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Minimum time generation of $SU(2)$ transformations with asymmetric bounds on the controls*

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We study how to generate in minimum time special unitary transformations for a two-level quantum system under the assumptions that: (i) the system is subject to a constant drift, (ii) its dynamics can be affected by three independent, bounded controls, (iii) the bounds on the controls are asymmetric, that is, the constraint on the control in the direction of the drift is independent of that on the controls in the orthogonal plane. Using techniques recently developed for the analysis of $SU(2)$ transformations, we fully characterize the reachable sets of the system, and the optimal control strategies for any possible target transformation.

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I. INTRODUCTION

The implementation in minimum time of specific transformations is a key ingredient of many protocols requiring the manipulation of two-level quantum systems (as in quantum information theory [1], in quantum optics, or in atomic and molecular physics). Usually, one is interested in mapping in minimum time an initial state to a final state under specific conditions (see e.g. [2–4] and references therein). However, a general and convenient approach to this problem consists in considering as control target the transformations themselves, rather than the states of the underlying physical system. In our context, the problem can be formulated as an optimal control problem on the Lie group $SU(2)$ of special unitary transformations in 2 dimensions [5–7]. An arbitrary element of this group can be written as

$$X = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}, \quad (1)$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$. The dynamics of X is given by the Schrödinger operator equation

$$\dot{X} = (\omega_0 J_z + u_x J_x + u_y J_y + u_z J_z)X, \quad X(0) = I, \quad (2)$$

where J_k ($k = x, y, z$) are the skew-Hermitian generators of $SU(2)$, that is, independent elements of the Lie algebra $\mathfrak{su}(2)$, ω_0 is an arbitrary real parameter characterizing a constant drift term in the dynamics, and $u_k = u_k(t)$ are possibly time-dependent control actions constrained by

$$u_x^2 + u_y^2 \leq \gamma_1^2, \quad u_z^2 \leq \gamma_2^2, \quad (3)$$

where $\gamma_1 \geq 0$, $\gamma_2 \geq 0$. In other words, we assume that the strengths of the controls affecting the dynamics through J_z , or rather through generators depending on J_x and J_y , are independent.

Bounds on the controls depending on their squares, as in (3), naturally arise when the control actions are physically realized through fields, with energy proportional to their square amplitude. The choice of bounds in (3) corresponds to different driving strengths along the z direction, or along directions in an orthogonal plane. This means that there is an anisotropy in the problem, with a privileged direction in space, determined by the specific apparatuses which are used to steer the system. Stronger constraints can be given by choosing independent bounds for the three controls, $u_x^2 \leq \gamma_1^2$, $u_y^2 \leq \gamma_2^2$ and $u_z^2 \leq \gamma_3^2$, and have been considered elsewhere (for instance, see [2] and references therein). From a theoretical point of view, the analysis of the specific constraints considered in this work is relevant because it represents an intermediate situation between the problem with independently constrained controls, which has not been solved in the general case, and the problem with the isotropic bound $u_x^2 + u_y^2 + u_z^2 \leq \gamma^2$, which has been fully investigated [6, 7]. Physically, bounds on the controls depending on their squares are generically relevant in quantum information processing [1], in atomic and molecular physics, and in Nuclear Magnetic Resonance (NMR) [8].

The generators satisfy the standard $SU(2)$ commutation relations

$$[J_j, J_k] = J_l, \quad (4)$$

where (j, k, l) is a cyclic permutation of (x, y, z) . More explicitly, $J_k = -\frac{i}{2}\sigma_k$, where σ_k are the Pauli matrices.

The target of the control action is to steer the identity $I = X(0)$ to an arbitrary final operator $X_f = X(t_f)$ in minimum time t_f , through a suitable optimal control strategy $u_k(t)$. To determine this strategy, we will use the necessary condition of optimality provided by the *Pontryagin Maximum Principle* (PMP) [9, 10], which we briefly review. We introduce an auxiliary variable, the so-called *costate* $M \in \mathfrak{su}(2)$, represented by the coefficients

$$b_k = -\langle M, X^\dagger J_k X \rangle, \quad k = x, y, z, \quad (5)$$

where $b_k = b_k(t)$, and $\langle A, B \rangle \equiv \text{Tr}(AB^\dagger)$. Then we

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define the *Pontryagin Hamiltonian* as

$$H(M, X, v_x, v_y, v_z) = \omega_0 b_z + v_x b_x + v_y b_y + v_z b_z. \quad (6)$$

The PMP says that, if a control strategy $u_k(t)$ satisfying the bounds (3) and the corresponding trajectory $\tilde{X}(t)$ are optimal, then there exists a costate $\tilde{M} \neq 0$ such that

$$H(\tilde{M}, \tilde{X}, u_x, u_y, u_z) \geq H(\tilde{M}, \tilde{X}, v_x, v_y, v_z) \quad (7)$$

for all v_k satisfying (3). The PMP is only a necessary condition for optimality, useful for finding extremal control strategies and trajectories. The optimal strategy and trajectory are determined by comparing the extremal ones, analytically or numerically.

For its relevance, the control of $SU(2)$ operations has been extensively studied, with several constraints on the control protocols. In this work we follow the approach presented in [6], which allows an analytical investigation of the problem. In that paper, the cases with three or two controls were considered, with bounds $u_x^2 + u_y^2 + u_z^2 \leq \gamma^2$ and $u_x^2 + u_y^2 \leq \gamma^2$ respectively. We refer to this work for several technical details which are omitted here for brevity. Note that the present framework reduces to the case with two controls in [6, 7] when $\gamma_2 = 0$. Moreover, some optimal solutions in the case with three controls in [6] are also optimal solutions in the present case, with values of γ_1 and γ_2 depending on the specific trajectory. Therefore, our analysis complements that presented in [6, 7].

II. DETERMINATION OF EXTREMAL TRAJECTORIES

The differential equations describing the costate dynamics can be found by using (4) in (5), and they are given by

$$\begin{aligned} \dot{b}_x &= -(\omega_0 + u_z)b_y + u_y b_z, \\ \dot{b}_y &= (\omega_0 + u_z)b_x - u_x b_z, \\ \dot{b}_z &= u_x b_y - u_y b_x. \end{aligned} \quad (8)$$

After defining $\mu = \sqrt{b_x^2 + b_y^2}$, we find that $\mu^2 + b_z^2$ is a constant, which cannot vanish because $M \neq 0$. By maximizing the Pontryagin Hamiltonian (6) with the constraints (3), we determine the form of the extremal controls. There are three possible cases: (i) $\mu \equiv 0$; (ii) $b_z \equiv 0$; (iii) neither of them. In case (i), it follows from (8) that $u_x \equiv 0$ and $u_y \equiv 0$, and then b_z is a non-zero constant. Therefore, it must be

$$u_z = \gamma_2 \text{sign}(b_z). \quad (9)$$

In case (ii), from maximization of H we determine the form of the extremal controls

$$u_x = \gamma_1 \frac{b_x}{\mu}, \quad u_y = \gamma_1 \frac{b_y}{\mu}, \quad (10)$$

and u_z has values in the interval $[-\gamma_2, \gamma_2]$. Finally, in case (iii) both (9) and (10) must hold.

Having the form of the extremal controls, we can integrate (2) and determine the trajectories in $SU(2)$. Case (i) is trivial, and we obtain $X(t) = e^{i(\omega_0 \pm \gamma_2)\tau} I$, where $\tau = \frac{t}{2}$. We do not further consider this situation which does not provide optimal solutions. Cases (ii) and (iii) can be jointly integrated, by remembering that $-\gamma_2 \leq u_z \leq \gamma_2$ or $u_z = \pm \gamma_2$ in the two cases, respectively. By using these controls in (8), it is possible to prove that

$$b_x = \mu \cos(\omega t + \phi), \quad b_y = \mu \sin(\omega t + \phi), \quad (11)$$

where ϕ is a constant, and ω is a possibly time-dependent function given by

$$\omega = \omega(t) = \omega_0 + \frac{1}{t} \int_0^t u_z(s) ds \quad (12)$$

in case (ii), and the constant

$$\omega = \omega_0 + u_z - \gamma_1 \frac{b_z}{\mu} \quad (13)$$

in case (iii). Its range depends on the case under investigation: $\omega < \omega_0 + \gamma_2$ if $b_z > 0$, $\omega > \omega_0 - \gamma_2$ if $b_z < 0$, and $\omega_0 - \gamma_2 \leq \omega(t) \leq \omega_0 + \gamma_2$ if $b_z = 0$. We shall use ω rather than b_z and μ to identify extremal trajectories. Notice that we have been able to integrate (8) even in the case of time-dependent u_z (and then ω) because of the simple form of this system. By substituting (11) into (10), we find

$$u_x = \gamma_1 \cos(\omega t + \phi), \quad u_y = \gamma_1 \sin(\omega t + \phi). \quad (14)$$

By using these expressions in (2), and considering the representation of X given in (1), following the steps detailed in [6] (which can be simply readapted when ω is a function of time), we find

$$\begin{aligned} \alpha &= e^{-i\omega\tau} \left(\cos a\tau - i \frac{b}{a} \sin a\tau \right) \\ \beta &= -i \frac{\gamma_1}{a} e^{i(\omega\tau + \phi)} \sin a\tau, \end{aligned} \quad (15)$$

where we have rescaled time as $\tau = \frac{t}{2}$, and defined

$$a = a(\omega) = \sqrt{b^2 + \gamma_1^2}, \quad (16)$$

with $b = 0$ in case (ii), $b = b(\omega) = \omega_0 + u_z - \omega$ in case (iii).

Because of the form of the drift term and the bounds on the controls, the problem has a natural cylindrical symmetry. This is apparent from (15), where the phase of β can be arbitrarily modified through the parameter ϕ . In other words, all operators X differing by an off-diagonal phase are completely equivalent in the framework adopted in this work, and they can be reached in the same optimal time [7]. Consequently, we can fully

describe the extreme trajectories in $SU(2)$ by considering the evolution of α , or, more conveniently, its real and imaginary parts x and y respectively, which must satisfy $x^2 + y^2 \leq 1$. Therefore, we can represent the relevant trajectories in the unit disk of \mathbb{R}^2 (or, equivalently, in \mathbb{C}). When $b_z \neq 0$ they are given by

$$\begin{aligned} x_{\pm} &= \cos \omega \tau \cos a \tau - \frac{b}{a} \sin \omega \tau \sin a \tau \\ y_{\pm} &= -\sin \omega \tau \cos a \tau - \frac{b}{a} \cos \omega \tau \sin a \tau, \end{aligned} \quad (17)$$

with ω as in (13); when $b_z = 0$ they are

$$x_0 = \cos \omega \tau \cos \gamma_1 \tau, \quad y_0 = -\sin \omega \tau \cos \gamma_1 \tau, \quad (18)$$

with $\omega = \omega(t)$ as in (12).

In the analysis of extremal trajectories, it is often important to consider the so-called *singular* trajectories, that is, extremal solutions such that the Pontryagin Hamiltonian is independent of the controls (for the relevance of extremal trajectories in concrete problems, see for instance [11, 12] and references therein). By jointly considering (6) and (5), we can conclude that, in the context considered in this work, these solutions do not exist, because they would require $b_x \equiv b_y \equiv b_z \equiv 0$, which is inconsistent with the requirement $M \neq 0$. Only *regular* trajectories (i.e., non singular) have to be taken into account. Using a similar argument, we observe that it is impossible to concatenate the extremal trajectories described before. Again, this would require the vanishing of all the b_j at the switching time, which is not admitted.

III. EVOLUTION OF THE REACHABLE SETS

Following [6], we define the *optimal front-line* as the set of terminal points for a candidate optimal trajectory at time τ . As we have seen, depending on b_z , there are three families of extremal trajectories. Correspondingly, there are three optimal front-lines in \mathbb{R}^2 ,

$$\begin{aligned} \mathcal{F}_+(\tau) &\equiv \{(x_+, y_+), -\infty < \omega < \omega_0 + \gamma_2\}, \\ \mathcal{F}_-(\tau) &\equiv \{(x_-, y_-), \omega_0 - \gamma_2 < \omega < \infty\}, \\ \mathcal{F}_0(\tau) &\equiv \{(x_0, y_0), \omega_0 - \gamma_2 \leq \omega \leq \omega_0 + \gamma_2\}, \end{aligned} \quad (19)$$

or similar definitions in \mathbb{C} , in terms of α_0, α_{\pm} . We remind that ω is a constant for a given trajectory in \mathcal{F}_+ or \mathcal{F}_- , a possibly time-dependent function for an extremal in \mathcal{F}_0 .

The reachable set at time τ is, by definition, the set of operators in $SU(2)$ which can be reached in time smaller or equal than τ . The evolution of the reachable set of the system is determined by the evolution of the optimal front lines (19), in particular, by their intersections, where the trajectories could lose optimality. In the setting considered in [6], there is a unique optimal front line $\mathcal{F}(\tau)$, and, if we work in \mathbb{C} and forget for a while the bounds on ω , $\mathcal{F}_{\pm}(\tau)$ can be expressed in terms of it as

$$\mathcal{F}_{\pm}(\tau) = e^{\mp i \gamma_2 \tau} \mathcal{F}(\tau) \quad (20)$$

by means of suitable shifts in ω . Therefore, assuming again that ω can be any real number, we can write

$$\mathcal{F}_-(\tau) = e^{2i \gamma_2 \tau} \mathcal{F}_+(\tau), \quad (21)$$

that is, at time τ the two sets are mapped into each other by a rotation of angle $2\gamma_2 \tau$ in the unit disk. We observe that $\mathcal{F}_+(\tau) = \mathcal{F}_-(\tau)$ when $\tau = \frac{k\pi}{\gamma_2}$, with $k \in \mathbb{Z}$.

The individual analysis of $\mathcal{F}_+(\tau)$ and $\mathcal{F}_-(\tau)$ follows from that of $\mathcal{F}(\tau)$. We summarize the main results. First of all, there is a one-to-one correspondence between values of ω and points on \mathcal{F}_+ and \mathcal{F}_- , that is, the associated trajectories do not intersect in optimal conditions. Moreover, for each locus there is a critical trajectory [15] spiraling around the center of the disk, modified with respect to that corresponding to $\mathcal{F}(\tau)$ according to (20), and parameterized by the critical frequencies

$$\omega_c = \frac{\gamma_1^2 + (\omega_0 \pm \gamma_2)^2}{\omega_0 \pm \gamma_2}, \quad (22)$$

and losing optimality at the critical times

$$t_c = \frac{\pi |\omega_0 \pm \gamma_2|}{\gamma_1 \sqrt{(\omega_0 \pm \gamma_2)^2 + \gamma_1^2}}. \quad (23)$$

These trajectories can be cut loci for the system, that is, special lines where optimal trajectories lose their optimality. Other trajectories lose optimality on the border of the unit disk, which is, then, a cut locus for the system. At time τ , the frequencies corresponding to these trajectories are given by

$$\omega_{c'}(\tau) = (\omega_0 \pm \gamma_2) \pm \sqrt{\left(\frac{\pi}{\tau}\right)^2 - \gamma_1^2}. \quad (24)$$

Note that, in (22) and (23), quantities with sign $+$ or $-$ refer to \mathcal{F}_+ or \mathcal{F}_- , respectively. The same applies for the first \pm sign in (24), but the second \pm sign depends on the specific case. It is possible to prove that the possible scenarios are $\omega_c > \omega_0 + \gamma_2 > 0$ or $\omega_c < \omega_0 + \gamma_2 < 0$ for \mathcal{F}_+ , and $\omega_c > \omega_0 - \gamma_2 > 0$ or $\omega_c < \omega_0 - \gamma_2 < 0$ for \mathcal{F}_- . Therefore, considering the allowed range of values for ω , we see that sometimes the critical trajectories are not extremal trajectories for the system. We can also refine the definition of the optimal front lines, neglecting contributions which are certainly sub-optimal. For instance, when $\omega_0 > 0$, the range of values of ω for $\mathcal{F}_+(\tau)$ is given by $\omega_{c'}(\tau) < \omega < \omega_0 + \gamma_2$. For $\mathcal{F}_-(\tau)$, it is $\omega_c < \omega < \omega_{c'}(\tau)$ when $\gamma_2 < \omega_0$, and $\omega_0 - \gamma_2 < \omega < \omega_{c'}(\tau)$ when $\gamma_2 > \omega_0$. Similar expressions can be found when $\omega_0 < 0$.

For small times (that is, in a neighborhood of $t = 0$) there is a one-to-one correspondence between values of ω and points in \mathcal{F}_0 . Nonetheless, from (12) we see that there are different control strategies $u_z = u_z(t)$ leading to the same ω , that is, those having the same time average. Therefore, in this case there are different trajectories converging to the same point of \mathcal{F}_0 , and they are all equivalent [16]. In other words, in the region spanned by

\mathcal{F}_0 , there are distinct extremal trajectories (corresponding to different control strategies) leading to the same final state in the same time, and remaining extremals after they intersect. This is not in contradiction with the results of [6], where the optimal solution is unique and u_z is constant, because, in general, the optimal solutions for the case of asymmetric bounds parameterized by γ_1 and γ_2 are not optimal solutions for the problem with symmetric bound given by $\gamma^2 = \gamma_1^2 + \gamma_2^2$.

Since $x_0^2 + y_0^2 = \cos^2 \gamma_1 \tau$, the optimal front-line $\mathcal{F}_0(\tau)$ is an arc of circle centered at the origin, with time-dependent radius $\cos \gamma_1 \tau$, and angle $2\gamma_2 \tau$. If $2\gamma_2 \tau \geq 2\pi$, there are several values of ω corresponding to the same point on the optimal front-line, and the corresponding extremal trajectories become equivalent, that is, they attain the same point at the same time even if the time average of $u_z(t)$ is different.

By considering the definition of $\mathcal{F}_0(\tau)$ and $\mathcal{F}_{\pm}(\tau)$, we see that these three loci are connected. Moreover, by using implicit differentiation, we find that

$$\frac{dy_0}{dx_0} = \frac{dy_+}{dx_+} = \frac{dy_-}{dx_-} = \cot \omega \tau, \quad (25)$$

therefore they are smoothly connected. It is possible to consider as optimal front line for this problem the union of these three loci.

To complete the analysis, we must consider the intersections between $\mathcal{F}_+(\tau)$, $\mathcal{F}_-(\tau)$ and $\mathcal{F}_0(\tau)$ at any time τ . It turns out that $\mathcal{F}_0(\tau)$ never intersects $\mathcal{F}_+(\tau)$ or $\mathcal{F}_-(\tau)$ unless it is sub-optimal (and then these intersections are irrelevant for the characterization of the evolution of the reachable sets of the system). The intersection of $\mathcal{F}_+(\tau)$ and $\mathcal{F}_-(\tau)$ can be found numerically. The two endpoints of \mathcal{F}_+ and \mathcal{F}_- , associated with $\omega = \omega_0 + \gamma_2$ and $\omega = \omega_0 - \gamma_2$ respectively, coincide in two cases: either when $\tau = \frac{\pi}{\gamma_2}$ (when the two endpoints of \mathcal{F}_0 overlap), or when $\tau = \frac{\pi}{2\gamma_1}$, when the radius of \mathcal{F}_0 vanishes.

IV. TYPOLOGIES OF EVOLUTION OF THE REACHABLE SETS AND EXAMPLES

We can sum up the previous results, and classify the systems in four classes, depending on the specific values of ω_0 , γ_1 and γ_2 . They correspond to different forms of the optimal trajectories, producing different time-evolutions of the reachable sets. A given target operator $X_f \in SU(2)$ will require different control strategies (and associated minimum time t_f) depending on the case at hand. For sake of simplicity we assume $\omega_0 \geq 0$, but a completely analogous classification can be given also in the case $\omega_0 < 0$.

First of all, we observe that, since $\omega_0 + \gamma_2 \geq 0$ the locus \mathcal{F}_+ rotates counter-clockwise in the unit disk. The sense of rotation of \mathcal{F}_- depends on the sign of $\omega_0 - \gamma_2$, therefore the evolution of the reachable set is radically different in the two cases $\gamma_2 > \omega_0$ and $\gamma_2 < \omega_0$. Similarly, the long-time evolution of the reachable set depends

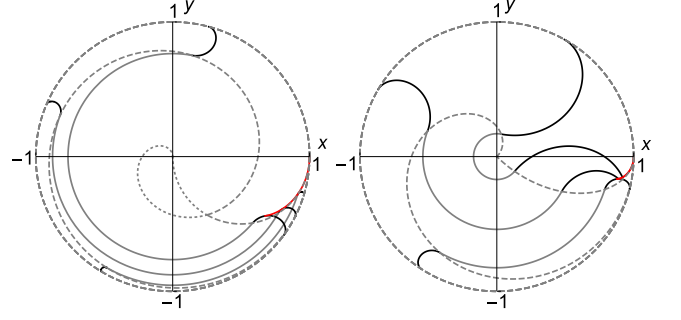


FIG. 1: (Color online) Time evolution of the reachable sets in the unit disk, with $\omega_0 = 4$, $\gamma_1 = 1$ (left plot) or 2 (right plot), $\gamma_2 = 3$. We have represented the optimal-front line at successive times $t = 0.6, 1.0$ and 1.4 . The gray curve is \mathcal{F}_0 , the dashed and dotted lines represent the evolution of its endpoints, before and after they converge, respectively. The region spanned by \mathcal{F}_0 and enclosed in the dashed lines contains points which can be reached via several equivalent optimal protocols (and, possibly, with different time average of $u_z(t)$ in the region enclosed in the dotted lines). The critical trajectory associated with \mathcal{F}_- is a cut locus for the system. The other cut loci, not shown in the plot, are the border of the unit disk and the set of intersections between \mathcal{F}_+ and \mathcal{F}_- .

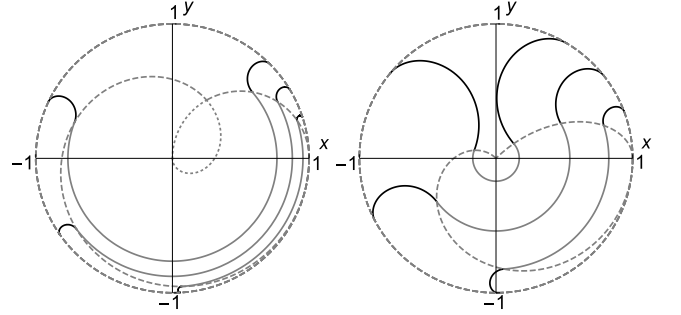


FIG. 2: Time evolution of the reachable sets in the unit disk, with $\omega_0 = 2$, $\gamma_1 = 1$ (left plot) or 2 (right plot), $\gamma_2 = 3$, at successive times $t = 0.6, 1.0$ and 1.4 . The cut loci for the system, not shown in the plot, are the border of the unit disk and the set of intersections between \mathcal{F}_+ and \mathcal{F}_- .

on the relative magnitude of γ_1 and γ_2 . Following the discussion of the previous section, if $2\gamma_1 \geq \gamma_2$ there are extremal trajectories with ending points on \mathcal{F}_+ and \mathcal{F}_- which can get arbitrarily close to the center of the unit disk (corresponding to the SWAP operator). Otherwise, when $2\gamma_1 < \gamma_2$ there is a disk of radius $\cos \pi \frac{\gamma_1}{\gamma_2}$, centered about the origin, whose points can only be reached by trajectories associated with \mathcal{F}_0 . The relative magnitude between ω_0 and γ_1 determines how many times the optimal front lines spiral around the origin before exhibiting the aforementioned features. For a graphical representations of the evolution of the reachable set with several choices of the parameters see Figs 1 and 2. In the first case we have $\gamma_2 > \omega_0$, in the second case $\gamma_2 < \omega_0$, with two different settings for $2\gamma_1$ and γ_2 in both cases.

Note that, if $\gamma_2 = 0$, the optimal front-line \mathcal{F}_0 collapses to a point and the analysis consistently reproduces that of [6] in the case of two independent controls.

To further illustrate the behavior of the reachable sets, we provide some examples of optimal synthesis of some standard unitary gates, and compare with the corresponding results in the case of symmetric bounds.

In the case of diagonal target operators, $X_f = e^{i\lambda\sigma_z}$ with $\lambda \in [0, 2\pi)$, since these are represented by points on the unit circle, which are reached by the trajectories forming \mathcal{F}_+ or \mathcal{F}_- , the optimal control strategies are given by controls u_x and u_y as in (14) with $\omega = \omega_{c'}$ as in (24), and $u_z = \pm\gamma_2$, for \mathcal{F}_\pm respectively. From (20) and the results in [6], the optimal time is given by

$$t_f = 2 \min_{u_z = \pm\gamma_2} \left(\frac{(\pi - \lambda)(\omega_0 + u_z) + \Omega}{(\omega_0 + u_z)^2 + \gamma_1^2} \right), \quad (26)$$

where $\Omega = \sqrt{\pi^2(\omega_0 + u_z)^2 + \lambda(2\pi - \lambda)\gamma_1^2}$. In particular, according to the former discussion on the evolution of the reachable sets, the minimum is obtained with $u_z = \gamma_2$ when $\gamma_2 < \omega_0$; when $\gamma_2 > \omega_0$ the situation is more complicated. The optimal time (26) and the corresponding strategies can be compared with analogous time and strategies in the case of a symmetric bound on the controls [6],

$$t_f = \begin{cases} \frac{4\pi - 2\lambda}{\gamma_1 + \omega_0}, & \text{if } \omega_0 \geq \frac{\pi - \lambda}{\pi} \gamma \\ \frac{2\lambda}{\gamma - \omega_0}, & \text{if } \omega_0 < \frac{\pi - \lambda}{\pi} \gamma \end{cases} \quad (27)$$

obtained with $u_x = u_y = 0$ and $u_z = \gamma$ or $-\gamma$. If we require $\gamma^2 = \gamma_1^2 + \gamma_2^2$ (that is, the total control strength is the same), we have that the time (27) is smaller than (26), since the scenario with asymmetric control bounds is compatible with a symmetric bound. As a check of consistency, this result can be proven by using the Lagrange multipliers method on (26), leading to the constrained minimum (27) obtained when $\gamma_1 = 0$ and $\gamma_2 = \gamma$.

As a second example, we choose as target operator the SWAP operator $X_f = i\sigma_y$, which represents the NOT operation in quantum information. In this case, the behavior of the optimal trajectories is described by the optimal front line \mathcal{F}_r , which we have analyzed in the previous section. The optimal strategies are given by u_x and u_y as in (14) with ω taking an arbitrary value in the interval $[\omega_0 - \gamma_2, \omega_0 + \gamma_2]$, possibly time-dependent. The control u_z can take any form, in particular $u_z = 0$ can be chosen, leading to $\omega = \omega_0$. For any choice of the parameters, the SWAP operator is attained in optimal time $t_f = \frac{\pi}{\gamma_1}$, which is independent on γ_2 , consistently with the fact that u_z is completely irrelevant for the optimal synthesis of this operator. In the case of a symmetric bound on the controls, the optimal control strategy has a similar structure [6], with $u_x = \gamma \cos(\omega_0 t + \varphi)$, $u_y = \gamma \sin(\omega_0 t + \varphi)$ and $u_z = 0$ (φ is a phase), and the optimal time is $t_f = \frac{\pi}{\gamma}$. Again, this optimal time is smaller than that with asymmetric bound on the controls, under the assumption that $\gamma^2 = \gamma_1^2 + \gamma_2^2$. It is clear that the case with symmetric bound is reproduced when $\gamma_2 = 0$ and $\gamma_1 = \gamma$.

V. CONCLUSIONS

We have fully characterized the evolution of the reachable sets of system (2) with controls subject to the asymmetric constraints (3). We have derived the optimal trajectories and the optimal control strategies for the system, and provided its classification in terms of the dynamical parameters, which are completely arbitrary: driftless dynamics or unbounded control actions are special cases of this treatment. By using this analysis, it is possible to compute, at least numerically, the minimum time for generating an arbitrary $SU(2)$ transformation, and the required strategy. For sake of clarity, we have analyzed the cases of diagonal operations, and the SWAP transformation of a two-level system. These examples clearly illustrate the role of the constraints on the controls in the study of time-optimal synthesis of quantum operations.

The main tool for studying the evolution of the reachable sets is the optimal front-line, which has already proven useful for the investigation of the minimum-time synthesis of $SU(2)$ operations. Its application to similar problems on different Lie groups is of great potential interest. It does not only provide a way to clearly visualize the behavior of the reachable sets, but also a simple approach to prove rigorous results, whose derivation when following the separate trajectories could be cumbersome in some regions of the space of dynamical parameters.

For instance, the results in $SU(2)$ can be used to perform a similar analysis in $SO(3)$ (because of the standard homomorphism connecting these groups) and are therefore related to the problem of attitude control of a rigid body. In this context, asymmetric bounds on the controls, as those considered in this work, are especially relevant for the treatment of rigid bodies with rotational symmetry about one axis.

Another problem where the specific investigations presented in this paper could be of relevance is the synthesis of $SU(2)$ operations with individual bounds on u_x , u_y and u_z . Also in this case the problem has more degrees of freedom, since the cylindrical symmetry of this work and [6, 7] is broken. However, the analysis of suitable optimal front lines could provide new insights for the investigation of regions, in space of dynamical parameters, which have not been studied so far.

Generalizations of this technique to problems characterized by an higher number of degrees of freedom seems a promising research line. Despite in these cases it seems difficult to obtain the particularly simple representation of the evolution of the reachable sets arising in $SU(2)$, we believe that an approach based on the study of the envelopes of the front lines is more promising than a direct analysis of the trajectories. A prospective direction for this line of research is the study of the optimal implementation of two-qubit gates, or, more generally, the simultaneous control of two spins (see [13, 14] for some recent applications of the PMP principle in this context).

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 - [15] When $\omega_0 = \pm\gamma_2$, there is only one critical trajectory.
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