  $T \equiv Tr(\mathbf{t}\mathbf{t}^\dagger)$  and  $R_{L,R} \equiv Tr(\mathbf{r}_{L,R}\mathbf{r}_{L,R}^\dagger)$  are both real [38].

We will refer to Eqs. (13) and (15) as the generalized conservation laws in a  $\mathcal{PT}$ -symmetric multimode waveguide. Note that they are valid in general and the details of the waveguide channels are not specified in its derivation, which can be, for example, sinusoidal waves in a ridge waveguide or cylindrical waves in an optical fiber. Therefore, the generalized conservation laws given by Eqs. (13) and (15) hold independent of the transverse geometry of the waveguide, as long as the waveguide is  $\mathcal{PT}$ -symmetric.

Eq. (13) reduces to its 1D form given by Eq. (1) once we replace  $\mathbf{t}, \mathbf{r}_{L,R}$  by their scalar form  $t, r_{L,R}$  in 1D and take the absolute value of both its left and right sides. In the special case that the waveguide channels do not couple (which occurs, for example, when the transverse and longitudinal coordinates are separable),  $\mathbf{t}, \mathbf{r}_{L,R}$  become diagonal matrices, and we again recover Eq. (1) for each waveguide channel, which acts as an independent 1D  $\mathcal{PT}$ -symmetric system. To exemplify how Eq. (1) breaks down for the diagonal elements of Eq. (13) in the general case, below we study the scattering of transverse electric (TE) waves in a two-dimensional (2D) waveguide and adopt Dirichlet boundary condition on the sidewalls in the transverse ( $y$ ) direction. Each channel can be labelled by a transverse mode number  $m = 1, 2, \dots, N$ , and the transverse mode profiles are given by  $\psi(y) = \sin(m\pi(y/d + 1/2))$  ( $y \in [-d/2, d/2]$ ). Here  $d$  is the width of the waveguide, and we have chosen the center axis of the waveguide as the origin of the  $y$ -axis and assumed  $n = 1$  in the leads.

Below we will refer to a system with separable longitudinal and transverse coordinates simply as a separable system. We start with such a separable system of  $N = 2$  (i.e., two channels for incoming/outgoing waves in both the left and right leads) as illustrated in Fig. 1(a), which has a uniform refractive index  $n = 3$  and length  $L = 2 \mu\text{m}$  before gain and loss represented by  $\text{Im}[n] = \mp 0.005$  are introduced to its two halves. We then increase the amplitude  $b$  of a stepwise index modulation in the transverse direction,  $\Delta n(y) = b(y < 0); -b(y > 0)$ , which introduces and changes the coupling between the two waveguide channels and the system becomes non-separable. As Fig. 2 shows, the generalized conservation law (13) holds independent of  $b$ , while Eq. (1) for a single channel no longer holds at a non-zero  $b$  and its behavior is correlated with the resonances of the system.

Next we turn to an  $\mathcal{RT}$ -symmetric multimode waveguide. It is straightforward to obtain the symmetry relation of the  $S$  matrix,

$$\mathcal{R}\mathcal{T}\mathcal{S}\mathcal{R}\mathcal{T} = \mathbf{S}^{-1}, \quad (19)$$

and its matrix representation,

$$\mathbf{R}\mathbf{S}^*\mathbf{R} = \mathbf{S}^{-1}, \quad (20)$$

which are similar to Eqs. (5) and (6) in the  $\mathcal{PT}$ -symmetric case. As we have mentioned, the  $\pi$ -rotation operator  $\mathcal{R}$

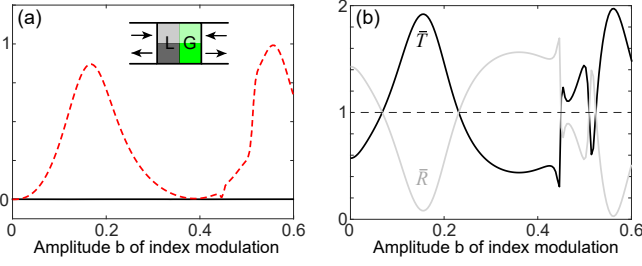


FIG. 2. Verification of generalized conservation laws in a 2D  $\mathcal{PT}$ -symmetric waveguide with two channels in both the left and right leads. (a) Solid line plots the norm of the difference between the two sides of Eq. (13), confirming the 2D conservation relation (13). Dashed line plots the absolute value of the difference of the two sides of Eq. (1) for the  $m = 2$  channel, showing the breakdown of the 1D conservation relation (1). Inset: the length  $L$  and width  $d$  of the waveguide are both  $2 \mu\text{m}$ , and the wavelength is  $1550 \text{ nm}$ . (b) Black and grey lines plot  $\tilde{T}$  and  $\tilde{R}$ . Their mirror symmetry about 1 (dashed horizontal line) verifies the 2D scalar conservation relation (15), where  $N = 2$ .

does not necessarily transform the incoming/outgoing channel  $m$  on the left to the same incoming/outgoing channel  $m$  on the right. Take the 2D waveguide shown in Fig. 1(a) for example, if we choose the channel wave functions in the left lead to be the same as before, i.e.,  $\psi(y) = \sin(m\pi(y/d + 1/2))$ , then they become  $(-)^{m+1} \sin(m\pi(y/d + 1/2))$  in the right leads because  $\mathcal{R}$  transforms  $y \rightarrow -y$  in addition to  $x \rightarrow -x$ . In other words, a negative sign is introduced for the even- $m$  channels (whose mode profiles are odd functions of  $y$ ), and  $\mathbf{R}$  takes the following form

$$\mathbf{R} = \begin{pmatrix} 0 & \tilde{\mathbf{I}}_N \\ \tilde{\mathbf{I}}_N & 0 \end{pmatrix}, \quad \tilde{\mathbf{I}}_N \equiv \begin{pmatrix} 1 & & 0 \\ & -1 & \\ & & \ddots \\ 0 & & & (-1)^{N+1} \end{pmatrix}. \quad (21)$$

We note that  $\mathbf{R}$  is symmetric but not symplectic (it satisfies  $\mathbf{R}^T \mathbf{\Omega} \mathbf{R} = -\mathbf{\Omega}$  instead of  $\mathbf{R}^T \mathbf{\Omega} \mathbf{R} = \mathbf{\Omega}$ , where  $\mathbf{\Omega} = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}$ ). Using the above block form of  $\mathcal{R}$ , we find that the generalized conservation law in an  $\mathcal{RT}$ -symmetric waveguide is

$$\mathbf{t} \tilde{\mathbf{I}}_N \mathbf{t}^* - \tilde{\mathbf{I}}_N = -\mathbf{r}_L \tilde{\mathbf{I}}_N \mathbf{r}_R^*. \quad (22)$$

By taking the trace of both sides of this conservation law and slightly re-arranging the result, we find  $\tilde{T} + \tilde{R} = \delta_{N,2M+1}$ , or equivalently

$$\text{Re}[\tilde{T}] + \text{Re}[\tilde{R}] = \begin{cases} 0, & N \text{ even} \\ 1, & N \text{ odd} \end{cases} \quad (23)$$

$$\text{Im}[\tilde{T}] + \text{Im}[\tilde{R}] = 0. \quad (24)$$

Here  $\tilde{T} \equiv \text{Tr}(\tilde{\mathbf{I}}_N \mathbf{t}^* \mathbf{t})$ ,  $\tilde{R} \equiv \text{Tr}(\tilde{\mathbf{I}}_N \mathbf{r}_R^* \mathbf{r}_L)$  are complex in general, and  $\delta_{N,2M+1}$  is the Kronecker delta where  $M$

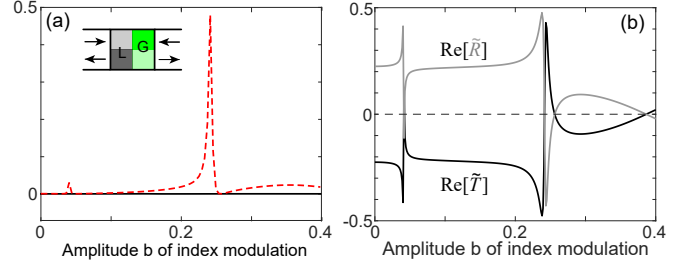


FIG. 3. Verification of generalized conservation laws in a 2D  $\mathcal{RT}$ -symmetric waveguide with two channels in both the left and right leads. (a) Solid line plots the norm of the difference between the two sides of Eq. (22), confirming the 2D conservation relation (22). Dashed line plots the absolute value of the difference of the two sides of Eq. (25) for the  $m = 2$  channel, showing the breakdown of the 1D conservation relation (25). (b) Black and grey lines plot  $\text{Re}[\tilde{T}]$  and  $\text{Re}[\tilde{R}]$ . Their mirror symmetry about 0 (dashed horizontal line) verifies the 2D scalar conservation relation (23), where  $N = 2$ . The parameters are the same as in Fig. 2.

represents any positive integer. If the system has an even number of channels, then these relations imply in particular that  $\tilde{T}$  and  $\tilde{R}$  have the same modulus but are  $\pi$  out of phase. We will refer to  $\tilde{T}$  and  $\tilde{R}$  as the generalized transmittance and reflectance in  $\mathcal{RT}$ -symmetric multimode waveguides.

If the incident light is only in a single channel and it does not couple to other channels, Eq. (22) then reduces to its 1D form

$$|T - 1| = \sqrt{R_L R_R}, \quad (25)$$

which is identical to that of the  $\mathcal{PT}$ -symmetric case with  $T \equiv |t_{mm}|^2$  and  $R_{R,L} \equiv |(t_{R,L})_{mm}|^2$ . This 1D form is independent of  $m$ , i.e., whether the channel is an even or odd function of  $y$ , and more importantly, it does not require the system to be  $\mathcal{RT}$ - and  $\mathcal{PT}$ -symmetric simultaneously.

In Fig. 3 we again introduce a transverse index modulation  $\Delta n$  to a half-gain-half-loss waveguide with a uniform  $\text{Re}[n]$ , similar to what we did in Fig. 2 except that  $\Delta n$  now depends on both  $x$  and  $y$  inside the waveguide,  $\Delta n(x, y) = b(xy > 0); -b(xy < 0)$ , which satisfies the  $\mathcal{RT}$  symmetry but not the  $\mathcal{PT}$ -symmetry. At  $b = 0$  the system is separable and Eq. (25) holds for both channels. This is no longer the case as  $b$  increases, and Eq. (25) breaks down while the generalized conservation law (22) always holds.

We end this section by discussing a few properties of a multimode waveguide that is simultaneously  $\mathcal{RT}$ - and  $\mathcal{PT}$ -symmetric. In this case both Eq. (13) and (22) hold, and by taking their difference we find that Eq. (25) also holds for each channel in a two-channel waveguide. It then indicates that there is no coupling between the channels, i.e.,  $\mathbf{t}$  and  $\mathbf{r}_{L,R}$  are all diagonal, even though the system is not necessarily separable (see the 2D example in Fig. 4). We note that this behavior only holds in the

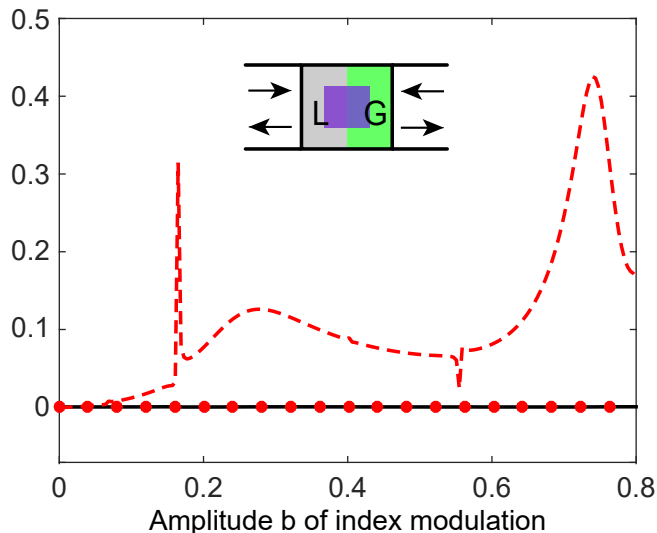


FIG. 4. Generalized conservation laws in a  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric waveguide that is non-separable in the longitudinal and transverse direction. Solid line shows the norm of the difference between the two sides of Eq. (13) (and that of Eq. (22) which is also zero). Dashed line and dots show the absolute value of the differences of the two sides of Eq. (25) for the  $m = 1$  and  $2$  channel respectively. The length  $L$  and width  $d$  of the waveguide are both  $3 \mu\text{m}$ , and the index modulation is given by  $\Delta n(x, y) = b(|x|, |y| < 0.75 \mu\text{m})$ . The other parameters are the same as in Fig. 2.

two-channel case: the system is symmetric about  $y = 0$ , which is implied by the simultaneous satisfaction of  $\mathcal{RT}$  and  $\mathcal{PT}$  symmetries ( $\mathcal{RTPT} = \mathcal{RP}$ , i.e.,  $y \rightarrow -y$ ); the two channels are even and odd functions about  $y = 0$  respectively, and they cannot couple as a result. If there are more channels, then the even- $m$  channels couple to each other and so do the odd- $m$  channels. For example, we plot the difference of the left and right hand sides of Eq. (25) for the  $m = 1$  and  $2$  channels in a 3-channel waveguide in Fig. 4(b). It is zero for the  $m = 2$  channel, which is the only channel of odd parity about  $y = 0$  and hence does not couple to the other two channels; it is nonzero for the  $m = 1$  channel, which couples to the other even-parity channel  $m = 3$ . Since now the even- $m$  channels and the odd- $m$  channels form two uncoupled  $\mathcal{PT}$ -symmetric systems, we can define the generalized transmittance  $\bar{T}$  and reflectance  $\bar{R}$  in them respectively (i.e.,  $\bar{T}_{e,o}$  and  $\bar{R}_{e,o}$ ), which are real and satisfy  $\bar{T}_{e,o} + \bar{R}_{e,o} = N_{e,o}$  according to Eq. (15), where  $N_{e,o}$  are the number of even- $m$  and odd- $m$  channels. This observation then implies that the other generalized transmittance  $\tilde{T}$  and reflectance  $\tilde{R}$  for the whole ( $\mathcal{RT}$ -symmetric) system are now also real valued and given by

$$\tilde{T} = \bar{T}_o - \bar{T}_e, \quad (26)$$

$$\tilde{R} = \bar{R}_o - \bar{R}_e, \quad (27)$$

which is the simplest scenario that leads to the conserva-

tion relation (23). We emphasize that these two relations above hold only when the system is simultaneously  $\mathcal{RT}$ - and  $\mathcal{PT}$ -symmetric.

### III. SPONTANEOUS SYMMETRY BREAKING OF THE $S$ MATRIX

Due to the symmetry relation (19) in a  $\mathcal{RT}$ -symmetric system, the  $S$  matrix can undergo a spontaneous symmetry breaking, similar to the  $\mathcal{PT}$ -symmetric case. If the system is simultaneously  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric, one may wonder whether one of these two symmetries can be broken while the other one is not. This situation does not occur for two reasons. First, if one symmetry is broken then some or all the eigenvalues of the  $S$  matrix are no longer unimodular, but if the other symmetry still holds, then all the eigenvalues should be unimodular, which imposes an obvious contradiction. The same argument applies to a specific pair of scattering eigenstates  $\varphi_1, \varphi_2$ , i.e., they cannot be  $\mathcal{PT}$ -broken but still  $\mathcal{RT}$ -symmetric (or vice versa). Another way to arrive at this observation is the following. If  $\varphi_1, \varphi_2$  are in the  $\mathcal{PT}$ -broken phase, then  $\mathcal{PT}\varphi_1 \propto \varphi_2$  holds [9]. As we mentioned at the end of the previous section, the system itself has parity symmetry about  $y = 0$  when it is both  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric, meaning that  $\varphi_1$  and  $\varphi_2$  are either even or odd functions of  $y$ . We then find

$$\mathcal{RT}\varphi_1 \propto \varphi_2, \quad (28)$$

using  $\mathcal{P}_y\mathcal{PT} = \mathcal{RT}$ , which indicates that  $\varphi_1$  and  $\varphi_2$  are also in the  $\mathcal{RT}$ -broken phase; otherwise we would find  $\mathcal{RT}\varphi_{1,2} \propto \varphi_{1,2}$  instead. Here  $\mathcal{P}_y$  is the parity operator about  $y = 0$ . Since this derivation is reversible, it gives the second reason why the two symmetries either hold or break simultaneously. As an example, we show in Figs. 5(a) and (b) such a pair of broken-symmetry scattering eigenstates, the intensity profile of which can be easily checked to satisfy  $\mathcal{P}|\varphi_1|^2 \propto |\varphi_2|^2$  and  $\mathcal{R}|\varphi_1|^2 \propto |\varphi_2|^2$ , as a consequence of  $\mathcal{PT}\varphi_1 \propto \varphi_2$  and  $\mathcal{RT}\varphi_1 \propto \varphi_2$ , respectively. Figs. 5(c) and (d) show a pair of scattering

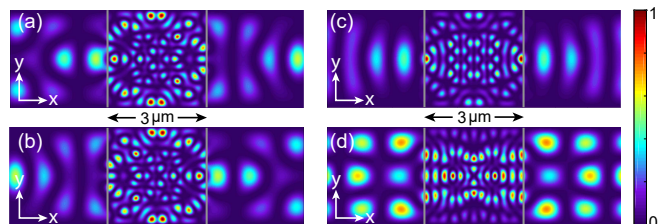


FIG. 5. (Color online) Intensity plot of scattering eigenstates in the  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric multimode waveguide shown in Fig. 4. (a,b) A pair of  $\mathcal{PT}$ - and  $\mathcal{RT}$ -broken eigenstates. (c,d) A pair in the  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric phase. All four eigenstates are the result of the coupling between  $m = 1, 3$  channels.  $\text{Im}[n] = \pm 0.05$ ,  $b = 0.8$ , and the grey vertical lines mark the left and right sides of the waveguide.

eigenstates that are both  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric, the intensity of which satisfies both  $\mathcal{P}|\varphi_{1,2}|^2 \propto |\varphi_{1,2}|^2$  and  $\mathcal{R}|\varphi_{1,2}|^2 \propto |\varphi_{1,2}|^2$ .

The simplest multimode waveguide that is simultaneously  $\mathcal{PT}$ - and  $\mathcal{RT}$ -symmetric is a separable half-gain-half-loss system depicted in Fig. 1(a). Below we discuss how spontaneous symmetry breaking of the  $S$  matrix can be observed in such a multimode waveguide, *without* the need to tune any system parameters. It is based on the observation that scattering in uncoupled channels of different transverse order  $m$ , at a given frequency, displays different thresholds for symmetry breaking in terms of the gain and loss strength. Therefore, if a system possesses  $\mathcal{PT}$ -symmetry at a particular frequency  $\omega_0$ , the scattering at this frequency will display two contrasting behaviors depending on the scattering channel: it can either be in the  $\mathcal{PT}$  symmetric phase with conserved flux in the corresponding scattering eigenstates, or in the broken symmetry phase with a pair of amplified and attenuated scattering eigenstates. The transverse order  $m$  then plays the role of the symmetry breaking parameter.

We have chosen the waveguide with uncoupled channels for a practical consideration: scattering eigenstates that exist in more than one channel is difficult to measure and tune in an experiment. Since we want to observe a clear transition between the symmetric and broken phases of the  $S$  matrix, we need at least a few channels in both phases.

The  $m$ -dependent symmetry breaking can be understood in two ways. First we note that the symmetry breaking threshold (and the EP) of the  $S$  matrix for a 1D half-loss-half-gain structure is given by the following expression [9]:

$$\frac{\omega_c}{c} L \approx \frac{1}{\tau} \ln \left( \frac{2n_0}{\tau} \right), \quad (29)$$

where  $n_0 \pm i\tau$  is the complex refractive index in the loss and gain regions and  $c$  is the speed of light in vacuum. For a given  $n_0$  and  $\tau$ , the broken symmetry phase lies in  $\omega_0 > \omega_c$  and the symmetric phase in  $\omega_0 < \omega_c$ . In the semiclassical regime where the wavelength is much shorter than the system size, one can apply the picture of ray optics, and propagating modes of different transverse order  $m$  experience a different length  $L \rightarrow L_m$  in the scattering region, because they propagate at different angles with respect to the sidewalls (e.g., see the wave vectors  $k_x, k_y$  in Fig. 1(b)). As a result, Eq. (29) shows that  $\omega_c$  at the symmetry breaking threshold is now  $m$ -dependent (denoted by  $\omega_c^{(m)}$ ) for a given  $n_0$  and  $\tau$ . Consequently, scattering in higher transverse channels (with a larger  $m$ , a longer  $L_m$ , and a lower  $\omega_c^{(m)}$ ) can be in the broken symmetry phase ( $\omega_0 > \omega_c^{(m)}$ ), while scattering in lower transverse channels can be in the  $\mathcal{PT}$ -symmetric phase ( $\omega_0 < \omega_c^{(m)}$ ) at the same frequency  $\omega_0$ .

To be more quantitative and able to address the regime where the system size is comparable to the wavelength, we offer an alternative (but equivalent) explanation of the

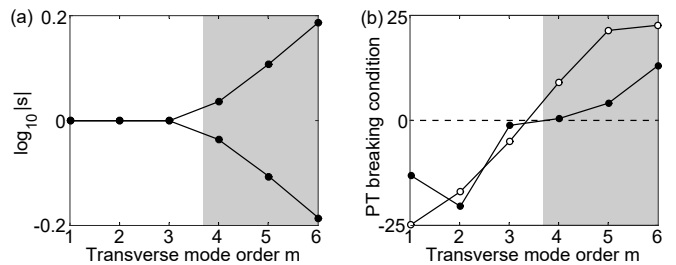


FIG. 6. (a) Spontaneous symmetry breaking of the eigenvalues  $s$  of the scattering matrix in a multimode waveguide. The parameters used are  $n_0 = 3$ ,  $\tau = 0.005$ ,  $L = 23 \mu\text{m}$ ,  $d = 5 \mu\text{m}$ , and the wavelength is  $1550 \text{ nm}$  ( $\omega_0/c = 4.05 \mu\text{m}^{-1}$ ). At this frequency the waveguide supports six channels ( $m = 1, 2, \dots, 6$ ), and the upper three are in the breaking symmetry phase (shaded area). (b) Filled circles: the difference of the left and right hand sides of the symmetry breaking condition (32), which predicts the same broken symmetry regime ( $m \geq 4$ ) as in (a). Open circles: 10 times the value of  $(R_L + R_R) - 2(T + 1)$  in each channel, the sign of which indicates whether the system is in the  $\mathcal{PT}$ -symmetric phase or broken symmetry phase.

channel-dependent symmetry breaking, by mapping the wave propagation to an effective 1D system where the criterion (29) holds. As we shall see, in this explanation the length  $L$  of the scattering region is the same in all channels, while the effective frequency and refractive index now become channel-dependent. Take the 2D waveguide shown in Fig. 1 for example, the transverse wave number is  $k_y = m\pi/d$ , ( $m = 1, 2, \dots$ ), and the incident light thus has an effective frequency

$$\omega_m = \sqrt{\omega_0^2 - c^2 k_y^2} \quad (30)$$

in the propagation direction  $x$ , and the effective index inside the gain and loss regions is

$$n_m = \frac{\sqrt{(n_0 \pm i\tau)^2 \omega_m^2 - c^2 k_y^2}}{\omega_m}, \quad (31)$$

which is still  $\mathcal{PT}$  symmetric. The criterion (29) now becomes

$$\frac{\omega_m}{c} L \approx \frac{1}{|\text{Im}[n_m]|} \ln \left( \frac{2\text{Re}[n_m]}{|\text{Im}[n_m]|} \right), \quad (32)$$

and it predicts a symmetry breaking threshold between  $m = 3$  and  $4$  for the case shown in Fig. 6, which is verified by directly calculating the eigenvalues of the  $S$  matrix.

In Ref. [11] a simplified signature of the symmetry breaking was given in terms of the transmittance and reflectances for a 1D  $\mathcal{PT}$ -symmetric structure: the system is in the symmetric phase if one measures  $(R_L + R_R) - 2(T - 1) < 0$ ; otherwise it is in the broken symmetry phase. We test this signature in each channel using  $T \equiv |\mathbf{t}_{mm}|^2$  and  $R_{L,R} \equiv |(\mathbf{r}_{L,R})_{mm}|^2$  in the example above, and indeed this quantity changes its sign between

$m = 3$  and 4 (see the open circles in Fig. 6(b)), which makes the detection of the symmetry breaking much easier than tuning into the scattering eigenstates and measuring their scattering eigenvalues. The latter can, of course, be measured indirectly by substituting the measured values of  $\mathbf{t}_{mm}, (\mathbf{r}_{L,R})_{mm}$  in the definition of the  $S$  matrix (Eq. (4) for each channel), which has the inconvenience of measuring the phases of  $\mathbf{t}_{mm}, (\mathbf{r}_{L,R})_{mm}$ .

#### IV. CONCLUSION

In summary, we have discussed the generalized conservation laws of scattering in  $\mathcal{PT}$ -symmetric and  $\mathcal{RT}$ -symmetric multimode waveguides. Not only do they exist in a matrix form for the transmission and reflection matrices, they also exist in a scalar form for real-valued quantities using generalized transmittances and reflectances. If different waveguide channels are decoupled, each channel is effectively a 1D system and we recover the generalized conservation law found in Ref. [11], now for both  $\mathcal{PT}$ -symmetric and  $\mathcal{RT}$ -symmetric cases. When this condition holds, we have shown that a simple half-gain-half-loss  $\mathcal{PT}$ -symmetric system offers a convenient setup to observe the spontaneous symmetry breaking of the  $S$  matrix, without the need to tune any system parameters. It is facilitated by the existence of a single “exceptional” value of the channel number  $m$  for a given incident light frequency and fixed gain and loss strength in this system. By performing scattering experiment in each channel of the waveguide, one observes the transition from the symmetric phase to the broken symmetry phase as  $m$  becomes larger than this exceptional value.

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#### APPENDIX: EIGENVALUES OF $\mathbf{AA}^*$

Interestingly, we find that the eigenvalues of the matrix  $\mathbf{AA}^*$  are either real or form complex conjugate pairs, where  $\mathbf{A}$  is an arbitrary square matrix. Here we provide a short proof. If  $\mathbf{AA}^*$  has an eigenvalue  $\lambda_n$  and the corresponding eigenvector is  $\psi_n$ , i.e.,

$$\mathbf{AA}^*\psi_n = \lambda_n\psi_n, \quad (33)$$

then by taking the complex conjugate of both sides and multiplying  $\mathbf{A}$  from left, we find

$$\mathbf{AA}^*\mathbf{A}\psi_n^* = \lambda_n^*\mathbf{A}\psi_n^*. \quad (34)$$

It indicates that  $\lambda_n^*$  is also an eigenvalue of  $\mathbf{AA}^*$ , with the corresponding eigenvector  $\mathbf{A}\psi_n^*$ . Therefore,  $\lambda_n$  is real if  $\mathbf{A}\psi_n^* \propto \psi_n$ , or a pair of complex conjugate eigenvalues  $\lambda_n, \lambda_{n'}$  exist when  $\mathbf{A}\psi_n^* \propto \psi_{n'}$  ( $n \neq n'$ ).

This property not only guarantees that  $Tr(\mathbf{AA}^*)$  (and more specifically  $Tr(\mathbf{tt}^*)$ ) is real, which we have used in the main text to derive the real-valued conservation law (15); it also suggests a general form of the effective Hamiltonian for a  $\mathcal{PT}$ -symmetric (and  $\mathcal{RT}$ -symmetric) system. For example, the toy model of two coupled  $\mathcal{PT}$ -symmetric resonators can be written as

$$\mathbf{H} = \begin{pmatrix} \omega + i\tau & g \\ g & \omega - i\tau \end{pmatrix}, \quad (35)$$

where  $\omega$  is the identical resonant frequency of both resonators,  $\pm\tau$  represent the loss and gain strength, and  $g \ll \omega$  is the coupling strength of these two resonators. The decomposition  $\mathbf{H} = \mathbf{AA}^*$  exists for multiple choices of  $\mathbf{A}$ , and one such choice is

$$\mathbf{A} = \begin{pmatrix} \frac{g}{2b} & b \\ b - i\frac{\tau}{b} & \frac{g}{2b} \end{pmatrix}, \quad (36)$$

where  $b$  is a real quantity satisfying  $b^2 = (\omega \pm \sqrt{\omega^2 - g^2})/2$ .

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