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In a three-body system of identical bosons interacting through a large S-wave scattering length a, there are several sets of Efimov features related by discrete scale invariance. Effective field theory was recently used to derive universal relations between these Efimov features that include the first-order correction due to a nonzero effective range r_s. We reveal a simple pattern in these range corrections that had not been previously identified. The pattern is explained by the renormalization group for the effective field theory, which implies that the Efimov three-body parameter runs logarithmically with the momentum scale at a rate proportional to r_s/a. The running Efimov parameter also explains the empirical observation that range corrections can be largely taken into account by shifting the Efimov parameter by an adjustable parameter divided by a. The accuracy of universal relations that include first-order range corrections is verified by comparing with various theoretical calculations using models with nonzero range.

Introduction. Dramatic experimental and theoretical progress in few-body physics has been stimulated by the realization that Feshbach resonances can be used to control the strength of interatomic interactions in ultracold atomic gases. The *low-energy universality* that arises when scattering lengths are much larger than the range of interactions is of particularly broad interest. It implies that for systems as disparate as atoms, hadrons, and nuclei, dimensionless combinations of few-body observables are the same, despite orders of magnitude differences in the length and energy scales.

A particularly fascinating class of low-energy universal behavior is *Efimov physics*, which is characterized by *discrete scale invariance* [1, 2]. The ability to control the strength of interatomic interactions has allowed the observations of various aspects of Efimov physics in ultracold atoms [3]. The simplest example of Efimov physics is the *Efimov effect*: in the unitary limit where the scattering length a is infinite, there are infinitely many three-body bound states with an accumulation point at the 3-particle scattering threshold [4]. These Efimov trimers have binding energies $E_{T,n}$ whose limiting behavior is

$$E_{T,n} \longrightarrow \lambda^{-2n} \hbar^2 \kappa_*^2 / m \quad \text{as } n \to \infty, \qquad (1)$$

where m is the particle mass and λ is called the *discrete scaling factor*. The limit in Eq. (1) defines a three-body interaction parameter κ_*. In the case of identical bosons,

$\lambda = e^{\pi/s_0} = 22.6944$ with $s_0 = 1.00624$. The discrete spectrum in the unitary limit in Eq. (1) reveals that low-energy physics in the three-particle sector depends not only on the scattering length a but also on κ_*. Efimov physics can also be revealed through discrete scale invariance at finite values of a. For example, the negative scattering lengths $a_{-,n}$ at which the Efimov trimers cross the three-boson threshold have the limiting behavior

$$a_{-,n} \longrightarrow \theta_- \lambda^n \kappa_*^{-1} \quad \text{as } n \to \infty, \qquad (2)$$

where $\theta_- = -1.50763$ [5] is a universal number.

The discrete scale invariance that characterizes Efimov physics becomes exact, with the limits in Eqs. (1) and (2) replaced by equalities, only in the limit of zero-range interactions. In that limit, few-body observables depend only on a and κ_*. The interactions in real physical systems always have nonzero range. Experiments naturally involve the deepest Efimov trimers, for which range corrections are largest. In order to make quantitative experimental tests of Efimov physics, it is important to understand the range corrections in detail.

Range corrections can be studied theoretically by calculating Efimov features in various models for interatomic interactions and comparing them with the universal zero-range predictions. Kievsky and Gattobigio discovered empirically that range corrections can be largely taken into account by making substitutions for a and κ_* in zero-range formulas [6]. Their substitution for the Efimov parameter is $\kappa_* \to \kappa_* + \Gamma/a$, where Γ is determined empirically for each observable and each system.

A systematic theoretical approach to the problem is to organize range corrections into an expansion in powers of the range r_0 [7]. In the two-body sector, the only new

parameter that enters through second order in r_0 is the S-wave effective range r_s. Bedaque et al. showed that three-body observables do not depend on any additional three-body parameter at first order in r_0 [8], but they made the implicit assumption that the scattering length a is fixed. Ji, Phillips, and Platter (JPP) showed that if variations in the scattering length are considered, there is an additional three-body parameter at first order in r_0 [9]. They developed an effective field theory (EFT) framework for calculating range corrections [9, 10], and used it to derive universal relations between Efimov features, such as $E_{T,n}$ and $a_{-,n}$, that are accurate to next-to-leading order (NLO) in r_0/a.

In this paper, we reveal a pattern in the first-order range corrections to Efimov features that was not recognized in Refs. [9, 10]. We demonstrate that the pattern has a simple renormalization-group interpretation in terms of a *running Efimov parameter* that runs with the momentum scale at a rate proportional to r_s/a. The empirical shift of the Efimov parameter used in Ref. [6] to incorporate range corrections into zero-range formulas can be identified as the expansion of the running Efimov parameter to first order in r_s/a. We compare the predictions of the NLO universal relations with results for Efimov features in specific models.

Efimov Features. The Efimov trimers can be labelled by an integer n. In the 3-atom sector, the most dramatic Efimov features associated with the nth branch of Efimov trimers are (a) the binding momentum $\kappa_{T,n} = (mE_{T,n}/\hbar^2)^{1/2}$ of the Efimov trimer in the unitary limit $a = \pm\infty$, (b) the negative scattering length $a_{-,n}$ at which the Efimov trimer crosses the 3-atom threshold, (c) the positive scattering length $a_{*,n}$ at which the Efimov trimer disappears through the atom-dimer threshold, and (d) the positive scattering length $a_{+,n}$ at which there is an interference minimum in the three-body recombination rate at threshold.

The three-body interaction parameter κ_* defined by Eq. (1) is approximately equal to the binding momentum of the Efimov trimer labelled $n = 0$. The binding momenta are $\kappa_{T,n} = \lambda^{-n}\kappa_*$ in the zero-range limit. JPP showed that this equation remains exact, with no range corrections, at first order in r_0/a [11]. In the universal zero-range limit, the ratio of any pair of Efimov features is a universal number. The leading order (LO) universal relations are

$$a_{i,n} = \lambda^n \theta_i \kappa_*^{-1}, \qquad (3)$$

where $\theta_- = -1.50763$, $\theta_+ = |\theta_-|/\sqrt{\lambda} = 0.316473$ [5], and $\theta_* = 0.0707645$ [1].

First-order Range Corrections. JPP developed an EFT framework for calculating range corrections as strict expansions in powers of r_0 [9, 10]. At first order in r_0, two Efimov features are required as inputs, with at most one being a trimer binding momentum (or κ_*). A simple choice for the two Efimov features is κ_* and $a_{-,0}$. The deviation from the zero-range prediction for $a_{-,0}$ can be expressed using Eq. (3) as $1/a_{-,0} = \theta_-^{-1}\kappa_* + \mathcal{I}\kappa_*^2 r_s$, which

defines a nonuniversal number $\mathcal{I}$. JPP showed that the range expansion to first order in the range for any other Efimov feature can then be expressed as

$$1/a_{i,n} = \lambda^{-n}\theta_i^{-1}\kappa_* + (\xi_{i,n} + \eta_{i,n}\mathcal{I})\,\kappa_*^2 r_s\,, \qquad (4)$$

where $\xi_{i,n}$ and $\eta_{i,n}$ are universal numbers. JPP calculated many such numbers to at least three digits [9, 10].

There is a pattern to the dependence of the universal numbers in Eq. (4) on the number n labelling the branch of Efimov trimers that was not identified in Ref. [10]. The range expansion in Eq. (4) can be expressed in the much simpler form

$$a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i - n\sigma)r_s, \qquad (5)$$

where $\sigma = 1.095$ is a universal number. The differences between the coefficients J_i are also universal numbers: $J_+ - J_- = 0.548$, $J_* - J_- = 1.250$. We will refer to Eq. (5) as the *NLO range expansion* for the Efimov feature.

Renormalization. The n in the range correction in Eq. (5) can be understood as a logarithmic dependence on the scale at which the observable $a_{i,n}$ is measured, suggesting a renormalization-group interpretation. We therefore discuss the renormalization of the EFT used to calculate range corrections in Refs. [9, 10]. The Lagrangian density for that EFT, which has an atom field ψ and a molecule field ϕ, is

$$\begin{aligned} \mathcal{L} = {} & \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m}\right)\psi - \phi^\dagger\left(i\partial_0 + \frac{\nabla^2}{4m} - \Delta\right)\phi \\ & - \frac{g}{\sqrt{2}}\left(\phi^\dagger\psi\psi + \text{h.c}\right) + h\phi^\dagger\phi\psi^\dagger\psi. \end{aligned} \qquad (6)$$

The parameters g and Δ can be tuned as functions of the ultraviolet cutoff, Λ, so that a and r_s have the desired values. In the three-body sector, the ultraviolet cutoff can be implemented as an upper limit Λ on the loop momentum in a modified Skorniakov–Ter-Martirosian equation [12] for the scattering of three bosons [13]. The three-body coupling constant can be expressed as $h = 2mg^2 H/\Lambda^2$, where H is a dimensionless log-periodic function of Λ. Its dependence on Λ in the zero-range limit was determined up to a numerical factor in Ref. [13]:

$$H_0(\Lambda/\kappa_*) = 0.879\frac{\sin[s_0\log(\Lambda/\Lambda_*) + \arctan s_0]}{\sin[s_0\log(\Lambda/\Lambda_*) - \arctan s_0]}\,, \quad (7)$$

where $\Lambda_* = 0.548\,\kappa_*$. The multiplicative constant in Eq. (7) was first determined in Ref. [14]. The dependence of H on Λ at first order in the range has the form [10]

$$\begin{aligned} H(\Lambda) = {} & H_0(\Lambda/\kappa_*) + h_{10}(\Lambda/\kappa_*)\Lambda r_s \\ & + \left[\gamma\, H_0'(\Lambda/\kappa_*)\log(\Lambda/\mu_0) + \tilde{h}_{11}(\Lambda/\kappa_*)\right]\frac{r_s}{a}, (8) \end{aligned}$$

where μ_0 is a momentum scale and $H_0' = (\Lambda d/d\Lambda)H_0$ is the logarithmic derivative of H_0 in Eq. (7). We will not need the analytic forms of the log-periodic functions h_{10} and $\tilde{h}_{11}$. JPP gave an analytic expression for γ which is

within 5% of the value we find numerically, $\gamma = 0.351$. In Ref. [10], μ_0 was chosen so that a second Efimov feature in addition to κ_*, such as $a_{-,0}$, had no first-order range corrections. The order of magnitude of the required value of μ_0 is then $1/|a_{-,0}|$.

Running Efimov Parameter. The term proportional to $\log(\Lambda/\mu_0)$ in Eq. (8) represents a logarithmic violation of discrete scale invariance. It can be absorbed into the zero-range coupling constant by changing the argument of H_0 to $(\Lambda/\kappa_*)(\Lambda/\mu_0)^{\gamma r_s/a}$. The power of μ_0 in the argument can be canceled by replacing κ_* by a *running Efimov parameter* defined by

$$\bar{\kappa}_*(\mu_0, a) \equiv (\mu_0/\kappa_*)^{-\gamma r_s/a} \kappa_* . \qquad (9)$$

Unnecessarily large logarithms in the range corrections to an observable dominated by the momentum scale Q can be avoided by expressing the observable in terms of $\bar{\kappa}_*(Q, a)$, a, and r_s instead of κ_*, a, and r_s.

In the NLO range expansion in Eq. (5), the term linear in n gives unnecessarily large range corrections if n is large. Natural scaling variables in the zero-range limit are the inverse scattering length $1/a$ and the energy variable $K = \text{sign}(E)(m|E|/\hbar^2)^{1/2}$. The momentum scale Q for an observable with energy E at scattering length a is $Q = (K^2 + 1/a^2)^{1/2}$. The momentum scales for the Efimov features $a_{-,n}$, $a_{+,n}$, and $a_{*,n}$ are $1/|a_{-,n}|$, $1/a_{+,n}$, and $\sqrt{2}/a_{*,n}$, respectively. Since a factor of $\sqrt{2}$ is not essential, these can be summarized via the LO universal relation as $Q \approx \lambda^{-n}\kappa_*/|\theta_i|$. If we replace κ_* in the LO universal relation in Eq. (3) by $\bar{\kappa}_*(Q, a)$ where $a = \lambda^n \theta_i \kappa_*^{-1}$ and expand in powers of r_s, we obtain a term $-n\gamma r_s \log \lambda$. This matches the $-n\sigma r_s$ term in Eq. (5) provided $\sigma = \gamma \log \lambda$. Our numerically determined values of γ and σ satisfy this condition to within numerical accuracy. Replacing κ_* in the LO universal relation by $\bar{\kappa}_*(\lambda^{-n}\kappa_*/|\theta_i|, \lambda^n\theta_i\kappa_*^{-1})$ therefore includes the NLO correction proportional to nr_s.

Similarly, renormalization-group improvement of the NLO range expansion can be obtained by eliminating κ_* in Eq. (5) in favor of the appropriate running Efimov parameter and demanding agreement to first order in r_s:

$$a_{i,n} = \lambda^n \theta_i \left(\lambda^n|\theta_i|\right)^{-\gamma r_s \kappa_*/(\lambda^n\theta_i)} \kappa_*^{-1} + \tilde{J}_i r_s, \qquad (10)$$

where $\tilde{J}_i = J_i + \gamma \log|\theta_i|$. The differences between the coefficients $\tilde{J}_i$ are universal numbers: $\tilde{J}_+ - \tilde{J}_- = 0.000$, $\tilde{J}_* - \tilde{J}_- = 0.177$. We will refer to Eq. (10) as the *RG-improved NLO range expansion* for the Efimov feature. For $n = 1$, the NLO range expansion in Eq. (5) and the RG-improved NLO range expansion in Eq. (10) have the same parametric accuracy: higher-order range corrections are suppressed by a factor of $(\kappa_* r_s)^2$. For large n, the NLO range expansion has corrections of order $n^2(\lambda^{-n}\kappa_* r_s)^2$. In the RG-improved NLO range expansion, those higher-order range corrections that are enhanced by a factor of n for every factor of r_s are summed up to all orders. Thus the higher-order range corrections to Eq. (10) are at most of order $n(\lambda^{-n}\kappa_* r_s)^2$.

Comparisons with Models. Successive Efimov features have been calculated in several models with nonzero range. We can use the results to illustrate the accuracy of the NLO range expansion in Eq. (5) and the RG-improved NLO range expansion in Eq. (10). The NLO range expansion can be used to express $a_{i,n+1}/(\lambda a_{n,i})$ as a linear combination of $\kappa_* r_s$ and $J_i\kappa_* r_s$ with universal coefficients. For any three ratios of successive Efimov features, there is a linear combination with universal coefficients in which $\kappa_* r_s$ is eliminated. We refer to this equation as an *NLO universal relation.* If two of the ratios are taken as inputs, the third can be predicted to NLO accuracy without knowing r_s. Similarly the RG-improved NLO range expansion can be used to derive a universal relation that expresses $\log(a_{i,n+1}/(\lambda a_{i,n}))$ as a linear combination of two other such expressions with universal coefficients.

TABLE I: Ratios of successive Efimov features $a_{-,n+1}/a_{-,n}$ divided by the discrete scaling factor λ. The ratios of the features calculated by Deltuva [15] are compared to the predictions of Eq. (5) (NLO) and Eq. (10) (RG-NLO) using the numbers in square brackets as inputs.

	$(a_{-,n+1}/a_{-,n})/\lambda$				
n	0	1	2	3	4
Ref. [15]	0.7822	0.9665	0.9976	0.9999	1.0000
NLO	[*0.7822*]	[*0.9665*]	0.9975	0.9998	1.0000
RG-NLO	[*0.7822*]	[*0.9665*]	0.9975	0.9998	1.0000

In Ref. [15], Deltuva calculated the scattering lengths at which universal tetramers cross the 4-boson threshold for identical bosons interacting through a separable Gaussian potential. He also gave accurate results for the ratios $a_{-,n+1}/a_{-,n}$ of the scattering lengths at which 6 successive Efimov trimers cross the three-boson threshold. These ratios divided by λ are given in Table I. They rapidly approach 1 as n increases. In Table I, we have taken the ratios $a_{-,n+1}/a_{-,n}$ for $n = 0$ and 1 as inputs, and then used NLO universal relations to predict the ratios for $n = 2$, 3, and 4. The predictions are in excellent agreement with the results calculated by Deltuva. The RG-improved NLO universal relation gives the same predictions to four digits.

Schmidt et al. calculated multiple Efimov features for a three-parameter model in which the interaction is a transition between an atom pair and a molecule with a Gaussian form factor [16]. The three independent parameters can be specified by a, r_s, and a dimensionless parameter $s_{\text{res}} = r_0/r_*$ obtained by dividing the range r_0 of the Gaussian form factor by a length r_* determined by its strength. They presented results for three sets of parameters with $s_{\text{res}} = 100$, 1, and 0.1. Their results for $a_{-,n+1}/a_{-,n}$ and $a_{*,n+1}/a_{*,n}$ divided by the discrete scaling factor λ are given in Table II. We have taken the ratios $a_{-,1}/a_{-,0}$ and $a_{-,2}/a_{-,1}$ as inputs, and then used NLO universal relations to predict the other ratios. All

TABLE II: Ratios of successive Efimov features $a_{i,n+1}/a_{i,n}$ divided by the discrete scaling factor λ. The ratios calculated by Schmidt et al. [16] are compared to the predictions of Eq. (5) (NLO) and Eq. (10) (RG-NLO) using the numbers in square brackets as inputs. The three blocks are for models with $s_{\text{res}} = 100$, 1, and 0.1.

	$(a_{-,n+1}/a_{-,n})/\lambda$			$(a_{*,n+1}/a_{*,n})/\lambda$		
n	0	1	2	0	1	2
Ref. [16]	0.753	0.962	0.998	0.175	1.764	1.029
NLO	[0.753]	[0.962]	0.997	−8.814	1.150	1.032
RG-NLO	[0.753]	[0.962]	0.997	0.0002	1.206	1.034
Ref. [16]	1.008	0.998	0.9998	0.757	0.983	1.001
NLO	[1.008]	[0.998]	0.9998	−0.431	0.986	1.002
RG-NLO	[1.008]	[0.998]	0.9998	0.240	0.986	1.002
Ref. [16]	1.156	1.012	1.0007	1.188	0.938	0.991
NLO	[1.156]	[1.012]	1.0007	0.449	0.869	0.990
RG-NLO	[1.156]	[1.012]	1.0008	0.916	0.887	0.990

predictions for $n = 2$ are in good agreement with the results of Ref. [16]. The predictions for $n = 1$ are also in reasonable agreement, except for the case $s_{\text{res}} = 100$. The large error in this case can be attributed to the feature $a_{-,0}$ being outside the window of universality. The RG-improved NLO universal relation gives slightly better predictions for $n = 1$ and dramatically better predictions for $a_{*,1}/a_{*,0}$, for which the NLO universal relation yields unphysical negative values in the $s_{\text{res}} = 100$ and 1 models.

Shift in Three-body Parameter. In Ref. [6], Kievsky and Gattobigio calculated the binding energies of Efimov trimers and the atom-dimer scattering length in two models with a Gaussian two-body potential and with or without a Gaussian three-body potential. They made the empirical observation that range corrections could be largely taken into account by making substitutions in the zero-range formulas, which are functions of a and κ_* only: (a) replace a by the inverse binding momentum of the universal dimer or virtual state, (b) replace the Efimov parameter κ_* by $\kappa_* + \Gamma/a$, where Γ is a parameter that is determined empirically for each observable and each system. In Ref. [17], Garrido et al. showed that this prescription also works for the three-body recombination rate at threshold. The accuracy of the prescription was verified only in limited regions of a and for a few specific models. The running Efimov parameter provides a theoretical justification for the prescription. The expansion of the running Efimov parameter in Eq. (9) to first order in the range is

$$\bar{\kappa}_*(Q,a) \approx \kappa_* \left[1 - \gamma \log(Q/\kappa_*) r_s/a\right]. \qquad (11)$$

If the slow logarithmic dependence of the momentum scale Q on a is ignored, this has the same form $\kappa_* + \Gamma/a$ as the empirical shift in the Efimov parameter introduced by Kievsky and Gattobigio.

The NLO range expansion can be used to identify universal relations between the empirical constants Γ for different 3-body observables. The prescription in Ref. [17] for Efimov features associated with the second Efimov trimer is $a_{i,1}\kappa_{T,1} + \Gamma_i = \theta_i$. According to the NLO range expansion in Eq. (5), the ratio $(\Gamma_i - \Gamma_j)/(\kappa_{T,1} r_s)$ should be the universal number $J_j - J_i$. The predicted universal value of $J_* - J_+$ is 0.702, while the two models considered in Ref. [17] give the results 0.671 and 0.753. We consider this to be reasonable agreement. The corresponding ratios involving Γ_- display large discrepancies between the two models, and do not agree as well with the universal predictions.

Experiment. By measuring the first Efimov feature $a_-^{(0)}$ at different Feshbach resonances in ^{133}Cs atoms and comparing with previous measurements with other atoms, the Innsbruck group discovered a correlation to within 20% between $a_-^{(0)}$ and the coefficient of the van der Waals tail $-C_6/r^6$ of the interatomic potential: $a_{-,0} \approx -9\,r_{\text{vdW}}$, where $r_{\text{vdW}} = \frac{1}{2}(mC_6/\hbar^2)^{1/4}$ [18]. This correlation, which can be called *van der Waals universality*, was subsequently verified theoretically [19, 20]. Van der Waals universality gives a sharp prediction for the effective range: $r_s = 2.79 r_{\text{vdW}}$ [21]. It thus predicts a narrow range of values for the coefficient J_- in Eq. (5).

To use the NLO range expansion to predict Efimov features for a specific bosonic atom, the required inputs are the scattering length a, the effective range r_s, and two measured Efimov features. There are not many atoms for which there are accurate measurements of three or more Efimov features. Several Efimov features have been observed for both ^{7}Li and ^{133}Cs, but there are complications in ^{7}Li from significant variation of r_s and in ^{133}Cs from multiple Feshbach resonances. Comparison of NLO predictions with experimentally measured Efimov features will be presented elsewhere [22].

Summary. We have shown that universal relations which account for a nonzero effective range are able to predict Efimov features with higher accuracy. A simple pattern in the NLO range expansion can be interpreted in terms of a running Efimov parameter. The running Efimov parameter also explains the empirical findings of Gattobigio et al. that range corrections can be incorporated into universal zero-range results through a shift in the Efimov three-body parameter.

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