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Bright solitons in a 2D spin-orbit-coupled dipolar Bose-Einstein condensate

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We study a two-dimensional spin-orbit-coupled dipolar Bose-Einstein condensate with repulsive contact interactions by both the variational method and the imaginary time evolution of the Gross-Pitaevskii equation. The dipoles are completely polarized along one direction in the 2D plane to provide an effective attractive dipole-dipole interaction. We find two types of solitons as the ground states arising from such attractive dipole-dipole interactions: a plane wave soliton with a spatially varying phase and a stripe soliton with a spatially oscillating density for each component. Both types of solitons possess smaller size and higher anisotropy than the soliton without spin-orbit coupling. Finally, we discuss the properties of moving solitons, which are nontrivial because of the violation of Galilean invariance.

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I. INTRODUCTION

Ever since the first achievement of Bose-Einstein condensates (BECs) in ultracold atomic gases [1–3], matter wave solitons in these systems have been the central focus of many experiments and theory [4]. Solitons that keep their shape while traveling are the result of the interplay between nonlinearity and dispersion. In BECs, nonlinearity usually originates from short-range collisional interactions between atoms, which can be readily tuned via Feshbach resonances [5]. In general, there are two types of solitons: a bright soliton with a density bump, and a dark soliton with a density notch as well as a phase jump [4]. For pure contact interactions, a bright and a dark soliton can emerge respectively when the interactions are attractive or repulsive. Both these solitons have been experimentally observed in cold atoms [6–14]. However, for these contact attractive interactions, bright solitons can only exist in one dimension (1D), but not in two dimensions (2D), where the states either collapse or expand [15].

Different from the local nonlinearity resulting from contact interactions, the non-local nonlinearity, in particular, the nonlinearity introduced by the dipole-dipole interaction, can stabilize a 2D bright soliton [16, 17]. The dipole interaction is long-range and anisotropic with the strength and sign (i.e. repulsive or attractive) depending on the dipole orientation. When an external rotating magnetic field is applied to reverse the sign of the interaction [18], or the dipoles are completely polarized in a 2D plane [19], the dipolar interaction can become attractive, and 2D bright solitons can be, therefore, stabilized under appropriate conditions (i.e. the ground state of the system is a bright soliton with a localized density profile). It is essential to note that although the relevant interaction in most experiments with cold atomic gases is contact, increasing interest has been focused on the atoms with

large magnetic moments that possess dipole-dipole interactions [17, 20, 21]. In fact, the Bose-Einstein condensation of several dipolar atoms such as Chromium [22–24], Dysprosium [25], and Erbium [26], as well as the degeneracy of a dipolar Fermi gas [27, 28] have been observed in experiments.

Recently, the spin-orbit coupling between two hyperfine states of cold atoms has been experimentally engineered [29–34]. This achievement has ignited tremendous interest in this field because of the dramatic change in the single particle dispersion (induced by spin-orbit coupling) which in conjunction with the interaction leads to many exotic superfluids [35–45] (also see [46–53] for review). Such change in dispersion also results in exotic solitons even when the interaction is contact (without dipole-dipole interactions), including 1D bright solitons [54–60] for a BEC with attractive contact interactions, 1D dark [61, 62] and gap solitons [63–65] for a BEC with repulsive contact interactions, as well as 1D dark solitons for Fermi superfluids [66, 67]. These solitons exhibit unique features that are absent without spin-orbit coupling, for instance, the plane wave profile with a spatially varying phase and the stripe profile with a spatially oscillating density for BECs, as well as the presence of Majorana fermions inside a soliton for Fermi superfluids. Also, the violation of Galilean invariance [55, 68, 69] by spin-orbit coupling dictates that the structure of solitons changes with their velocities.

On the other hand, spin-orbit-coupled BECs with dipole-dipole interactions [70–73] have also been explored, and intriguing quasicrystals [74] as well as meron states [75] have been found (these ground states are extended and not localized bright solitons). However, whether a soliton can exist in such BECs in 2D with both long-range dipole-dipole interactions and spin-orbit dispersion has not yet been investigated.

In this paper, we examine the existence and properties of a bright soliton in a two species spin-orbit-coupled dipolar BEC in 2D with repulsive contact interactions via both the variational method and the imaginary time evolution of the Gross-Pitaevskii equation (GPE). The

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dipoles are completely oriented along the y direction in the 2D plane (i.e. (x,y) plane) in order to provide an effective attractive dipole-dipole interaction. Thanks to such attractive interactions, we find two types of solitons: a plane wave soliton (when the repulsive intraspecies contact interaction is larger than the repulsive interspecies one) and a stripe soliton (when the interspecies one is larger). These 2D solitons are the ground states of the system, and they cannot exist as the ground states of a system with purely attractive contact interactions (no dipole) and spin-orbit coupling. Such solitons are highly anisotropic, and their size is also reduced by spin-orbit coupling. Finally, we study the moving solitons, which are nontrivial because of the lack of Galilean invariance. The size of a soliton first increases and then decreases with the rise of the velocity, and this change is anisotropic. The moving soliton also tends to be plane wave even when its stationary counterpart has the stripe structure.

The paper is organized as follows. In Sec. II, we introduce the energy functional and the time-dependent GPE, which are used to describe a spin-orbit-coupled dipolar BEC. In Sec. III, we calculate the ground states that are bright solitons by performing the minimization of the energy of the variational ansatz wave functions and an imaginary time evolution of the GPE. The properties of such soliton are also explored by the former method. Then, we study the nontrivial moving solitons in Sec. IV. Finally, we conclude in Sec. V.

II. MODEL

We consider a Rashba-type spin-orbit-coupled BEC [46, 50] and write its single particle Hamiltonian as

$$H_s = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2}m\omega_\perp^2\rho^2 + \frac{1}{2}m\omega_z^2z^2 + \lambda(\hat{\mathbf{p}} \times \boldsymbol{\sigma}) \cdot \mathbf{e}_z, \quad (1)$$

where $\rho = \sqrt{x^2 + y^2}$, $\hat{\mathbf{p}} = -i\hbar\nabla$ is the momentum operator, m is the atom mass, λ is the spin-orbit coupling strength, $\boldsymbol{\sigma}$ are Pauli matrices, and \mathbf{e}_z is a unit vector along the z direction. ω_\perp (ω_z) is the trap frequency in the (x,y) plane (along the z direction). Here, we assume that $\hbar\omega_z$ is much larger than $\hbar\omega_\perp$ and the mean-field interaction so that the atoms are frozen to the ground state in the z direction. Given that a soliton is studied, we thus set $\omega_\perp = 0$.

When the s -wave contact and dipole-dipole interactions are involved, the energy functional of a 2D condensate can be written as

$$E = \int d\mathbf{r} \left[\Psi(\mathbf{r})^\dagger H_s \Psi(\mathbf{r}) + \frac{1}{2}g(|\Psi_\uparrow|^4 + |\Psi_\downarrow|^4) + g_{12}|\Psi_\uparrow|^2|\Psi_\downarrow|^2 \right] + E_{dd}, \quad (2)$$

where the condensate wave function $\Psi(\mathbf{r}) = [\Psi_\uparrow(\mathbf{r}), \Psi_\downarrow(\mathbf{r})]^T$ with two pseudo-spin components $\Psi_{\uparrow(\downarrow)}(\mathbf{r})$, $g = 4\pi\hbar^2 a/(\sqrt{2\pi}l_z m)$ and

$g_{12} = 4\pi\hbar^2 a_{12}/(\sqrt{2\pi}l_z m)$ are the intraspecies and interspecies contact interaction strength respectively with the intraspecies and interspecies s -wave scattering length being a and a_{12} and the characteristic length along z being $l_z = \sqrt{\hbar/(m\omega_z)}$. Here, $H_s = -\hbar^2(\partial_x^2 + \partial_y^2)/(2m) - i\hbar\lambda(\partial_x\sigma_y - \partial_y\sigma_x)$ is the 2D single particle Hamiltonian, and the third dimension has been integrated out. For dipole-dipole interactions, we only consider the density-density interaction which is dominant when a two subspace (i.e. two pseudo-spin states) of a large spin atom (e.g. dysprosium) is considered. We also assume that the dipoles are all oriented along the y direction, thus

$$E_{dd} = \frac{g_d}{2} \frac{1}{(2\pi)^2} \int d\mathbf{k} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} U_d(\mathbf{k}l_z), \quad (3)$$

where the Fourier transform of the total density is $\rho_{\mathbf{k}} = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} (|\Psi_\uparrow|^2 + |\Psi_\downarrow|^2)$ and $U(\mathbf{k})$ is given by [18]

$$U_d(\mathbf{k}l_z) = -\sqrt{2\pi} + \frac{3\pi l_z k_y^2 e^{k^2 l_z^2/2} \text{erfc}(kl_z/\sqrt{2})}{k}, \quad (4)$$

with erfc being the complementary error function. Here, $g_d = \mu_0\mu^2/(6\pi l_z)$ characterizes the strength of the dipole-dipole interaction where μ is the magnetic dipolar moment and μ_0 is the permeability of the free space. We note that this head-to-tail dipole configuration that can be attractive is different from the head-to-head configuration in Ref. [74] that is repulsive.

The dynamical behavior of a BEC can be described by the time-dependent GPE

$$i\hbar \frac{\partial \Psi(\mathbf{r})}{\partial t} = H_s \Psi(\mathbf{r}) + gG\Psi(\mathbf{r}) + g_d U_d(\mathbf{r})\Psi(\mathbf{r}), \quad (5)$$

where the contact interaction matrix is

$$G = \begin{pmatrix} |\Psi_\uparrow|^2 + \frac{g_{12}}{g}|\Psi_\downarrow|^2 & 0 \\ 0 & |\Psi_\downarrow|^2 + \frac{g_{12}}{g}|\Psi_\uparrow|^2 \end{pmatrix}, \quad (6)$$

and the dipolar interaction potential is

$$U_d(\mathbf{r}) = \frac{1}{(2\pi)^2} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \rho(\mathbf{k}) U_d(\mathbf{k}l_z). \quad (7)$$

For numerical simulation, we choose $\hbar\omega_z$, l_z , and $1/\omega_z$ as the units of energy, length, and time, respectively, and the dimensionless energy per atom hence reads

$$\mathcal{E} = \int d\mathbf{r} \left[\Phi(\mathbf{r})^\dagger \mathcal{H}_s \Phi(\mathbf{r}) + \frac{1}{2}\gamma(|\Phi_\uparrow|^4 + |\Phi_\downarrow|^4) + \gamma_{12}|\Phi_\uparrow|^2|\Phi_\downarrow|^2 \right] + \frac{\gamma_d}{2(2\pi)^2} \int d\mathbf{k} n_{\mathbf{k}} n_{-\mathbf{k}} U_d(\mathbf{k}), \quad (8)$$

where $\mathcal{H}_s = -(\partial_x^2 + \partial_y^2)/2 - i\alpha(\partial_x\sigma_y - \partial_y\sigma_x)$, $\alpha = \lambda/(\omega_x l_z)$, $\gamma = 2\sqrt{2\pi}N_0 a/l_z$, $\gamma_{12} = 2\sqrt{2\pi}N_0 a_{12}/l_z$ with the total particle number N_0 , $\gamma_d = 2N_0 a_d/l_z$ with $a_d = m\mu_0\mu^2/(12\pi\hbar^2)$, and $n_{\mathbf{k}} = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} (|\Phi_\uparrow|^2 + |\Phi_\downarrow|^2)$.

The wave function is normalized to 1 (i.e. $\int d\mathbf{r}(|\Phi_\uparrow|^2 + |\Phi_\downarrow|^2) = 1$).

The dimensionless time-dependent GPE reads

$$i\frac{\partial\Phi(\mathbf{r})}{\partial t} = \mathcal{H}_s\Phi(\mathbf{r}) + \gamma\mathcal{G}\Phi(\mathbf{r}) + \frac{\gamma_d}{(2\pi)^2} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} n(\mathbf{k}) U_d(\mathbf{k}) \Phi(\mathbf{r}), \quad (9)$$

where

$$\mathcal{G} = \begin{pmatrix} |\Phi_\uparrow|^2 + \frac{\gamma_{12}}{\gamma} |\Phi_\downarrow|^2 & 0 \\ 0 & |\Phi_\downarrow|^2 + \frac{\gamma_{12}}{\gamma} |\Phi_\uparrow|^2 \end{pmatrix}. \quad (10)$$

III. STATIONARY BRIGHT SOLITONS

To shed light on the structure of a soliton, we start from a Rashba spin-orbit-coupled single particle system in the absence of interactions in a free space and write its momentum space dispersion as

$$E(\mathbf{k}) = \frac{\mathbf{k}^2}{2} \pm \alpha k, \quad (11)$$

with two branches labeled by the helicity \pm which is the eigenvalue of $(k_x\sigma_y - k_y\sigma_x)/k$ where $k = \sqrt{k_x^2 + k_y^2}$. Clearly, the ground state is degenerate with the energy being $-\alpha^2/2$ when the momenta lie in the $k = |\alpha|$ ring. This is different from the case without spin-orbit coupling where the ground state only occurs at $k = 0$. In this single particle case, any superposition of the states in the ring is also its ground state. Yet, this is not the case when the repulsive contact interaction is involved. The ground state either possesses a single momentum (i.e. plane wave phase) when $\gamma_{12}/\gamma < 1$ or two opposite momenta (i.e. stripe phase) when $\gamma_{12}/\gamma > 1$ [36]. When the dipolar interaction is turned on, one may expect two types of ground states — plane wave and stripe solitons — due to this effective long-range attractive interaction along with contact repulsive interaction.

To examine whether the ground state can indeed be a bright soliton in the spin-orbit-coupled dipolar BECs, we first consider a plane wave soliton variational ansatz

$$\Phi_P = \begin{pmatrix} \Phi_0(x_0/2) \\ -\Phi_0(-x_0/2) \end{pmatrix} \exp(-i\mathbf{J}_p \cdot \mathbf{r}), \quad (12)$$

where

$$\Phi_0(x_0) = \frac{(a_x a_y)^{1/4}}{\sqrt{2\pi}} e^{-\frac{1}{2}[a_x(x-x_0)^2 + a_y y^2]}. \quad (13)$$

Here $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$ with unit vectors \mathbf{e}_x and \mathbf{e}_y along x and y directions respectively, $\mathbf{J}_p = J_p\mathbf{e}_y$ is the wave vector of the plane wave soliton, $l_\nu = 1/\sqrt{a_\nu}$ with $\nu = x, y$ is the size of the soliton, and x_0 is the separation distance between two components. When $x_0 = 0$, this state is an eigenstate of $p_y\sigma_x$ multiplied by a Gaussian

profile $\Phi_0(0)$, and $J_p = \alpha$ yields the minimum energy. In fact, x_0 is usually nonzero because of a force acting on the BEC by spin-orbit coupling $\mathbf{F} = \alpha^2(\mathbf{p} \times \mathbf{e}_z)\sigma_z$ [40, 76], which is opposite along the x direction when \mathbf{p} (here \mathbf{J}_p) is along the y direction.

In writing down the ansatz (12), we have assumed that the wave vector of the plane wave soliton is in the y direction. The prerequisite of this assumption is that the rotation symmetry [40, 41] about the z axis has been broken by the dipole-dipole interaction. Indeed, without the dipole-dipole interaction, this state with \mathbf{J}_p along y is not special, and other states with \mathbf{J}_p along other directions are degenerate with it. For example, the state with \mathbf{J}_p along y has the same energy as a state with \mathbf{J}_p along x . Yet, with the specific dipole-dipole interaction arising from the dipoles entirely oriented along y , the symmetry is broken, and the ground state should be elongated along y ($a_x > a_y$) so as to provide an effective attractive interaction because of the head-to-tail configuration of polarized dipoles. This elongated configuration allows the existence of a 2D soliton [19] and also requires the wave vector to be along y [77].

Although the wave vector \mathbf{J}_p of the ground state is along y , there are still two options: negative and positive directions in terms of the time-reversal symmetry \mathcal{T} (i.e. $-i\sigma_y\mathcal{K}$ with the complex conjugate operator \mathcal{K}). Specifically, the state $\Phi_{P2} = \mathcal{T}\Phi_P$ is degenerate with Φ_P . In the absence of interactions, all superposition states of Φ_P and Φ_{P2} ,

$$\Phi_{PS} = |\cos\theta|\Phi_P + |\sin\theta|e^{i\varphi}\Phi_{P2}, \quad (14)$$

are degenerate. This degeneracy may be broken by the interaction so that the ground state is either Φ_P or Φ_{P2} , or a certain superposition state of them. But this degeneracy breaking should not happen at $\gamma_{12}/\gamma = 1$ since the interaction energy only depends on the total density which is independent of θ and φ . This gives us an intuitive understanding that $\gamma_{12}/\gamma = 1$ may separate the plane wave state ($|\cos\theta| = 0$ or 1) and the stripe one ($|\cos\theta| = |\sin\theta|$), similar to the homogenous spin-orbit-coupled BEC [36] without dipole-dipole interactions. For the stripe state, we note that $\varphi = 0, \pi$ corresponds to the ground state as the energy contributed by φ is $-\gamma_{12}\sqrt{a_x a_y} e^{-(a_x x_0^2/2 + 2J_p^2/a_y)} \cos(2\varphi)/(16\pi)$ [78].

To evaluate the variational parameters a_x, a_y, x_0, J_p , and θ , we minimize the energy \mathcal{E} after substituting Φ_{PS} in Eq. (14) to Eq. (8). Indeed, the calculated variational solutions reveal that there are two types of ground states: plane wave states when $\gamma_{12}/\gamma < 1$ and stripe states when $\gamma_{12}/\gamma > 1$. These ground states are bright solitons with localized density profiles as shown in Fig. 1; we therefore call these ground states plane wave solitons and stripe solitons respectively. In the first panel of this figure, the density and phase profiles of a typical plane wave soliton (we choose $\theta = \varphi = 0$) are presented, and the stripe structure of the phase of both two components reveals the plane wave feature. The soliton is highly elongated along the y direction, and the centers of two components

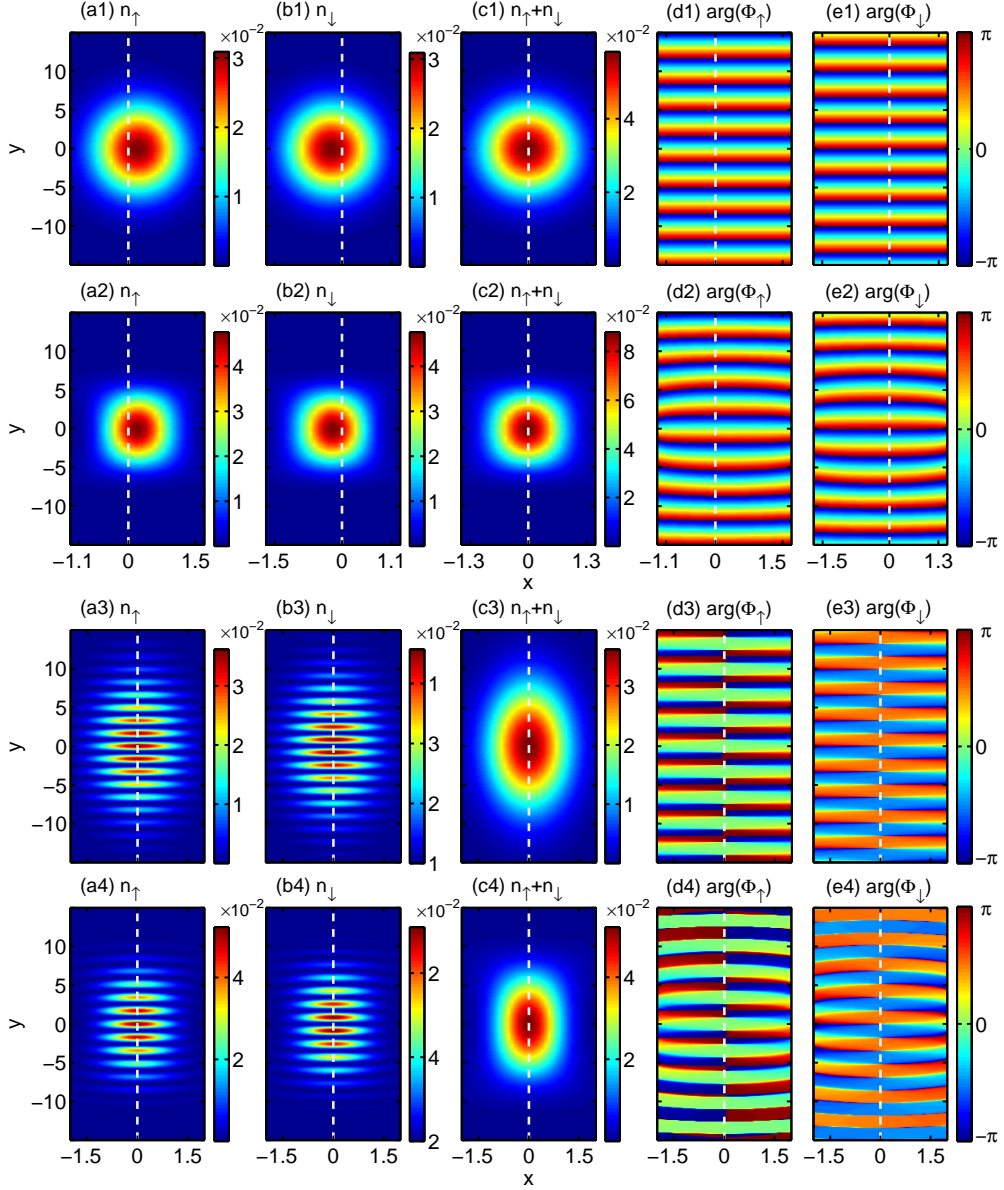


FIG. 1. (Color online) Profiles of the density $n_{\uparrow,\downarrow} = |\Phi_{\uparrow,\downarrow}|^2$ of spin $\uparrow(\downarrow)$ in (a)((b)), the total density $n_{\uparrow} + n_{\downarrow}$ in (c), the phase of spin $\uparrow(\downarrow)$ in (d)((e)) for a plane wave soliton (the first two panels) with $\gamma_{12} = 6$ and a stripe soliton (the last two panels) with $\gamma_{12} = 10$. The solitons in the first and third panels are obtained by the variational method, while the solitons in the second and fourth panels are calculated by the imaginary time evolution of the GP Eq. (9). The dashed white line labels the $x = 0$ line. Here, $\gamma = 8$, $\gamma_d/\gamma = 0.67$, and $\alpha = 2$.

are spatially separated along the x direction because of nonzero x_0 . To confirm that this variational solution can qualitatively characterize the ground state of the system, we numerically compute the ground state by an imaginary time evolution of the GP Eq. (9). This exact numerical solution also concludes that $\gamma_{12}/\gamma < 1$ yields the plane wave soliton, while $\gamma_{12}/\gamma > 1$ the stripe one. In the second panel of Fig. 1, we also plot the corresponding density and phase profiles of the GP obtained plane wave soliton. The variational ansatz is in qualitative agreement with it given the separated centers and the plane

wave varying phase that both states possess. Yet, the shape of the soliton obtained by the imaginary time evolution deviates slightly from the Gaussian, and the size is also slightly smaller.

When $\theta = \pi/4$ and $\varphi = 0$, Φ_{PS} is a stripe soliton state with a density oscillation along the y direction for each component. And there is no stripe for the total density. Along the x direction, two components are not spatially separated, and the phase for the spin \uparrow reverses suddenly across $x = 0$. Following these properties by replacing $(\Phi_0(x_0/2) + \Phi_0(-x_0/2))/\sqrt{2}$ with $\cos(J_x x)\Phi_0(0)$

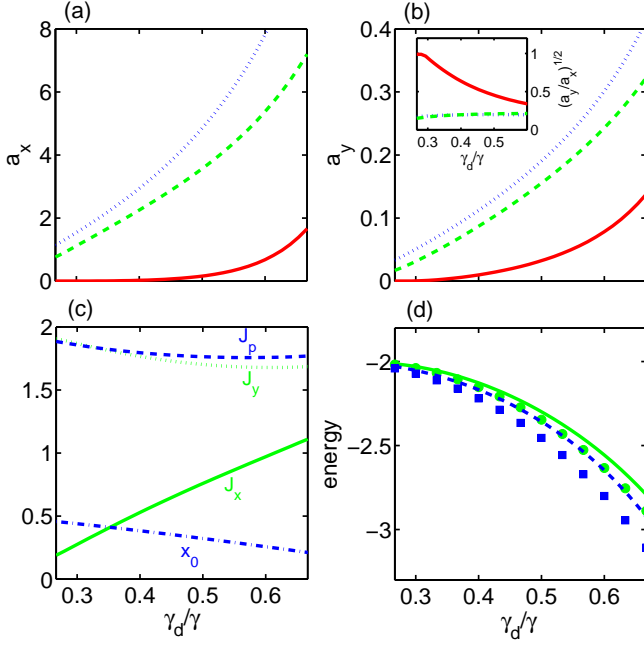


FIG. 2. (Color online) Plot of a_x in (a) and a_y in (b) as a function of γ_d/γ for the plane wave solitons (dotted blue line), stripe solitons (dashed green line), and traditional solitons (solid red line) without spin-orbit coupling. The aspect ratio $\sqrt{a_y/a_x}$ of a soliton is displayed in the inset of (b). Variational parameters with respect γ_d/γ are plotted in (c) associated with x_0 (dash-dot blue line) and J_p (dashed blue line) for the plane wave soliton, and J_x (solid green) and J_y (dotted green) for the stripe soliton. In (d), the total energy of the variational ansatz wave function and the wave function numerically obtained by the imaginary time evolution of the GPE for both plane wave and stripe solitons is shown. The solid green line (stripe soliton) and dashed blue line (plane wave soliton) correspond to the variational results, while the green circles and blue squares correspond to the GP results. Here, $\alpha = 2$, $\gamma = 8$, $\gamma_{12} = 6$ ($\gamma_{12} = 10$) for the plane wave (stripe) soliton.

and $(\Phi_0(x_0/2) - \Phi_0(-x_0/2))/\sqrt{2}$ with $\sin(J_x x)\Phi_0(0)$ in Eq. (14), we obtain another better variational ansatz for the stripe soliton

$$\Phi_S = \Gamma\Phi_0(0), \quad (15)$$

where

$$\Gamma = \begin{pmatrix} \cos(J_y y) \cos(J_x x) - i \sin(J_y y) \sin(J_x x) \\ \cos(J_y y) \sin(J_x x) + i \sin(J_y y) \cos(J_x x) \end{pmatrix}, \quad (16)$$

with the variational parameters J_x and J_y . The period of the stripe along the y direction is π/J_y . Interestingly, this stripe soliton corresponds to four points $(\pm J_x, \pm J_y)$ in momentum space instead of traditional two points [55] when $J_x = 0$, if we do not consider the Gaussian profile Φ_0 .

We calculate the variational parameters of stripe solitons by performing the minimization of the energy \mathcal{E} in

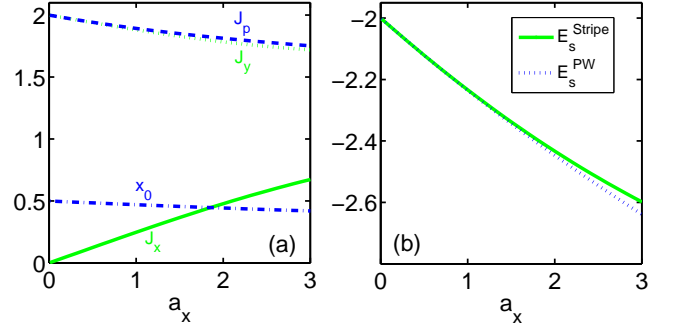


FIG. 3. (Color online) Plot of x_0 (dash-dot blue line) and J_p (dashed blue line) for the plane wave variational ansatz, and J_x (solid green line) and J_y (dotted green line) for the stripe variational ansatz with respect to a_x by the minimization of the energy E_s^{PW} and E_s^{Stripe} in (a). (b) shows the minimum energy of E_s^{PW} (dotted blue line) and E_s^{Stripe} (solid green line) as a function of a_x . Here, $\alpha = 2$.

Eq. (8) where Φ is replaced with Φ_S . The density and phase profiles of a typical stripe soliton calculated by this method are displayed in the third panel of Fig. 1. Evidently, the density of each component exhibits the stripe structure, while the total density does not. The phase of spin \uparrow along the y direction varies like a plane wave, but reverses across $x = 0$ due to the presence of $\sin(J_x x)$ in the imaginary part. The phase of spin \downarrow exhibits the phase rotation like vortices around $x = 0$ and $y = n\pi/J_y$ with integer n ; around these points, the wave function $\Phi_{S\downarrow}$ is proportional to $(-1)^n (J_x x + i(J_y y - n\pi))$, and the corresponding density of spin \downarrow is extremely low. Moreover, in the last panel of Fig. 1, we plot the density and phase profiles of the corresponding stripe soliton obtained by the imaginary time evolution of the GPE; comparing this figure with the third panel of Fig. 1 implies that the stripe variational ansatz is qualitatively consistent with the GP results.

To study the properties of a soliton with respect to dipole-dipole interactions γ_d , we evaluate the variational parameters of both the plane wave and stripe solitons by the variational method and plot them in Fig. 2 as γ_d/γ varies. Clearly, with increasing γ_d/γ , a_x and a_y increase monotonously because of the enhanced effective attractive interaction, indicating that the size l_x and l_y of the soliton decrease monotonously. We note that as γ_d/γ increases further, the soliton can collapse so that both a_x and a_y diverge. For the plane wave soliton, a_x and a_y are slightly larger than the stripe soliton because of the smaller contact interaction of the former. Moreover, compared with the soliton without spin-orbit coupling (red line in Fig. 2(a) and (b)), a_x and a_y for both the plane wave and stripe solitons are much larger, implying that the size of solitons can be reduced by spin-orbit coupling. Also, these solitons are highly anisotropic with the much smaller aspect ratio $\sqrt{a_y/a_x}$ as shown in the inset of Fig. 2(b). To elucidate the reason, we explicitly write

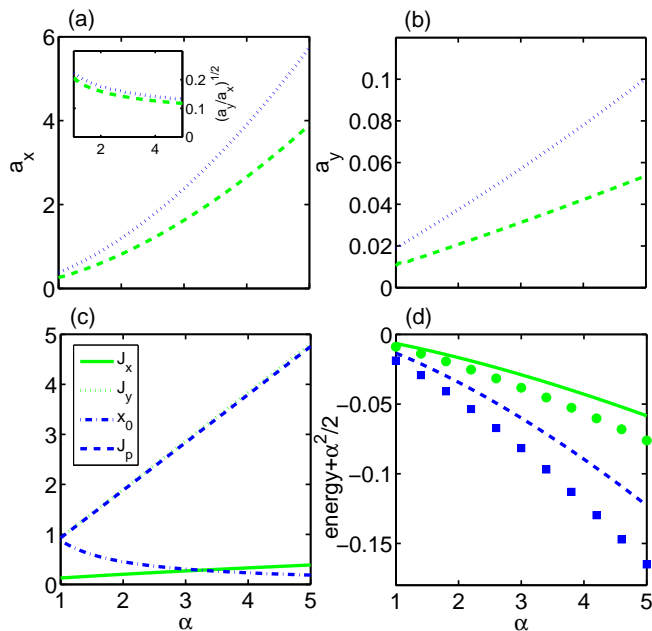


FIG. 4. (Color online) Plot of a_x in (a) and a_y in (b) as a function of the spin-orbit coupling strength α for the plane wave soliton (dotted blue line) and stripe soliton (dashed green line). The aspect ratio $\sqrt{a_y/a_x}$ of a soliton is plotted in the inset of (a). (c) illustrates the change of x_0 (dash-dot blue line), J_p (dashed blue line) for the plane wave soliton and J_x (solid green line) and J_y (dotted green line) for the stripe soliton with respect to α . In (d), the total energy plus $\alpha^2/2$ is plotted as a function of α . The solid green (for a stripe soliton) and dashed blue (for a plane wave soliton) are calculated by the variational method, while the green circles (for a stripe soliton) and blue squares (for a plane wave soliton) are numerically obtained by the imaginary time evolution of the GPE. Here, $\gamma = 8$, $\gamma_d/\gamma = 0.67$, and $\gamma_{12} = 6$ ($\gamma_{12} = 10$) for the plane wave (stripe) soliton.

that single particle energy of the plane wave variational ansatz in Eq.(12) which results from the presence of x_0 and J_p

$$E_s^{\text{PW}} = \frac{1}{2}J_p^2 - \alpha e^{-\frac{x_0^2 a_x}{4}} \left(J_p + \frac{1}{2}a_x x_0 \right). \quad (17)$$

The minimization of E_s^{PW} with respect to x_0 and J_p for fixed a_x yields

$$x_0 = \frac{-J_p + \sqrt{J_p^2 + 2a_x}}{a_x} \quad (18)$$

$$J_p = \alpha e^{-x_0^2 a_x / 4}. \quad (19)$$

For $a_x = 0$, the energy is independent of x_0 and $J_p = \alpha$, while for $a_x \neq 0$, both x_0 and J_p decrease slightly with increasing a_x as shown in Fig. 3(a) with the asymptotic $x_0 = 1/\alpha$ and $J_p = \alpha$ as a_x goes zero. The energy E_s^{PW} is also a monotonously decreasing function of a_x . And this energy decline combined with the reduced dipole-dipole interaction energy competes with the rise of the

kinetic energy (when $x_0 = J_p = 0$) and contact interaction energy, leading to an increased a_x and a_y compared with the soliton without spin-orbit coupling. This is also consistent with Fig. 2(c), showing that with increasing the dipole-dipole interaction, a_x increases, and both x_0 and J_p , therefore, decrease so as to lower E_s^{PW} . It is important to note that although E_s^{PW} is not a function of a_y , other energy such as the kinetic energy (when $x_0 = J_p = 0$), the contact and dipolar interaction energy depends on it.

For the stripe soliton, the single particle energy due to the presence of J_x and J_y is

$$E_s^{\text{Stripe}} = \frac{1}{2}(J_x^2 + J_y^2) - \alpha(J_x + J_y e^{-J_x^2/a_x}). \quad (20)$$

Similar to the plane wave case, this energy is independent of a_y . For fixed a_x , the minimization of this energy yields both J_x and J_y as a function of a_x as shown in Fig. 3(a). When a_x moves towards zero, the solution approaches $(J_x = \alpha, J_y = 0)$ or $(J_x = 0, J_y = \alpha)$; when it moves away from zero, there is only one solution where J_y decreases from α while J_x increases from zero with the rise of a_x . Also, the energy E_s^{Stripe} decreases as a_x increases. Analogous to the plane wave soliton, the total energy decrease resulted from spin-orbit coupling and dipole-dipole interactions as a_x and a_y increase from the value without spin-orbit coupling exceeds the energy gain of the kinetic (when $J_x = 0$ and $J_y = 0$) and contact interaction; this leads to the increased a_x and a_y compared with the soliton without spin-orbit coupling. This picture is also consistent with Fig. 2(c) where J_x increases while J_y decreases with respect to γ_d/γ .

To explicitly demonstrate the effect of the spin-orbit coupling on the properties of a soliton, we plot the variational parameters as a function of the spin-orbit coupling strength α for both the plane wave and stripe solitons in Fig. 4. Consistent with the aforementioned feature that spin-orbit coupling can reduce the size of the soliton, both Fig. 4(a) and Fig. 4(b) display a monotonous increasing behavior of a_x and a_y as a function of α . Also, the aspect ratio $\sqrt{a_y/a_x}$ is decreased by spin-orbit coupling. Similar to Fig. 2(a) and Fig. 2(b), a_x and a_y for the plane wave soliton are slightly larger than the stripe soliton in that the former has a smaller contact interaction. For the plane wave soliton, J_p (determined mainly by the spin-orbit coupling strength) increases with respect to α while x_0 decreases; for the stripe soliton, both J_x and J_y increase.

In Fig. 2(d) and Fig. 4(d), for both plane wave and stripe solitons, we compare their energy obtained by the variational procedure with the one obtained by the imaginary time evolution of the GPE. Both figures show that the energy calculated by the imaginary time evolution is lower as expected. Yet, the difference between these two energy is not large (no more than 10%), suggesting that the variational ansatz can qualitatively characterize the solitons. We note that in Fig. 4(d), the energy is shifted by $\alpha^2/2$ in order to clearly present the different results

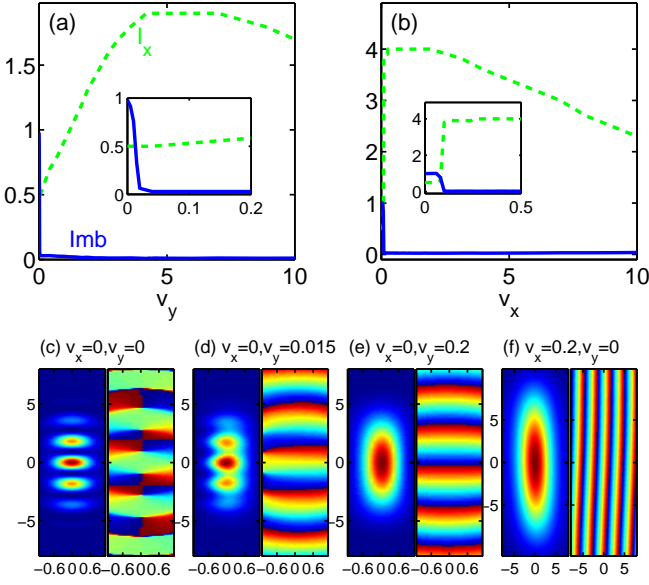


FIG. 5. (Color online) Imbalance Imb and width l_x of spin \uparrow of the solitons with respect to the velocity along the y (x) direction in (a) ((b)). The insets display the enlarged figure in a small velocity region. Density and phase profiles of four typical moving solitons for spin \uparrow corresponding to the different velocities in (a) and (b) are plotted in (c-f) where the horizontal and vertical coordinates are x' and y' respectively. Here, $\alpha = 2$, $\gamma = 8$, $\gamma_{12} = 10$, and $\gamma_d/\gamma = 1$, corresponding to a stripe soliton when stationary.

of the two methods, which could be smeared by the large value of $\alpha^2/2$.

IV. MOVING BRIGHT SOLITONS

Generally, the wave function of a moving soliton with the velocity \mathbf{v} can be simply written as $\exp(i\mathbf{v} \cdot \mathbf{r})\Phi_s(\mathbf{r} - \mathbf{v}t)$ (the density does not disperse) where Φ_s is the wave function of a stationary soliton. But this is only valid for a system respecting Galilean transform invariance. In fact, Galilean invariance is broken in a spin-orbit-coupled BEC [68], and this violation dictates that the shape of a soliton depends on its velocity strength [55]. Here, for a soliton in a spin-orbit-coupled dipolar BEC in 2D, we assume that a moving soliton can be written as

$$\Phi_M(\mathbf{r}, t) = \Phi_v(\mathbf{r}', t) \exp(i\mathbf{v} \cdot \mathbf{r} - i\frac{1}{2}v^2t), \quad (21)$$

where Φ_v is a localized function (the density still does not disperse), and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ is the coordinate in the moving frame. Plugging $\Phi_M(\mathbf{r}, t)$ into Eq. (9) yields

$$i\frac{\partial\Phi_v(\mathbf{r}', t)}{\partial t} = \mathcal{H}_s(\mathbf{v})\Phi_v(\mathbf{r}', t) + \gamma\mathcal{G}\Phi_v(\mathbf{r}', t) + \frac{\gamma_d}{(2\pi)^2} \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}} n(\mathbf{k}) U_d(\mathbf{k}) \Phi_v(\mathbf{r}', t), \quad (22)$$

where $\mathcal{H}_s(\mathbf{v}) = \mathcal{H}_s + \alpha(\mathbf{v} \times \boldsymbol{\sigma}) \cdot \mathbf{e}_z + \mathbf{v}^2/2$. Compared to Eq. (9), this dynamical equation has an additional term $\alpha(\mathbf{v} \times \boldsymbol{\sigma}) \cdot \mathbf{e}_z$ ($\mathbf{v}^2/2$ term has no effects on the dynamics), acting as a Zeeman field; this additional term implies the violation of Galilean invariance. This violation means that it is no longer a trivial task to find a moving bright soliton for a BEC with spin-orbit coupling; we need to perform an imaginary time evolution of the Eq. (22), but not Eq. (9). Furthermore, such a 2D moving soliton should be different for different velocity directions even if their amplitude is the same, in contrast to a 1D soliton which can only move in one direction.

To examine how the shape of a soliton changes with respect to the velocities along x and y directions, we plot the imbalance Imb and the width l_x of a soliton of spin \uparrow as a function of the velocities v_x and v_y in Fig. 5. Here, the imbalance for spin \uparrow is defined as

$$Imb = \frac{|\Phi_{\uparrow}(0)|^2 - |\Phi_{\uparrow}(\pi/(2\alpha))|^2}{|\Phi_{\uparrow}(0)|^2 + |\Phi_{\uparrow}(\pi/(2\alpha))|^2}, \quad (23)$$

which characterizes a stripe soliton (as shown in Fig. 5(c) and Fig. 5(d)) when it approaches one and a plane wave soliton (as shown in Fig. 5(e) and Fig. 5(f)) when it approaches zero. Fig. 5(a) and Fig. 5(b) demonstrate that Imb suffers a sharp decline from one to near zero as v_x and v_y increase, indicating that a moving soliton tends to be a plane wave state. The reason is the broken rotation symmetry of the single particle Hamiltonian by the velocity induced Zeeman field, giving rise to a ground state of the single particle system lying at one momentum point located along the x (y) direction when the velocity is along that direction. This also explains why the phase of a moving plane wave soliton with the velocity along the x (y) direction varies along that direction.

Furthermore, Fig. 5(a) demonstrates that the width of the soliton gradually grows when the velocity along the y direction is enlarged. To explain the growth, we consider the plane wave ansatz in Eq. (12) which yields an additional term $-\alpha v_y e^{-a_x x^2/4}$ for the single particle energy when a soliton moves; this energy decrease enlarges exponentially with the decline of a_x (i.e. increase of the width), leading to an expanded soliton with the rise of the velocity. However, this is not a monotonous behavior, and the soliton begins shrinking when the velocity goes larger, due to the enlarged J_p by the velocity induced Zeeman field, similar to increasing spin-orbit coupling. On the other hand, when the velocity is along the x direction, the width of the soliton gains a sudden rise as the velocity varies, as shown Fig. 5(b). This corresponds to a change from a stripe soliton with the wave vector along the y direction to a plane wave soliton with the wave vector along the x direction. For the stationary solitons, the soliton with the wave vector mainly along the y direction has lower energy than the one with the wave vector mainly along the x direction as the dipoles are completely oriented along y . But the Zeeman field induced by the presence of a velocity along the x direction gives rise to the single particle ground state that

possesses the wave vector along x . The two states with the wave vector along these two directions compete and change from the former to the latter (i.e. first order phase transition). For the decrease of the width when v_x goes even larger, the reason is the same as the case for v_y . When a stationary soliton is plane wave, the moving behavior is similar except that the moving soliton is always the plane wave soliton.

V. CONCLUSION

We have studied the bright solitons as the ground states in a spin-orbit-coupled dipolar BEC in 2D with dipoles completely polarized along one direction in the 2D plane. It is important to note that the solitons are the ground states in 2D, but they are the metastable states in quasi-2D where the true ground state would collapse and there is an energy barrier between the soliton state and this ground state. Two types of solitons have been found: a plane wave soliton and a stripe soliton. The former has the plane wave phase variation, and its two components are slightly spatially separated; while for the latter, the density of each component is spatially oscillating, and the variational ansatz suggests that four points in momentum space are involved. Both plane wave and stripe solitons are highly anisotropic, and their size is decreased by spin-orbit coupling. These solitons cannot

exist as the ground states in a 2D system with purely attractive contact interactions and spin-orbit coupling. Moreover, the shape of these solitons changes with their velocities due to the absence of Galilean invariance, and this change is anisotropic.

The 2D bright soliton, albeit mainly plane wave soliton, can also exist when equal Rashba and Dresselhaus spin-orbit coupling is considered. In experiments, this type of spin-orbit coupling has been engineered by coupling two hyperfine states of atoms through two counterpropagating Raman laser beams [29–34], and such setup could be employed to realize this spin-orbit coupling in Dysprosium [71] with large dipole-dipole interactions. Also, the large magnetic moment in Dysprosium atoms may permit the realization of Rashba spin-orbit coupling [79].

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