# Continuous decomposition of quantum measurements via Hamiltonian feedback

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# Continuous decomposition of quantum measurements via Hamiltonian feedback

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We characterize the set of generalized quantum measurements that can be decomposed into a continuous measurement process using a stream of probe qubits and a tunable interaction Hamiltonian. Each probe in the stream interacts weakly with the target quantum system, then is measured projectively in a standard basis. This measurement result is used in a closed feedback loop to tune the interaction Hamiltonian for the next probe. The resulting evolution is a stochastic process with the structure of a one-dimensional random walk. To maintain this structure, and require that at long times the measurement outcomes be independent of the path, the allowed interaction Hamiltonians must lie in a restricted set, such that the Hamiltonian terms on the target system form a finite dimensional Jordan algebra. This algebraic structure of the interaction Hamiltonians yields a large class of generalized measurements that can be continuously performed by our scheme, and we fully describe this set.

Keywords: quantum continuous measurement, quantum feedback control, Jordan algebras

## I. INTRODUCTION

Many quantum systems either exhibit slow measurement read-out times or can only be probed weakly [1, 2]. Under such conditions, it is natural to monitor the systems continuously while simultaneously exerting some closed-loop feedback. Experiments can already be performed with such low latency that feedback can be performed continuously in real time [3–5]. While generalized continuous measurements have been studied [6, 7], in most systems the diffusive weak measurements [8] that constitute the continuous process must be applied via coupling to a probe system. Previously, we've studied the setting of closed-loop feedback applied to a stream of probe qubits interacting with the system by a fixed Hamiltonian [9]. Here, we investigate the possibilities that arise from closed-loop feedback when the interaction Hamiltonian is itself subject to control.

A key feature of [9] was the derivation of a reversibility equation which was used to characterize the class of measurements that admitted a continuous decomposition. This equation is necessary again in this work. The reversibility equation arises from the need to ensure that the a continuous decomposition of a quantum measurement is a faithful implementation of its instantaneous counterpart. Since quantum measurements are stochastic instantaneous processes, it follows that continuous quantum measurements must be stochastic continuous processes. Of course, stochastic continuous processes have an infinite number paths, not all of which match the action of an instantaneous measurement. The reversibility equation effectively eliminates the dependence of the path on time. In our construction, the time to complete a continuous decomposition will be random and so we must guarantee that repeated use of our scheme yields measurement statistics that are dependent on the quantum state rather than the time.

This reversibility condition cannot be generally satisfied without the use of continuous feedback. The one exception to this fact is the case of decomposing projective measurements, which can be shown to be path-independent [6]. Furthermore, although we'll restrict our analysis here to qubit probes and two-outcome measurements, we note that general two-outcome measurements are sufficient building blocks for $n$-outcome measurements [6, 10].

This paper is organized as follows, in section II we will describe our model for continuously decomposing a quantum measurement using a stream of probes and a linearly controllable interaction Hamiltonian. In section III we will prove our main result about which measurements can be decomposed given a linearly controllable interaction Hamiltonian. In sections IV and V we will give illustrative examples of the methods described in section III and in section VI we will conlude by comparing this new scheme to our previous work [9].

## II. MODEL

We will treat our scheme using discrete timesteps with the implicit understanding that in the limit of infinitesimal timesteps our scheme converges to continuous stochastic process. Consider a quantum system $S$ undergoing a stochastic evolution driven by two-outcome diffusive weak measurements. The outcome of any particular step during the evolution is one of two *weak measurement step operators* $M_\pm(x)$. These step operators are functions of a pointer variable $x$ which updates with each outcome. The exact feedback scheme is illustrated in Figure 1, and the process terminates when $x$ reaches

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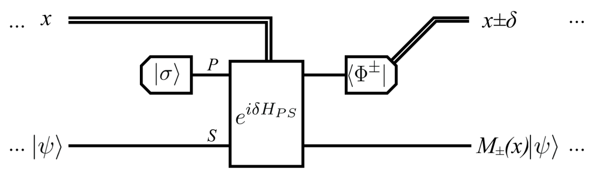


FIG. 1. The system $S$ is continuously measured. At each timestep, we perform a weak measurement by preparing the probe $|\sigma\rangle$ and tuning the interaction Hamiltonian $H_{PS}(x)$ based on a pointer variable $x$. The system and probe interact for a short time $\delta$ and the probe is measured in an orthogonal detector basis $\langle\Phi^{\pm}|$. The measurement result from the detector is used to update the pointer variable from $x$ to $x \pm \delta$ and the procedure is repeated with the new value.

a fixed constant $\pm X$. The reversibility condition can be written

$$M_{\mp}(x \pm \delta) M_{\pm}(x) \propto I. \tag{1}$$

From the above equation, two consecutive outcomes that step "forward" from $x$ to $x+\delta$ then "backward" from $x+\delta$ to $x$ have no net effect on $|\psi\rangle$. However, rather than track the evolution of $|\psi\rangle$ directly, we can express the total action of our procedure as the *total walk operator*

$$M(x) = \begin{cases} \lim_{\delta\to 0} \prod_{j=1}^{\lfloor x/\delta \rfloor} M_{+}(j\delta) & x \geq 0 \\ \lim_{\delta\to 0} \prod_{j=-1}^{\lfloor x/\delta \rfloor} M_{-}(j\delta) & x < 0 \end{cases}, \tag{2}$$

We identify the *endpoint* operators $M_1 \propto M(X)$ and $M_2 \propto M(-X)$ with the instantaneous measurement being decomposed by this continuous process.

We will consider a simple model of probe-state interaction that will generate $M_{\pm}(x)$. In [9] we found that weak measurements with qubit probes had to form a *probe-basis* on the qubit Hilbert space. In particular, we required that the probe state, the orthogonal quantum states of the detector, and the eigenstates of the interaction Hamiltonian on the probe subsystem, have mutually orthonormal representations on the Bloch sphere. For this reason, we choose the interaction Hamiltonian to be $H_{PS} = Y_P \otimes \hat{\varepsilon}(x)$, the probe state to be $|0\rangle$, and the detector states to be $\langle\pm|$. The operator acting on the system $S$ is $\hat{\varepsilon}$ and is defined to be an $x$-dependent linear combination of $d$ constant Hamiltonian terms,

$$\hat{\varepsilon}(x) = \sum_{i=0}^{d} p_i(x) H_i. \tag{3}$$

The weak measurement step operators of Figure 1 then become

$$\begin{aligned} M_{\pm}(x) &= \langle\pm| e^{i\delta H_{PS}(x)} |0\rangle \qquad (4) \\ &\approx \frac{1}{\sqrt{2}} I \mp \frac{\delta}{\sqrt{2}} \hat{\varepsilon}(x) - \frac{\delta^2}{2\sqrt{2}} \hat{\varepsilon}^2(x). \end{aligned}$$

The reversibility condition of Eq. 1 can now be rewritten in terms of $\hat{\varepsilon}(x)$. Note that this condition need only be satisfied up to $O(\delta^2)$ since the random walk induced on the pointer variable $x$ will take $O(N^2)$ steps to converge when $N = \lfloor X/\delta \rfloor$. Collecting terms by orders of $\delta$ yields

$$\begin{aligned} M_{\mp}(x \pm \delta) M_{\pm}(x) &= \frac{I}{2} \mp \frac{\delta}{2} \left( \hat{\varepsilon}(x) - \hat{\varepsilon}(x \pm \delta) \right) \qquad (5) \\ &\quad - \frac{\delta^2}{2} \hat{\varepsilon}^2(x) - \frac{\delta^2}{2} \hat{\varepsilon}^2(x) \qquad (6) \\ &= \frac{I}{2} + \frac{\delta^2}{2} \left( \partial_x \hat{\varepsilon}(x) - 2\hat{\varepsilon}^2(x) \right) + O(\delta^3). \qquad (7) \end{aligned}$$

Let $\alpha(x)$ be the $O(\delta^2)$ coefficient of proportionality in Eq. (1). We find that the reversibility equation reduces to

$$\partial_x \hat{\varepsilon}(x) = 2\hat{\varepsilon}^2(x) + \alpha(x) I. \tag{8}$$

In the derivations that follow, we will temporarily ignore the $\alpha(x)I$ term, as it will not change the class of measurements that satisfy the reversibility equation.

Consider the set of controls that appear in Eq. (3). Without loss of generality, we can always assume that $H_0 = I$ since the action of $I$ is equivalent to an overall phase on the probe system. The reversibility equation in Eq. (8) can then be rewritten as

$$\sum_{k=0}^{d} \partial_x p_k(x) H_k = \sum_{i,j=0}^{d} p_i(x) p_j(x) \frac{1}{2} \{H_i, H_j\}. \tag{9}$$

where $\{\cdot,\cdot\}$ is the anti-commutator. It will be useful to introduce the tensor $\Gamma_{ij}^{k}$ for expressing the action of the anti-commutator on the matrices $H_i$. In particular,

$$\frac{1}{2} \{H_i, H_j\} = \sum_{k=0}^{n(n-1)/2} \Gamma_{ij}^{k} H_k. \tag{10}$$

We choose the matrices $H_i$ for $i > d$ such that they form a basis for $\mathcal{H}_n(\mathbb{C})$, the space of all $n$-dimensional complex Hermitian matrices. We will use $\Gamma^{(k)}$ to denote the matrix resulting from fixing the index $k$. The reversibility equation Eq. (8) can then be read as

$$\begin{cases} \partial_x p_k = \vec{p}^T \Gamma^{(k)} \vec{p} & 0 \leq k \leq d \\ 0 = \vec{p}^T \Gamma^{(k)} \vec{p} & d < k. \end{cases} \tag{11}$$

## III. MAIN RESULT

We now present our main result which characterizes solutions to the above equations. Let us denote $\mathbb{F} = \operatorname{span}\{H_i\}$ so that $\hat{\varepsilon} \in \mathbb{F}$. We prove the following lemma about solutions to Eq. (11).

**Lemma 1.** *Any solution $\hat{\varepsilon}(x)$ to Eq.* (11) *must lie entirely in $\mathbb{V}$, a subspace of $\mathbb{F}$ that is closed under anti-commutation.*

*Proof.* We note that if $\mathbb{F}$ is already closed under anti-commutation, then the reversibility equation reduces to an initial value problem in terms of the control coefficients $\vec{p}(x)$ at $x = 0$. However if $\mathbb{F}$ is not closed under anti-commutation, then we must characterize the set of vectors $\vec{p}$ such that Eq. (11) is satisfied. To do so, consider choosing any $k > d$ and solving the associated equation

$$\vec{p}^T \Gamma^{(k)} \vec{p} = 0. \tag{12}$$

Note that the matrix $\Gamma^{(k)}$ is symmetric and defines a quadratic space over $\mathbb{R}^n$, denoted by $\left(\Gamma^{(k)}, \mathbb{R}^n\right)$. We know that every quadratic space admits a Witt decomposition [11]. That is, the quadratic space is isomorphic to the direct product of three types of subspaces,

$$\left(\Gamma^{(k)}, \mathbb{R}^n\right) \cong \bigoplus_{i=0}^{N} W_i \oplus V_0 \oplus V'. \tag{13}$$

In the above, $W_i$ are hyperbolic planes, $V_0$ is the nullspace of $\Gamma^{(k)}$, and $V'$ is an anisotropic subspace of $\mathbb{R}^n$. The solutions $\vec{p}$ can also be written in terms of these subspaces. We begin by solving Eq. (12) in the subspace formed by the hyperplanes $W_i$. Solutions to $\vec{x}^T W_i \vec{x} = 0$ are span $\{[1,1]\}$ or span $\{[1,-1]\}$ for each $i$. By definition, there are no non-zero vectors in the anisotropic subspace $V'$ which satisfy $\vec{x}^T V' \vec{x} = 0$ and so we set all components of $\vec{p}$ in this subspace to 0. Let $T^{(k)}$ be the isomorphism that describes

$$(I_n, \mathbb{R}^n) \quad \underset{T_k}{\longrightarrow} \quad \left(\Gamma^{(k)}, \mathbb{R}^n\right) \tag{14}$$

where $I_n$ is the identity matrix. Let also $\vec{p} = T^{(k)}\vec{q}$. We deduce that all possible solutions to Eq. (12) must lie in the direct product of the hyperplane subspace solutions and the nullspace. The solution space $V$ is then

$$V = T^{(k)} \left( \bigoplus_{i=0}^{N} \text{span}\{[1, x_i]\} \oplus V_0 \right). \tag{15}$$

Notice that in this expression we've assumed a fixed choice of $x_i = \pm 1$. Varying any of these values gives rise to an entirely different space of solutions.

To fully solve Eq. (11) we must now recurse the above procedure. At each step we restrict $\vec{p}$ to lie in the subspace $V$ defined by a particular choice of $x_i$. We then define a new matrix basis for the controls restricted to $V$ and generate a new set of $\Gamma^{(k)}$ matrices. We then choose a new $k$ and decompose $V$ using $\Gamma^{(k)}$. Since the order in which the $k$ are chosen will affect the form of $V$, it is also important to enumerate all sequences of choices of $k$ and $x_i$ if one wishes to identify *all* solution spaces. This procedure terminates when the vector space of Hermitian matrices $\mathbb{V}$ formed from $V$ is closed under anti-commutation.

Since the Witt decomposition is unique (up to isometries of $V'$), we can guarantee that this procedure lists all closed subspaces contained in $\mathbb{F}$. It remains only to show that if $\vec{p}(0) \in V$ for a particular sequence of choices of $k$ and $x_i$, then $\vec{p}(x)$ will remain in the same subspace for all other values of $x$. This follows directly, however, from the fact that if $\hat{\varepsilon}(x) \in \mathbb{V}$ then $\hat{\varepsilon}^2(x) \in \mathbb{V}$ and so $\partial_x \hat{\varepsilon} \in \mathbb{V}$. □

Lemma 1 establishes that in order to solve the reversibility equation, one must use a set of controls whose span is closed under anti-commutation. The proof of the lemma also includes an implicit algorithm for finding closed subspaces given a set of Hermitian matrices. The next lemma gives the structure of the subspaces enumerated by lemma 1.

**Lemma 2.** *The $\hat{\varepsilon}(x)$ operator has the form*

$$\hat{\varepsilon}(x) = \bigoplus_{l=1}^{S(\mathbb{V})} U_l(x) D_l(x) U_l^\dagger(x). \tag{16}$$

*where $S(\mathbb{V})$ is the number of simple components of the algebra $\mathbb{V}$ (with anti-commutation as its product), and $D_l(x)$ and $U_l(x)$ correspond to the $l^{\text{th}}$ simple component and are given by Table I.*

*Proof.* We begin by identifying $\mathbb{V}$ as a finite-dimensional Jordan algebra. Every such algebra accepts a Wedderburn-type decomposition [12, 13],

$$\mathbb{V} \cong \bigoplus_{l=1}^{S(\mathbb{V})} \mathbb{B}_l, \tag{17}$$

where $S(\mathbb{V})$ is the number of simple components $\mathbb{B}_l$ of $\mathbb{V}$. A classification of all finite-dimensional simple Jordan algebras was given by Jordan, von Neumann, and Wigner [14]. The three types of Jordan algebras that can be found in our decomposition are the self-adjoint real, complex, and quaternionic matrices. The isomorphism in Eq. (17) leaves a lot of freedom in terms of how to represent each of these simple components by Hamiltonian terms and so we summarize the possible representations in Table I. Note that the exceptional Albert algebra is absent, since octonions do not have a matrix representation over $\mathbb{R}$ or $\mathbb{C}$ [15].

Since $\mathbb{V}$ can be written as a direct sum of the simple algebras in Table I, we can also write the operator $\hat{\varepsilon}$ in this decomposition,

$$\hat{\varepsilon} = \bigoplus_{l=1}^{S(\mathbb{V})} \hat{\varepsilon}_l(x). \tag{18}$$

Each operator in the direct sum can, in turn, be diagonalized with the unitary $U_l(x)$ and the diagonal matrix $D_l(x)$ to yield the form in the statement of the lemma. □

Before we proceed to the final lemma that will complete our main result, we will give a few illustrative examples of the last three representations found in Table I. First, we consider the matrix algebra resulting from the 2-dimensional embedding of $\mathbb{C}$ into $\mathbb{R}$, i.e.

| Block $\mathbb{B}_l$ | $D_l(x)$ | $U_l(x)$ |
|---|---|---|
| $\mathcal{H}_n(\mathbb{R})$ | diag $(\mathbb{R}^n)$ | $SO(n)$ |
| $\mathcal{H}_n(\mathbb{C})$ | diag $(\mathbb{R}^n)$ | $SU(n)$ |
| $\mathcal{H}_n(\mathbb{C}) \cong \mathcal{H}_{2n}(\mathbb{R})$ | diag $(\mathbb{R}^n) \otimes I_2$ | $SO(n) \otimes SO(2)$ |
| $\mathcal{H}_n(\mathbb{H}) \cong \mathcal{H}_{2n}(\mathbb{C})$ | diag $(\mathbb{R}^n) \otimes I_2$ | $SU(n) \otimes SU(2)$ |
| $\mathcal{H}_n(\mathbb{H}) \cong \mathcal{H}_{4n}(\mathbb{R})$ | diag $(\mathbb{R}^n) \otimes I_4$ | $SO(n) \otimes SO(4)$ |

TABLE I. We list all rank-$n$ representations of Jordan algebras that can be embedded into a span of Hermitian matrices. The third representation corresponds to the 2-dimensional embedding of $\mathbb{C}$ into $\mathbb{R}$. The fourth and fifth representations correspond to 2- and 4-dimensional embeddings of $\mathbb{H}$ into $\mathbb{C}$ and $\mathbb{R}$. The notation diag $(\mathbb{R}^n)$ refers to the algebra of $n$-dimensional diagonal matrices with real coefficients.

$\mathcal{H}_n(\mathbb{C}) \cong \mathcal{H}_{2n}(\mathbb{R})$. Let the coeffecients of a matrix in $\mathcal{H}_n(\mathbb{C})$ be $u_{jk} = a_{jk} + ib_{jk}$, then

$$\begin{bmatrix} u_{00} & \dots & u_{0n} \\ \vdots & \ddots & \vdots \\ u_{n0} & \dots & u_{nn} \end{bmatrix} \cong \begin{bmatrix} a_{00} & -b_{00} & \dots & a_{0n} & -b_{0n} \\ b_{00} & a_{00} & \dots & b_{0n} & a_{0n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n0} & -b_{n0} & \dots & a_{nn} & -b_{nn} \\ b_{n0} & a_{n0} & \dots & b_{nn} & a_{nn} \end{bmatrix} \tag{19}$$

describes the embedding $\mathcal{H}_n(\mathbb{C}) \cong \mathcal{H}_{2n}(\mathbb{R})$. For the fourth and fifth representations in Table I, we replace each quaternionic element $h = a + b\hat{i} + c\hat{j} + d\hat{k}$ by one of the following two submatrices. For $\mathcal{H}_{2n}(\mathbb{C})$ we use

$$h_{jk} \cong \begin{bmatrix} a_{jk} + ib_{jk} & c_{jk} + id_{jk} \\ -c_{jk} + id_{jk} & a_{jk} - ib_{jk} \end{bmatrix}, \tag{20}$$

and for $\mathcal{H}_{4n}(\mathbb{R})$ we use

$$h_{jk} \cong \begin{bmatrix} a_{jk} & b_{jk} & c_{jk} & d_{jk} \\ -b_{jk} & a_{jk} & -d_{jk} & c_{jk} \\ -c_{jk} & d_{jk} & a_{jk} & -b_{jk} \\ -d_{jk} & -c_{jk} & b_{jk} & a_{jk} \end{bmatrix}. \tag{21}$$

Note that up to this point, we've ignored the term $\alpha(x)I$ in the reversibility equation Eq. (8). We were able to ignore it in lemma 1 because the matrix $\Gamma^{(0)}$ that corresponds to $H_0 = I$ never appears as one of the matrices we use during the recursive part of the proof. We were also able to ignore it in lemma 2 since all of the simple Jordan algebras listed above always contain a full set of rank-1 idempotents. We will now reintroduce the term $\alpha(x)I$ as it will play an important role in regularizing the behaviour of the differential equations in lemma 3.

**Lemma 3.** *The $\hat{\varepsilon}(x)$ operator and the total walk operator $M(x)$ are simultaneously diagonalizable.*

*Proof.* We begin by noting that Eq. (8) can be solved for individual blocks $\hat{\varepsilon}_l(x)$, yielding

$$\partial_x \left(U_l(x) D_l(x) U_l^\dagger(x)\right) = 2\left(U_l(x) D_l(x) U_l^\dagger(x)\right)^2 + \alpha(x) I_l.$$

where $I_l$ is the identity on the block $(l)$. A few simple manipulations result in the following equivalent expression (for clarity, we've omitted the $x$-dependence)

$$\partial_x U_l D_l U_l^\dagger + U_l \partial_x D_l U_l^\dagger + U_l D_l \partial_x U_l^\dagger = 2 U_l D_l^2 U_l^\dagger + \alpha I_l.$$

Applying $U_l^\dagger$ and $U_l$ from the left and the right gives

$$U_l^\dagger \partial_x U_l D_l + \partial_x D_l + D_l \partial_x U_l^\dagger U_l = 2D_l^2 + \alpha I_l.$$

Since $U_l$ is a unitary matrix we can write it as the exponent of a Hermitian matrix $G_l$ and we note that $U_l^\dagger \partial_x U_l = i\partial_x G_l$. This reduces the above equation to

$$i\left[\partial_x G_l, D_l\right] + \partial_x D_l = 2D_l^2 + \alpha I_l. \tag{22}$$

The entries of the commutator term are

$$\left(\left[\partial_x G_l, D_l\right]\right)_{ij} = \partial_x g_{ij}^{(l)} \left(d_i^{(l)} - d_j^{(l)}\right) \tag{23}$$

from which we can infer that the diagonal entries of the commutator term are 0 if $d_i^{(l)} \neq d_j^{(l)}$. Thus, the equation for any $d_i^{(l)}(x)$ reduces to

$$\partial_x d_i^{(l)}(x) = \left(d_i^{(l)}(x)\right)^2 + \alpha(x) \tag{24}$$

which is a special case of the Ricatti first-order non-linear differential equation. Without loss of generality however, we can simply consider the case where $\alpha(x)$ is fixed at a strictly positive real constant $\alpha$. Although a different choice of $\alpha(x)$ will ultimately lead to different solutions for the functions $d_i^{(l)}(x)$, this will not expand the class of measurements possible with our scheme. Thus, fixing $\alpha(x) = \alpha$, Eq. (24) has the solution

$$\sqrt{\alpha}\tanh\left(\sqrt{\alpha}\left(x - c_i^{(l)}\right)\right). \tag{25}$$

This form immediately implies that for any $i, j$ such that $c_i^{(l)} \neq c_j^{(l)}$, $g_{ij}^{(l)}$ is constant. Furthermore, we can even prove that for other solutions where $\alpha(x)$ has a more general form, $g_{ij}^{(l)}$ is constant when $d_i^{(l)} \neq d_j^{(l)}$. This is because the general solution $d_i^{(l)}(x)$ to the special Ricatti equation has only one free parameter $c_i^{(l)}$,

$$d_i^{(l)}(x) = f(x) + \frac{e^{2\int f(x)dx}}{c_i^{(l)} - \int e^{2\int f(x)dx} dx}. \tag{26}$$

where $f(x)$ is a known solution to Eq. (24). It can be seen, from this general solution that $d_i^{(l)} = d_j^{(l)}$ only in the case where $c_i^{(l)} = c_j^{(l)}$, just as with the tanh solution.

Finally, for the case of $\alpha(x) = \alpha$ Eq. (22) has the solution

$$\begin{cases} d_i^{(l)}(x) = \sqrt{\alpha}\tanh\left(\sqrt{\alpha}\left(x - c_i^{(l)}\right)\right) & \forall\, i, \\ g_{ij}^{(l)}(x) = g_{ij}^{(l)}(0) & \forall\, i,j \;:\; c_i^{(l)} \neq c_j^{(l)}, \\ g_{ij}^{(l)}(x) = g_{ij}^{(l)}(x) & \forall\, i,j \;:\; c_i^{(l)} = c_j^{(l)}. \end{cases}$$

We note that in the cases where $c_i^{(l)} = c_j^{(l)}$, $g_{ij}^{(l)}$ need not be constant. However, in these cases, the $x$-dependent sub-block of $G_l(x)$ is acting on a sub-block of $D_l(x)$ that is proportional to the identity. Thus, this freedom in $G_l(x)$ does not affect the form of $\hat{\varepsilon}$, or of $M_1$, $M_2$.

We now turn our attention to the total walk operator given in Eq. (2) which obeys the following differential equation (up to a normalization factor):

$$\partial_x M(x) = -\hat{\varepsilon}(x) M(x). \tag{27}$$

We can write $M(x)$ in the diagonal basis of $\hat{\varepsilon}(x)$ by introducing the operator

$$N(x) = \left( \bigoplus_{k=1}^{S(\mathbb{V})} U_l^\dagger \right) M(x) \left( \bigoplus_{l=1}^{S(\mathbb{V})} U_l \right). \tag{28}$$

Eq. (27) can then be rewritten as

$$\partial_x N(x) = -\bigoplus_{l=1}^{S(\mathbb{V})} D_l(x) N(x) - i \bigoplus_{l=1}^{S(\mathbb{V})} \left[ \partial_x G_l(x), N_l(x) \right]. \tag{29}$$

Note that since $M(0) = I$ then $N(0) = I$ and so the commutator term above disappears for all $x$. This immediately imples that $N(x)$ must be diagonal and so the total walk operator and the $\hat{\varepsilon}(x)$ operator are diagonal in the same basis. □

Lemmas 1, 2, and 3 combined give the full characterization of $M_1$ and $M_2$ operators achievable by our scheme:

**Theorem 1** (Main result)**.** *A continuous measurement using qubit probes and closed-loop feedback on the interaction Hamiltonian (as in Fig. 1) can realize any measurement $\{M_1, M_2\}$ of the form*

$$M_1 = \bigoplus_{l=1}^{S(\mathbb{V})} U_l^\dagger \left( \bigoplus_{i=1}^{\mathrm{rank}(\mathbb{B}_l)} \lambda_i^{(l)} \Pi_i^{(l)} \right) U_l, \tag{30}$$

*where $M_2 = (I - M_1^\dagger M_1)^{1/2}$ is diagonal in the same basis. The parameters $\lambda_i^{(l)}$ are real and contained in $(0, 1)$ and $\Pi_i^{(l)}$ is a projector onto 1, 2, or 4 basis states.*

*Proof.* Recall that the number of distinct diagonal entries possible in $D_l(x)$ is $\mathrm{rank}\,(\mathbb{B}_l)$. However, each distinct entry can appear 1, 2, or 4 times depending on the particular representation from Table I. Using lemma 3 we can plug our solution for $D_l(x)$ into Eq. (29) to find that the diagonal entries of $N(x)$ are

$$\lambda_i^{(l)}(x) = \exp\left( \int_0^x \sqrt{\alpha} \tanh\left( \sqrt{\alpha} \left( y - c_i^{(l)} \right) \right) dy \right). \tag{31}$$

The total walk operator $M(x)$ must then be

$$M(x) \propto U_l^\dagger \left( \bigoplus_{i=1}^{\mathrm{rank}(\mathbb{B}_l)} \lambda_i^{(l)}(x) \Pi_i^{(l)} \right) U_l. \tag{32}$$

The endpoint operators $M_1$ and $M_2$ are proportional to $M(X)$ and $M(-X)$. Their diagonal entries are $\lambda_i^{(l)}$, which after renormalization approach 0 when $c_i^{(l)} \to \infty$ and 1 when $c_i^{(l)} \to -\infty$. □

Note that in theorem 1 the eigenvalues of $M_1$ and $M_2$ are restricted to lie in the open set $(0, 1)$, not the closed set $[0, 1]$. This is a consequence of the reversibility condition at the points $x = X - \delta$ and $x = -X + \delta$. At these points, setting any eigenvalue of the total walk operator to 0 would be effectively a projection, which is an irreversible operation for the random walk. However we can approach arbitrarily close to any such projective measurement.

To allow for direct comparisons with the scheme of [9], we provide the following corollary.

**Corollary 1** (Spectrum of the measurement)**.** *Given the ability to perform any unitary transformations directly before and after the continuous process of theorem 1, one can continuously decompose any measurement with* $\sum_{l=1}^{S(\mathbb{V})} \mathrm{rank}\,(\mathbb{B}_l)$ *distinct singular values.*

*Proof.* The endpoint measurement operators $M_1$, $M_2$ in theorem 1 can have up to $\sum_{l=1}^{S(\mathbb{V})} \mathrm{rank}\,(\mathbb{B}_l)$ distinct eigenvalues. We can decompose any pair of general endpoint operators $M_1$, $M_2$ using their polar decompositions $M_i = W_i (M_i^\dagger M_i)^{1/2}$. Then, we can use a procedure like that of Figure 1 to measure the positive Hermitian operators $(M_i^\dagger M_i)^{1/2}$ and subsequently apply $W_i$ depending on the measurement result. □

Whereas in [9] we were able to decompose, with a fixed interaction Hamiltonian, any measurement operator with 2 singular values, using the scheme presented here we are able to increase the number of singular values much higher, depending on the number and types of controls available.

## IV. FOUR-DIMENSIONAL EXAMPLE

We will now demonstrate how our technique can be used to characterize the continuous measurements possible with an interaction Hamiltonian of five ($d = 5$) controllable terms acting on a four-dimensional Hilbert space. We denote by $S$ the set of controllable terms $H_i$ in $H_{PS}$,

$$S = \{II, ZI, IZ, ZZ, XX\}. \tag{33}$$

Note that this set is *not* closed under anti-commutation since $\{XX, ZZ\} = 2YY$. We will use a simplified version of the procedure described in lemma 1 to find closed Jordan algebras contained in $S$. Strictly speaking, the optimal procedure in lemma 1 finds all algebras contained in $\mathbb{F} = \mathrm{span}\,\{S\}$ and our example below will ignore possible linear combinations of terms or any rewriting of the control set in a new basis. Nonetheless, dropping individual

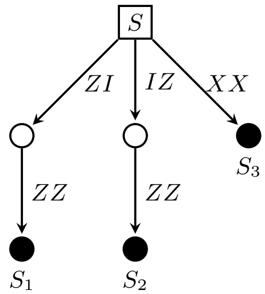


FIG. 2. The subsets of Hamiltonian terms resulting from the search for closed algebras as it progressively removes elements of $S$. Each node is an attempt to solve Eq. (11) and the labels on the edges are the elements removed from $S$ as a result.

terms from $S$ demonstrates the branching nature of the recursive search for closed algebras. The result of the search yields the following three subsets, each of which is closed under anti-commutation.

$$S_1 = \{II, IZ, XX\}, \tag{34}$$
$$S_2 = \{II, ZI, XX\}, \tag{35}$$
$$S_3 = \{II, ZI, IZ, ZZ\}. \tag{36}$$

Note that the identity $II$ along with any one of the Pauli operators in $S$ is also a closed algebra but since these are smaller than these above, we do not list them. The process of finding $S_1$, $S_2$, and $S_3$ is illustrated in Figure 2.

## V. THREE-DIMENSIONAL EXAMPLE

In the following example we illustrate how knowledge of the simple algebras listed in Table I can help reveal structure contained within a set of controllable Hamiltonian terms. Consider two nonorthogonal quantum states $|1\rangle$ and $|2\rangle$ such that $\langle 1|2\rangle = a$ with $a \in \mathbb{R}$. We consider the following set of control terms

$$\begin{aligned} H_0 &= I \\ H_1 &= |1\rangle\langle 1| \\ H_2 &= |2\rangle\langle 2| \\ H_3 &= |1\rangle\langle 2| + |2\rangle\langle 1| \\ S &= \{H_0, H_1, H_2, H_3\} \end{aligned}$$

One can check that $\mathbb{F} = \text{span}\{S\}$ is closed under anti-commutation. However, it is not clear which finite-dimensional Jordan algebra $\mathbb{F}$ is a representation of. To reveal the structure, consider rewriting $\mathbb{F}$ in the following basis

$$H_0' = H_1 \tag{37}$$
$$H_1' = \frac{a^2 H_1 + H_2 - aH_3}{1 - a^2} \tag{38}$$
$$H_2' = \frac{-2aH_1 + H_3}{\sqrt{1 - a^2}} \tag{39}$$
$$H_3' = H_0 - \frac{H_1 + H_2 - aH_3}{1 - a^2} \tag{40}$$

In this form, when we calculate the product $\frac{1}{2}\{\cdot, \cdot\}$, we find that

$$\begin{aligned} &\tfrac{1}{2}\{H_0', H_0'\} = H_0' && \tfrac{1}{2}\{H_3', H_0'\} = 0 \\ &\tfrac{1}{2}\{H_0', H_1'\} = 0 && \tfrac{1}{2}\{H_3', H_1'\} = 0 \\ &\tfrac{1}{2}\{H_1', H_1'\} = H_1' && \tfrac{1}{2}\{H_3', H_2'\} = 0 \\ &\tfrac{1}{2}\{H_2', H_0'\} = H_2' && \tfrac{1}{2}\{H_3', H_3'\} = H_3' \\ &\tfrac{1}{2}\{H_2', H_1'\} = H_2' && \\ &\tfrac{1}{2}\{H_2', H_2'\} = H_0' + H_1' && \end{aligned}$$

From these relations it is clear to see that $H_0'$, $H_1'$, and $H_3'$ are idempotents and that

$$\mathbb{F} = \text{span}\{H_0', H_1', H_2'\} \oplus \text{span}\{H_3'\} \tag{41}$$
$$\cong \mathcal{H}_2(\mathbb{R}) \oplus \mathbb{R}. \tag{42}$$

Armed with this decomposition, one can quickly and easily characterize the full set of continuous measurement decompositions possible using $S$. In this case, one can decompose any $M_1$ of the form

$$M_1 = \begin{bmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{bmatrix} \tag{43}$$

with $a, b, c, d \in \mathbb{R}$ and $M_2 = \sqrt{I - M_1^2}$.

## VI. CONCLUSIONS

In this work we've characterized the full class of continuous measurements achievable using a stream of probe qubits and a tunable interaction Hamiltonian. Given a set of linearly controlled Hamiltonian terms we provide a method to exhaustively list all continuous decompositions achievable with the control set. The class we find has a simple block-diagonal form, but results from a nontrivial application of the reversibility condition. Notably, measurements in this class have a quantifiably broader spectrum than in the case of a fixed interaction Hamiltonian.

Our work makes critical use of finite-dimensional Jordan algebras. This is surprising since these algebras have had little application elsewhere in quantum mechanics. Our model for continuous measurements does not include internal dynamics $H_S$ for the system or the probe, nor does it account for environment noise. In the presence of $H_S$, successive realizations of the continuous decomposition would yield inconsistent results unless $H_S$ commutes with the measurement operators.

The model presented here is still not the most general description of all continuous measurements realizable with a stream of probes. A completely general description would have to consider higher-dimensional probes, multiple outcomes to the weak measurement steps (as well as the endpoint measurements), and a more general reversibility condition. This is the subject of ongoing work. If Jordan algebras reappear in that scenario, then they will have found renewed application in quantum mechanics.

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