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Contagious error sources would need time travel to prevent quantum computation

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We consider an error model for quantum computing that consists of “contagious quantum germs” that can infect every output qubit when at least one input qubit is infected. Once a germ actively causes error, it continues to cause error indefinitely for every qubit it infects, with arbitrary quantum entanglement and correlation. Although this error model looks much worse than quasi-independent error, we show that it reduces to quasi-independent error with the technique of quantum teleportation. The construction, which was previously described by Knill, is that every quantum circuit can be converted to a mixed circuit with bounded quantum depth. We also consider the restriction of bounded quantum depth from the point of view of quantum complexity classes.

1. INTRODUCTION

Is quantum computation realistic even in principle? If we accept quantum mechanics (more precisely, quantum probability), then at the theoretical level this question is usually interpreted as the fault tolerance problem: Can a quantum computer still work if all of its gates and qubits are noisy? There are by now various fault-tolerance theorems for quantum computation, which establish that reliable quantum computation is indeed possible in principle assuming that the noise present in different qubits or gates is quasi-independent [1, 8, 11], and is below some threshold error rate. This threshold is called the fault tolerance constant. Thus, any remaining doubt that quantum computation is possible in principle reduces to one of three possibilities:

1. Quantum probability is not exactly true.
2. The fault tolerance constant is unattainable.
3. The quasi-independence assumption is too optimistic.

In this note, we will consider noise models with a relaxed version of the quasi-independence assumption, namely *contagious noise*. It seems possible that each qubit in a quantum computer might not just be noisy, but carry with it a noise source, a contagious “bug”, that spreads to all of the output qubits of each quantum gate. Each bug could get worse over time. Worse still, the descendants of the bug could be correlated and thus violate the quasi-independence assumption. If a quantum gate has two different bugs among its inputs, the bugs might also interact and make new bugs. Such possibilities come to mind given that one of the first bugs in the history of modern computing was an actual bug, a small moth [5]. That bug was no longer interacting with anything other than the relay switch where it had died. A “bug” can also mean a germ; at least in biological computers, germs can both replicate and affect data.

More realistically, contagious error is related to some forms of leakage error, where what was the state of a qubit leaves the

qubit Hilbert space and enters a larger Hilbert space. Knill has noted that leakage error is implicitly solved by teleportation [10], which is also the method that we will use. Leakage error is generally thought of as a measured error; if it is measured, it amounts to a qubit erasure and the qubit can be reset. However, before it is measured, leakage error can be contagious, since the effect of a quantum gate is undefined for leaked states.

In this article, we propose a mutual generalization of contagious germs and leakage error which we call *contagious quantum germs*. If a qubit has a Hilbert space $\mathcal{H}_Q \cong \mathbb{C}^2$, then we attach to it another Hilbert space \mathcal{H}_G , the germ state space, so that its total state space is $\mathcal{H}_Q \otimes \mathcal{H}_G$. At each time step, each qubit interacts with its germ and its germ evolves. At each gate, all of the germs of the input qubits interact in some way to make the germs of the output qubits. The only restrictions are that each fresh qubit is created with an independent germ, that the effect of a germ on its qubit is bounded above for the first few steps of its life, and that classical bits do not carry germs. Note that quantum germs not only strictly generalize leakage error, but can also do things that would be peculiar for leakage error. For instance, once activated, they can spread insidiously with no possibility of being detected, and then hit hard everywhere that they have spread. We have no argument that all types of quantum germs are realistic, only that they include various possibly realistic noise models as special cases.

Theorem 1.1. *With a constant overhead factor, every quantum circuit can be re-encoded so that noise from contagious quantum germs becomes quasi-independent.*

As a corollary, we can then apply the standard fault-tolerance theorem to conclude that quantum computation is possible in our model with polylogarithmic overhead.

A more detailed version of Theorem 1.1 is stated as Theorem 4.1 and Corollary 4.2, using the formalism defined in Section 2.

A closely related result is the “one-way” quantum fault tolerance proposal of Raussendorf, Harrington, and Goyal [14, 15]. This is a specific encoded computation with a high fault-tolerance constant, and which is one-way in the sense that after an initial encoded state is created, all of the computation is carried out by (adaptive) one-qubit measurements. Since in addition all of the parity checks of their code have bounded weight, their circuits automatically have bounded depth. Our Theorem 1.1 is both weaker and more general

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than the RHG construction. Without obtaining any new bound on the fault tolerance constant, it implies that any method of quantum fault tolerance whatsoever can be re-encoded in bounded quantum depth.

The idea of our proof of Theorem 1.1 is not original. The basic construction uses quantum teleportation [2]; it might first have been published by Knill [9]. We state the construction in terms of mixed quantum-classical circuits.

Theorem 1.2 (Knill). *With a constant overhead factor, every quantum circuit (or mixed classical-quantum circuit) can be re-encoded as a mixed classical-quantum circuit with bounded quantum depth.*

The surprising property of a mixed circuit with quantum teleportation is that, if we orient all of the qubit edges forward in time, then the qubit subgraph only needs to be weakly connected from its input to its output in order to transmit quantum information. (Recall that directed graph is *strongly* connected from a to b if it has a directed path from a to b ; and *weakly* connected if it merely has some connecting path with no restriction on the orientations of the edges.) In a sense, quantum information can travel backward in time, as long as classical information transports the result forward in time. Indeed, quantum teleportation is a reasonable description of *any* mixed circuit that transmits quantum information via a weakly connected quantum subgraph; in this sense, quantum information that travels backwards in time is just a slogan for teleportation. Theorems 1.1 and 1.2 then say that if we imagine that qubits (but not bits) are infected with contagious germs, then the germs would have to travel backward in time to prevent fault-tolerant quantum computation.

Given that we can loosely interpret a teleportation circuit as allowing quantum information to travel backwards in time, the question arises whether quantum germs could travel backwards in time in the same sense of teleportation. The answer is no, unless either:

1. Germs can infect classical bits.
2. Germs can travel on paths that are not in the circuit.

Given the many different ways that classical bits are encoded and communicated in practice, the first possibility seems implausible or at least avoidable. The second possibility certainly can happen in the sense that noise may not be limited to a quantum circuit in example physical implementations. In this case, the noise does not need to travel backwards in time either. However, if correlated noise can spread anywhere with no causal restriction other than that it travels forward in time, then it is well-known that both classical and quantum fault tolerance are impossible.

The authors were led to consider the constructions considered here by alternative error models proposed by the first author in which errors are convolved, or smoothed, in time [6]. Our basic observation led the first author to change his model to require convolution both forward and backward in time [7]. This leads to issues regarding causality that we will not discuss here.

Remark. If a mixed circuit is not even weakly quantumly connected, then it is equivalent to a model in quantum information theory known as “local operations and classical communication” [13, §12.5]. In particular, it is immediate that LOCC is weaker than full quantum communication, since it leaves no way to create quantum entanglement between weakly connected components of the circuit, and therefore no way to violate Bell-type inequalities.

To understand our hypothesis and our conclusion, it is important to distinguish between sources of error and erroneous qubits (or bits). An erroneous qubit is one whose state is different from what is intended in a quantum algorithm. If the intended state is a pure state $|\psi\rangle$ (or a density operator ρ), then the actual state might be some other state $|\psi'\rangle$ (or a density operator ρ'). Erroneous states propagate through gates and through quantum teleportation. In computer science in general, this is called *error propagation* and it is why computers need classical or quantum error correction. In fact, an error can propagate through a mixed path of classical and quantum edges in a mixed circuit; in particular, an error can propagate through the classical bits used in quantum teleportation. However, while errors can be corrected, by our rules error-causing germs cannot be removed. (Or, they can only be removed indirectly with teleportation.)

We will prove Theorem 1.2 in Section 3 and Theorem 1.1 in Section 4. Finally, in Section 5, we give a complexity theory interpretation of Theorem 1.2. One way to limit the power of quantum computation is to only allow bounded-time layers of it in between classical computation layers. (We do not mean pseudo-classical operators that are quantum but in the computational basis. Rather we mean classical data processing revealed to the environment.) We remark that if this is done asynchronously, then Theorem 1.2 implies that the resulting polynomial complexity class is exactly BQP.

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2. RIGOROUS DEFINITIONS

2.1. Mixed circuits

We consider the circuit model of computation. As usual, a circuit is a kind of acyclic, directed graph with labelled vertices which are called gates. In defining classical circuits carefully enough to generalize them to quantum circuits, we have to count bit copying as a gate with 1 input and 2 outputs. Also, every type of circuit that we consider in this paper can be assumed to be in a uniform circuit family, created by a classical Turing machine or similar.

The circuits of interest to us have two kinds of circuit edges, classical or bit edges, and quantum or qubit edges. In order to

understand what a general mixed gate can do with a combination of bit and qubit inputs and outputs, we can consider the *hybrid quantum memory* model [12]. In practice, we can simplify the definition of a mixed circuit to the following types of gates:

1. Deterministic classical gates acting on bits.
2. A measurement gate that converts a qubit to a bit.
3. Unitary quantum gates that may have classical control bits.
4. A gate that creates a fresh qubit in the state $|0\rangle$.

Since our circuits are uniformly generated, and for other reasons, we also want a finite set of quantum gates that densely generate unitary groups acting on qubits, such as the Hadamard and Toffoli gates. However, rather than proving something for every gate set, we interpret Theorem 1.1 as saying that there exists a set of gates such that the result holds with a constant overhead factor for those gates. (Changing gate sets requires the Solovay-Kitaev theorem, which has polylogarithmic overhead; we do not know that the stringent constant overhead factor can be satisfied by every universal gate set.) The only standard gates that we need for quantum teleportation (Figure 1), which is the only idea we need to prove Theorem 1.2, are Hadamard and CNOT gates, and 1-qubit Pauli gates with classical control bits.

Figure 1 has an example of a mixed circuit, with the qubit edges in black and the bit edges in gray. In general a circuit has a *total depth*, which is the length of its longest directed path; and a *quantum depth*, which is the length of its longest directed path following only qubit edges. The graph of a mixed circuit has a *quantum subgraph* consisting only of its qubit edges. (The specific gates used in the circuit are defined in Section 3.)

2.2. Quantum germs

As usual $U(n)$ is the group of unitary $n \times n$ matrices; let $M(n)$ be the vector space of all $n \times n$ matrices. If \mathcal{H} is a Hilbert space, then we let $U(\mathcal{H})$ be the corresponding abstract unitary group, and we let $M(\mathcal{H})$ be the abstract space of all operators on \mathcal{H} . The algebra $M(\mathcal{H})$ comes with an operator norm; by definition

$$\|A\| = \sup_{\langle \psi | \psi \rangle = 1} \langle \psi | A | \psi \rangle.$$

If \mathcal{H} is infinite-dimensional, then technically we take $M(\mathcal{H})$ to be the bounded operators, meaning those with finite operator norm. Recall in this case that a state ρ on \mathcal{H} is a positive semi-definite trace-class operator. (More precisely, if ρ is positive semi-definite, then it is *trace class* if the trace of its matrix defined using any orthonormal basis of \mathcal{H} is a convergent series.)

As mentioned in the introduction, each qubit has a Hilbert space $\mathcal{H}_Q \cong \mathbb{C}^2$, and a separate germ Hilbert space \mathcal{H}_G that could even be infinite-dimensional. If C is a quantum circuit

(or a mixed circuit), we expand it to an infected circuit C' as follows: If C creates a qubit with the state $|0\rangle \in \mathcal{H}_Q$, then C' also creates a germ in some initial state $|g_0\rangle \in \mathcal{H}_G$. For each gate

$$G \in U(\mathcal{H}_Q^{(1)} \otimes \mathcal{H}_Q^{(2)} \otimes \cdots \otimes \mathcal{H}_Q^{(k)})$$

that arises in C , there is a corresponding germ-mixing operator

$$M \in U(\mathcal{H}_G^{(1)} \otimes \mathcal{H}_G^{(2)} \otimes \cdots \otimes \mathcal{H}_G^{(k)})$$

that mixes the germ states. Finally, each edge of the circuit C is replaced with an error operator

$$E \in U(\mathcal{H}_G \otimes \mathcal{H}_Q).$$

Also, we do not assume that the operators M and E and the states $|g_0\rangle$ are the same at different positions in C . They must satisfy the error bound (4) below, but otherwise they can be different each time and they can be chosen adversarially rather than randomly.

The operators E are subject to an error bound which we explain carefully. Recall the relation

$$M(\mathcal{H}_G \otimes \mathcal{H}_Q) \cong M(\mathcal{H}_G) \otimes M(\mathcal{H}_Q).$$

Recall that $M(\mathcal{H}_Q) \cong M(2)$ can be given a Pauli basis

$$\begin{aligned} P_0 = I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & P_1 = X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ P_2 = Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & P_3 = Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

using any isomorphism $\mathcal{H}_Q \cong \mathbb{C}^2$. Then we can think of an operator $E \in M(\mathcal{H}_G \otimes \mathcal{H}_Q)$ as a superposition of operators acting only on \mathcal{H}_G :

$$E = \sum_{j=0}^3 E_j \otimes P_j. \quad (1)$$

The fact that E is unitary implies that $\|E_j\| \leq 1$, which we can read as saying that each E_j is subunitary; this will be useful in the proof of Theorem 1.1.

As a warm-up to the main argument, suppose that the germ at a given qubit edge has a pure state $|g\rangle \in \mathcal{H}_G$. Then the partial evaluation of E on $|g\rangle$ gives us a vector

$$\vec{F} \in \mathcal{H}_G \otimes M(\mathcal{H}_Q),$$

which then decomposes as

$$\vec{F} = \sum_{j=0}^3 |f_j\rangle \otimes P_j. \quad (2)$$

Here each $|f_j\rangle$ is a non-normalized state representing a vector-valued amplitude of the error mode. Following Knill, Laflamme, and Zurek [11, §I.B], we can define the size of this error as the sum of the norms of the output germ states $|f_j\rangle$

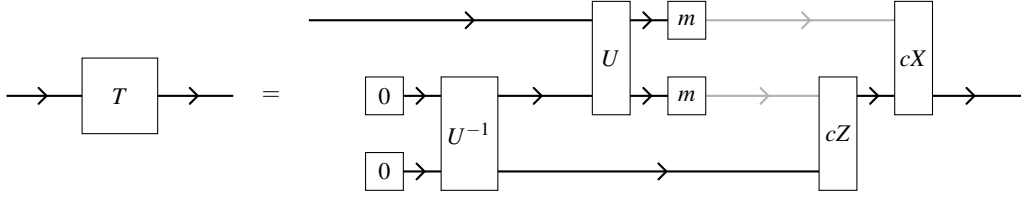


Figure 1. The teleportation circuit T , with qubit edges in black and bit edges in gray. The output is connected to the input by a path of qubit edges, but not by an *oriented* path.

other than the term for the identity. In other words, we can define an error seminorm

$$\|\vec{F}\| = \sum_{j=1}^3 \sqrt{\langle f_j | f_j \rangle}.$$

One subtle but standard point, which will be relevant in all of our error bounds, is that the vectors such as $|f_j\rangle$ need not be orthogonal. If they are orthogonal, then the different errors to which they are attached are stochastic; if they are parallel, then the errors are coherent or “stoquastic”.

If $\|\vec{F}\|$ is large, it means that the error operator E has a large effect on its qubit Q . We would like to bound $\|\vec{F}\|$; however for two reasons, we will not do this in all cases. The first reason is that the state ρ_G in general comes from a pure state that is entangled between many germs and computational qubits as well. The second reason is that we assume that if a germ creates an error in a qubit, then it is activated and can cause later errors with high probability.

We address the first issue, and clarify the second one, by passing to a multilinear expansion of all of the error operators using (2). Instead of directly considering the full vector state of all of the germs and the errors they cause, we can instead consider the amplitude contribution of any particular pattern of Pauli errors. The total error is a superposition of all of these patterns. To prove Theorem 1.1, we will bound each term of the superposition separately, and then sum to get the total bound.

If there are N edges, then we expand all possible errors across the circuit C :

$$\vec{F} = \sum_{j=0}^{4^N-1} |f_j\rangle \otimes P_j. \quad (3)$$

Here P_j is a *multi-Pauli* operator, a tensor product of Pauli operators including the identity. We consider the partial ordering on qubit edges in which $q_1 \prec q_2$ if there is a directed path from q_1 to q_2 . Then with respect to this partial ordering, some of the Pauli factors of P_j are the earliest among those that are not the identity. We call these qubit edges *locally first diseased* (in superposition).

If q is locally first diseased, then all of the germs that ever interacted with the one at q have an entangled state

$$|g\rangle \in \mathcal{H}_G^{(1)} \otimes \mathcal{H}_G^{(2)} \otimes \dots \otimes \mathcal{H}^{(k)}.$$

The state $|g\rangle$ is formed from various initial states $|g_0\rangle$ with germ-mixing operators M and error components E_j acting on

them. Since each M is unitary and each E_j is subunitary, we learn that $|\langle g|g\rangle| \leq 1$. We can make an error vector \vec{F} (2) in the same way as in the warmup case, except using a state $|g\rangle$ of many germs rather than one germ. We assume an upper bound

$$\|\vec{F}\| < \varepsilon(n), \quad (4)$$

if the quantum depth of the part of the circuit C that leads to q is at most n . We assume that $\varepsilon(n)$ is a small number when n is small. Otherwise, if the best upper bound $\varepsilon(n)$ is large for small n , then even small quantum circuits are unreliable; with enough such noise, there is no clear reason to expect quantum computation to be possible.

3. PROOF OF THEOREM 1.2

Proof. The theorem reduces to the existence of quantum teleportation. Quantum teleportation is a mixed quantum-classical circuit T that has one qubit input and one qubit output, and no quantum path from the input to the output. Moreover, the circuit computes the identity: The output agrees with the input and it even inherits any entanglement that the input had with other qubits. The teleportation circuit is given in Figure 1; a simplified version is given in Figure 2.

As mentioned, our convention for all diagrams is that qubit edges are black and bit edges are gray. The gates used in the expanded circuit in Figure 1 are as follows:

1. The gate 0 creates a qubit in the state $|0\rangle$.
2. The gate m measures a qubit in the computational basis and outputs a bit. The qubit input is destroyed.
3. The gate U is unitary with the following action:

$$\begin{aligned} U|00\rangle &= \frac{|00\rangle + |01\rangle}{\sqrt{2}} & U|01\rangle &= \frac{|10\rangle - |11\rangle}{\sqrt{2}} \\ U|10\rangle &= \frac{|10\rangle + |11\rangle}{\sqrt{2}} & U|11\rangle &= \frac{|00\rangle - |01\rangle}{\sqrt{2}}. \end{aligned}$$

It can be created as a CNOT gate

$$|x, y\rangle \mapsto |x + y, y\rangle$$

followed by a Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

applied to the second qubit.

4. The gate cX applies the one-qubit operator X if its bit input is 1, and the identity I if it is 0.
5. The gate cZ applies the one-qubit operator Z if its bit input is 1, and the identity I if it is 0.

If the circuit T is inserted at every edge of a circuit C to make C' , then the longest directed qubit path in C' is 6 edges. An initialized qubit lasts for 4 edges inside T . The output of T can be the input to some gate G in C , and then the output of G can become the input to another copy of T and last for 2 more edges to make 6 total. \square

It is a celebrated result that the teleportation circuit T computes the identity, even when its input is entangled. In order to prove this, it suffices to check that it is the identity for any spanning set of density operators. For example if the circuit preserves the vector states $|0\rangle$, $|1\rangle$, $|\pm\rangle$, and $|\pm i\rangle$, then it also preserves all six corresponding density operators, which implies that it preserves all density operators. It is easy to check that these states are indeed preserved by T .

4. PROOF OF THEOREM 1.1

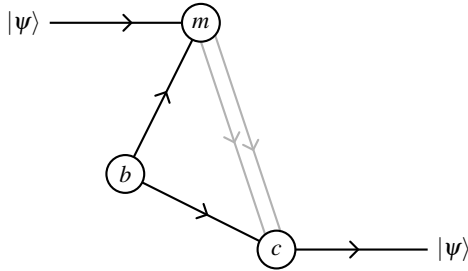


Figure 2. A simplified diagram of the teleportation circuit in Figure 1, in which the gates are combined as much as possible. The gate b creates a Bell pair; the gate m measures two qubits in an entangled basis, and the gate c is a unary quantum gate controlled by two bits.

We will actually prove two different results that both fit the words of Theorem 1.1.

Theorem 4.1. *Let C be a circuit with bounded quantum depth n and gates that act on at most k qubits. If C is infected with quantum germs with error bound $\delta(n)$ at depth at most n , then C has quasi-independent error with bound ϵ depending on n , k , and $\delta(n)$. Moreover, $\epsilon \rightarrow 0$ as $\delta(n) \rightarrow 0$.*

Corollary 4.2. *If C is a circuit with unbounded quantum depth, then it can be replaced by an equivalent circuit C' , so that if C' is infected with quantum germs, then the result is equivalent to quasi-independent noise in C .*

Corollary 4.2 is important because C could be constructed according to a fault tolerance threshold theorem that assumes quasi-independent error.

Proof of Theorem 4.1. We review the definition of quasi-independent error [11]. We assume that the edges of the circuit C are subject to error. Then we consider the multilinear expansion (3). The operator P_J has a weight $w(J)$, which is the number of tensor factors that are not the identity. The quasi-independent error condition says that

$$\sqrt{\langle f_J | f_J \rangle} = O(\epsilon^{w(J)}).$$

for some error bound $0 < \epsilon < 1$. (The fault tolerance theorem says that fault tolerance with polylog overhead is possible if ϵ is small enough. Note that Knill, Laflamme, and Zurek call this type of error *monotonic* quasi-independent error.)

Let P_J be a multi-Pauli operator that arises in the multilinear expansion of (2). Then we claim that

$$\sqrt{\langle f_J | f_J \rangle} < \delta(n)^{m(J)},$$

where $m(J)$ is the number of locally first errors in P_J . In each term J , the state $|f_J\rangle$ is actually the entangled state $|g\rangle$ of all of the germs at the end of the computation. At each position that is a locally first error, the norm of $|g\rangle$ decreases by a factor of $\delta(n)$ by (4). At every other position, the norm does not increase because each operator E_j is subunitary.

Finally, each edge with a locally first error has at most

$$1 + k + \dots + k^{n-1} < k^n$$

edges above it that could have errors. It follows that

$$w(J) < k^n m(J).$$

Thus C has quasi-independent error with bound

$$\epsilon < \delta(n)^{k^n},$$

as desired. \square

Proof of Corollary 4.2. The original purpose of the quasi-independent error model is that it renormalizes to itself under any map that changes a circuit C to an equivalent circuit C' made by replacing gates and qubit edges by gadgets. This is the technique to prove the fault tolerance theorem using concatenated quantum codes. Such an analysis applies in our case because we can replace each edge by a teleportation gadget, as we did in the proof of Theorem 1.2. Indeed, the analysis is particularly simple because every multi-Pauli error in the teleportation gadget in Figure 1 in the circuit C' is equivalent to a Pauli error or non-error in the original edge in C . (This is a standard fact and is left as an exercise to the reader. Note that even though multi-Pauli errors reduce to Pauli errors, the relative phase of two multi-Pauli errors might change.)

Suppose that the circuit C' has quasi-independent error ϵ by Theorem 1.1. Suppose that each gate in Figure 1 is available as a single gate in the gate set. Then 1 edge in C is replaced by 8 edges in C' . We suppose that C' has quasi-independent error with bound ϵ , say by Theorem 1.1. We suppose for simplicity that the total error amplitude of a multi-Pauli is at most $\epsilon^{w(J)}$ rather than $O(\epsilon^{w(J)})$, although the calculation works either way. The total amplitude of all 4^8 multi-Pauli operators

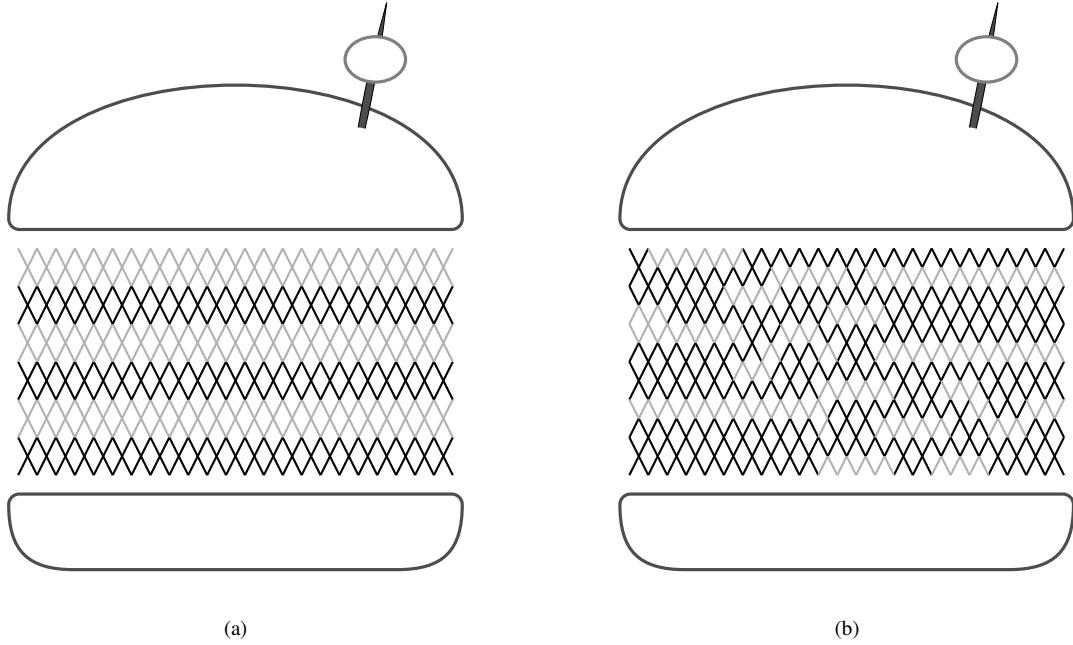


Figure 3. A schematic comparison of two kinds of mixed quantum-classical circuits: (a) SQCL: layered classical-quantum sandwiches; (b) AQCL, asynchronous Sloppy Joes^a with bounded quantum depth.

^a A *Sloppy Joe* is a popular American sandwich with loose ground beef and wet seasoning.

on the edges of a teleportation gadget is at most $(1 + 3\epsilon)^8$. One of these is the term in which all edges are assigned $Z_0 = I$, the non-error; the other errors are bounded by $(1 + 3\epsilon)^8 - 1$. It follows that if we interpret C' as an encoding of C , then C has quasi-independent error with bound

$$\delta < (1 + 3\epsilon)^8 - 1.$$

(In fact, we can divide the right side by 3 by symmetry between the non-trivial Pauli errors, but it is not necessary.) \square

5. A COMPLEXITY CLASS INTERPRETATION

It is interesting to consider complexity classes with a bounded amount of available quantum computation. One example of such a class is the *Fourier hierarchy* FH [16]. In the Fourier hierarchy, the entire circuit is quantum in the sense that it consists of qubits, but only a bounded number of layers of Hadamard gates are allowed. In between these layers the circuit is *pseudoclassical*, meaning that it is a unitary dilation of classical circuit.

Here we define two other mixed quantum-classical classes, even though we do not know whether they are actually useful. (They should not necessarily be added to the Complexity Zoo [17].) First, we can consider the class SQCL, or

sandwiched quantum and classical layers (Figure 3(a)). We represent this class by a uniform family of polynomial-sized quantum-classical circuits $\{C_n\}$. We assume that qubits are only allowed in the circuit in disjoint, global layers $[t, t + b]$ that are bounded in depth by a constant b . In between the layers, all edges have to bit edges. Even though a circuit in SQCL can have a polynomial number of layers, the fact that no quantum coherence connects any two of the layers is a much more severe restriction than in FH. A single quantum layer is a functional class known as QNC_0 , and surprisingly even this class seems different from classical computation [3, 4]. The class SQCL can be viewed as $\text{BPP}^{\text{QNC}_0}$, or BPP with oracle access to QNC_0 , except that it is a semantic type of oracle access in which the oracle output is a probability distribution.

Remark. We do not know whether QNC_0 is weaker with noisy gates. More precisely, whether there is a fault tolerance noise threshold below which QNC_0 is no weaker than before.

We alter the definition of SQCL subtly but dramatically. We define the class AQCL, or *asynchronous quantum and classical layers* with quantum circuits as follows: Each qubit path in the circuit has depth at most b , but the qubits do not have to disappear at the same time. (See Figure 3(b).) This definition matches the conclusion of Theorem 1.2, so we obtain this corollary:

Corollary 5.1. $\text{AQCL} = \text{BQP}$.

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- [1] Dorit Aharonov and Michael Ben-Or, *Fault-tolerant quantum computation with constant error rate*, STOC '97 (El Paso, TX), ACM Press, New York, 1999, arXiv:quant-ph/9906129, pp. 176–188.
 - [2] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters, *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*, Phys. Rev. Lett. **70** (1993), 1895–1899.
 - [3] Michael J. Bremner, Richard Jozsa, and Dan J. Shepherd, *Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2011), no. 2126, 459–472, arXiv:1005.1407.
 - [4] Stephen Fenner, Frederic Green, Steven Homer, and Yong Zhang, *Bounds on the power of constant-depth quantum circuits*, Fundamentals of computation theory, Lecture Notes in Comput. Sci., vol. 3623, Springer, Berlin, 2005, arXiv:quant-ph/0312209, pp. 44–55.
 - [5] Grace Hopper, *The first bug*, Ann. Hist. Comput. **3** (1981), no. 3, 285–286.
 - [6] Gil Kalai, *Quantum computers: noise propagation and adversarial noise models*, 2009, arXiv:0904.3265.
 - [7] ———, *How quantum computers fail: quantum codes, correlations in physical systems, and noise accumulation*, 2011, arXiv:1106.0485.
 - [8] Alexei Kitaev, *Quantum error correction with imperfect gates*, Quantum communication, computing, and measurement (Osamu Hirota, Alexander S. Holevo, and Carlton Caves, eds.), Springer, 1997, pp. 181–188.
 - [9] Emanuel Knill, *Fault-tolerant postselected quantum computation: Schemes*, 2004, arXiv:quant-ph/0402171.
 - [10] ———, *Scalable quantum computing in the presence of large detected-error rates*, Phys. Rev. A **71** (2005), 042322, arXiv:quant-ph/0312190.
 - [11] Emanuel Knill, Raymond Laflamme, and Wojciech H. Zurek, *Resilient quantum computation: error models and thresholds*, Proc. R. Soc. Lond. A **454** (1998), no. 1969, 365–384, arXiv:quant-ph/9702058.
 - [12] Greg Kuperberg, *The capacity of hybrid quantum memory*, IEEE Trans. Inform. Theory **49** (2003), no. 6, 1465–1473, arXiv:quant-ph/0203105.
 - [13] Michael A. Nielsen and Isaac L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000.
 - [14] Robert Raussendorf, Jim Harrington, and Kovid Goyal, *A fault-tolerant one-way quantum computer*, Ann. Physics **321** (2006), no. 9, 2242–2270, arXiv:quant-ph/0510135.
 - [15] ———, *Topological fault-tolerance in cluster state quantum computation*, New J. Phys. **9** (2007), no. 6, 199, arXiv:quant-ph/0703143.
 - [16] Yaoyun Shi, *Quantum and classical tradeoffs*, Theoret. Comput. Sci. **344** (2005), no. 2-3, 335–345, arXiv:quant-ph/0312213.
 - [17] *The Complexity Zoo*, <http://www.complexityzoo.com/>.