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Optimal Mode Transformations for Linear Optical Cluster State Generation

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Abstract. We analyze the generation of linear optical cluster states (LOCS) via sequential addition of one and two qubits. Existing approaches employ the stochastic linear optical two-qubit *CZ* gate with success rate of 1/9 per operation. The question of optimality of the *CZ* gate with respect to LOCS generation has remained open. We report that there are alternative schemes to the *CZ* gate that are exponentially more efficient and show that sequential LOCS growth is indeed globally optimal. We find that the optimal cluster growth operation is a state transformation on a subspace of the full Hilbert space. We show that the maximal success rate of post-selected entangling *n* photonic qubits or *m* Bell pairs into a cluster is $(1/2)^{n-1}$ and $(1/4)^{m-1}$ respectively with no ancilla photons, and give an explicit optical description of the optimal mode transformations.

I. INTRODUCTION

Cluster states [1] of photonic qubits are a fundamental resource for quantum information processing [2]. Constructing these states presents a major experimental and theoretical challenge because known physical implementations of optical multi-qubit transformations are intrinsically probabilistic [3]. To evaluate the efficiency of such transformations the success probability of a desired measurement outcome is routinely used [3]. Thus, finding entangling photonic transformations that achieve the best possible success probability is of critical importance for progress in the field. Unfortunately, the problem of designing an optimal linear optical transformation is at least #P-complete [4]. Here we construct an analytical solution to this problem for the special case of linear optical cluster state generation via sequential addition of one- and two-qubit states.

Optimization of Linear Optical Transformations. We define a linear optical device operating on M modes as a unitary transformation $\boldsymbol{U}$ of photon creation operators from the input modes $a_i^\dagger$ to the output modes $a_i^\dagger$

$$a_i^\dagger \to \sum_{j=1}^{M} \boldsymbol{U}_{ij} \tilde{a}_j^\dagger \; . \qquad (1)$$

The mode transformation matrix $\boldsymbol{U}$ induces a state transformation $\boldsymbol{\Omega}$ which generally is a high-dimensional unitary representation of $\boldsymbol{U}$ [5]. For a multi-photon product input state $\left|\Psi^{(\mathrm{in})}\right\rangle = \prod_{i=1}^{M}\left(a_i^{\dagger n_i} \big/ \sqrt{n_i!}\right)|0\rangle$ the action of $\boldsymbol{\Omega}$ will result in the output state $\left|\Psi^{(\mathrm{out})}\right\rangle = \boldsymbol{\Omega}\left|\Psi^{(in)}\right\rangle = \prod_{i=1}^{M}\left[\sum_{j=1}^{M}\left(\boldsymbol{U}_{ij}\tilde{a}_j^\dagger\right)^{n_i} \Big/ \sqrt{n!}\right]|0\rangle$.

Next, we define logical qubit states $|\uparrow\rangle$ and $|\downarrow\rangle$ using two-mode Fock states $|\uparrow\rangle = |1,0\rangle$ and $|\downarrow\rangle = |0,1\rangle$ which may conveniently be implemented as horizontal $|H\rangle$ and vertical $|V\rangle$ photon polarizations in a single spatial mode. For an arbitrary N-qubit input state, the output state produced by the action of $\boldsymbol{\Omega}$ will not lie in the computational space $span\left[\{|H\rangle,|V\rangle\}^{\otimes N}\right]$ but rather span a larger space of N photons distributed over $M = 2N$ modes, i.e.,

$$\left|\Psi^{(\mathrm{out})}\right\rangle = \alpha\left|\Psi_I^{(\mathrm{out})}\right\rangle + \beta\left|\Psi_{II}^{(\mathrm{out})}\right\rangle, \quad |\alpha|^2 + |\beta|^2 = 1, \qquad (2)$$

where $\left|\Psi_I^{(\mathrm{out})}\right\rangle$ and $\left|\Psi_{II}^{(\mathrm{out})}\right\rangle$ belong to the computational space and its complement, respectively.

Our goal in general is to find a mode transformation matrix $\boldsymbol{U}$ such that the output state $\left|\Psi_I^{(\mathrm{out})}\right\rangle$ is as close as possible to a target state

$\left|\Psi^{(\mathrm{tar})}\right\rangle$. The fidelity of the state transformation $\left|\Psi^{(\mathrm{in})}\right\rangle \to \left|\Psi^{(\mathrm{tar})}\right\rangle$ is defined as $f(\boldsymbol{U}) = \left|\left\langle \Psi_I^{(\mathrm{out})} \middle| \Psi^{(\mathrm{tar})}\right\rangle\right|^2$. When the fidelity is equal to one the success probability of the post-selection onto the computational space can be defined as $s(\boldsymbol{U}) = |\alpha|^2$. Physically this means that a photon coincidence pattern corresponding to the target state $\left|\Psi^{(\mathrm{tar})}\right\rangle$ is detected in dual-rail modes with the rate $s(\boldsymbol{U})$. The objective can then be restated as finding a mode transformation $\boldsymbol{U}$ that maximizes success probability while ensuring $f(\boldsymbol{U}) = 1$ for the desired logical target state transformation. As an example, we consider a trivial two-qubit case where the input state $\left|\Psi^{(\mathrm{in})}\right\rangle = |H\rangle_1 \otimes |H\rangle_2 = |1_1 0_2 1_3 0_4\rangle$ is transformed by a 50/50 beam splitter acting on dual-rail modes 2 and 3. This mode transform results in the output state $\left|\Psi^{(\mathrm{out})}\right\rangle = 1/\sqrt{2}\,|1_1 0_2 1_3 0_4\rangle + 1/\sqrt{2}\,|1_1 1_2 0_3 0_4\rangle$. Here the state $|1_1 0_2 1_3 0_4\rangle$ is in the computational space, whereas the state $|1_1 1_2 0_3 0_4\rangle$ is outside the computation space and $\alpha = \beta = 1/\sqrt{2}$. If we define $\left|\Psi^{(\mathrm{tar})}\right\rangle$ as $|1_1 0_2 1_3 0_4\rangle$, which is equivalent to saying we post select on detectors 1 and 3 firing, then the fidelity of the transformation is $f(\boldsymbol{U}) = 1$ and the success rate is $s(\boldsymbol{U}) = |\alpha|^2 = 1/2$.

Formation of Linear Cluster States. Here we consider the problem of joining two linear cluster states C_n and C_m containing qubits $(1,\ldots,n)$ and $(n+1,\ldots,n+m)$. In particular, we are interested in implementing the state transformation $|C_n\rangle|C_m\rangle \to |C_{n+m}\rangle$, equivalent to a maximally entangling operation, that maximizes the success probability of the transformation and achieves unit fidelity. Existing experimental cluster state generation schemes closely follow the original proposal of Raussendorf, Browne, and Briegel [6] where separate photonic qubits or small clusters are fused into a larger cluster state by a sequence of probabilistic optical *CZ* gates [7]. The *CZ* gate is conditioned on simultaneous detection of all photons with an overall probability of success 1/9, implying the success probability of $(1/9)^{n-1}$ for joining n unentangled optical qubits into a linear cluster state. However, the optimality of the *CZ* gate for generating linear optical cluster states has not been analyzed previously. Our recent results revealed that this method is far from optimal [8]. Employing numerical optimization based on methods developed in [8,9], we have identified a technique for constructing a linear cluster state C_{n+1} out of $n+1$ unentangled qubits with maximal success probability of $(1/2)^n$ while maintaining unit fidelity. This is in stark contrast to previous analytical results on the optimization of the *CZ* gate [10] where it was demonstrated that the maximal success rate of the *CZ* gate is 1/9. In resolving this paradox, we introduce a new theoretical approach for constructing linear optical gates for creating arbitrary-length clusters.

As revealed by numerical analysis [8], there exist various multi-mode (i.e. n-qubit) transformations that produce linear cluster states from an initial multi-qubit product state with the same maximal success probability. The structure of these solutions is too complex to allow a straightforward decomposition into a sequence of two-qubit and single-qubit transformations. However, we theorized that such structure may exist for two special cases: the addition of a single qubit and the addition of a Bell pair to a linear cluster state. This supposition was based primarily on the power-law dependence of the success probability first found in our previous work [8]. We now demonstrate this sequential decomposition.

II. ADDING A SINGLE QUBIT TO A LINEAR CLUSTER

First, let us consider the case when the first linear cluster $|C_n\rangle$ contains n qubits and the second cluster $|C_m\rangle$ is just a single qubit i.e., $|C_m\rangle \equiv |+\rangle$, where $|\pm\rangle \equiv (|H\rangle \pm |V\rangle)/\sqrt{2}$ (see Fig. 1). In general, the optimization of a linear optical transformation that joins $|C_n\rangle$ and $|C_1\rangle$ into $|C_{n+1}\rangle$ involves the entire set of $2(n+1)$ dual-rail modes of $n+1$ qubits. However, the numerical power-law result for the scaling of the success probability, $s = (1/2)^n$, found in [8] for the bulk transform $|+\rangle_1 \ldots |+\rangle_{n+1} \to |C_{n+1}\rangle$ with $n = 1,\ldots,7$ indicates that a concatenatable transformation acting only on two qubits at a time may exist. Each such operation acts only on the modes asscoiated with a single unentangled qubit and those associated with the last qubit of an existing cluster. We denote this operation as $C\tilde{Z}_1$, where the tilde indicates that this is not the standard *CZ* of [7] and the subscript stands for joining a single qubit to an existing cluster of arbitrary length. In other words, the addition of $n+1$ unentangled qubits into a linear cluster state is a product of n identical two-qubit four-mode $C\tilde{Z}_1$

operations. From the above scaling, the success probability of the $C\tilde{Z}_1$ operation must equal ½, which is significantly greater than the maximum success probability for the optical *CZ* gate.

It may come as a surprise that the *CZ* gate and $C\tilde{Z}_1$ operation perform an identical task with different success probability. An explanation can be found in quantum control theory [11] where one distinguishes two types of control problems. The first type is aimed at constructing a desired transformation (generally referred to as an *operator*) acting on entire Hilbert space. For example, in $\mathbf{C}^4$ the action of a *CZ* operator is formulated as $|H,H\rangle \to |H,H\rangle$, $|H,V\rangle \to |H,V\rangle$, $|V,H\rangle \to |V,H\rangle$, $|V,V\rangle \to -|V,V\rangle$ with the Hilbert space being $span\left[\{|H\rangle,|V\rangle\}^{\otimes N}\right]$. The second type (called *state control*) requires designing a quantum transformation affecting only a specific initial state of the system. In contrast, we consider a hybrid type of transformation acting on a *subspace* of the entire Hilbert space. The $C\tilde{Z}_1$ operation is of the hybrid kind while the canonical *CZ* gate is an operator acting on the full two-qubit $\mathbf{C}^4$ space, and the success rates are manifestly different.

Why is the full *CZ* gate not needed when adding a *single* qubit to an n-qubit cluster? Since the state of the $(n+1)^{\text{st}}$ qubit can be arbitrarily fixed, for example, to the $|+\rangle$ state, all transformations, including *CZ* and $C\tilde{Z}_1$, are acting only on C^2_{in}, the *subspace* of $\mathbf{C}^4$ spanned by states $|V\rangle_n|+\rangle$ and $|H\rangle_n|+\rangle$. The $\mathbf{C}^2_{in}$ space is being mapped onto two-dimensional subspaces of the full $\mathbf{C}^4$ space. If the action of a transformation on $\mathbf{C}^2_{in}$ is identical to the action of a *CZ* gate on $\mathbf{C}^2_{in}$ then the transformation will add one qubit to any C_n cluster, forming a C_{n+1} cluster state. This transformation must satisfy the set of equations determining its action on $\mathbf{C}^2_{in}$:

$$C\tilde{Z}_1|H\rangle|+\rangle = CZ|H\rangle|+\rangle = |H\rangle|+\rangle,$$
$$C\tilde{Z}_1|V\rangle|+\rangle = CZ|V\rangle|+\rangle = |V\rangle|-\rangle. \qquad \text{(3a,b)}$$

In the context of linear optical entangling gates conditioned on coincidence multimode photon detection, one should further relax the requirement on the $C\tilde{Z}_1$ operation by adding scaling factors α and β to reproduce Eq. (2). To distinguish an abstract $C\tilde{Z}_1$ operation satisfying equations (3a,b) from a linear optical transformation given by Eq. (2) we use the notation $C\tilde{Z}_1^{LO}$:

$$C\tilde{Z}_1^{LO}|H\rangle|+\rangle = \alpha|H\rangle|+\rangle + \beta\left|\Psi^{(\text{out})}_{II,H}\right\rangle,$$
$$C\tilde{Z}_1^{LO}|V\rangle|+\rangle = \alpha|V\rangle|-\rangle + \beta\left|\Psi^{(\text{out})}_{II,V}\right\rangle. \qquad \text{(4a,b)}$$

To find an optical mode transformation matrix $\boldsymbol{U}$ generating $C\tilde{Z}_1^{LO}$, Eqs. (4a,b) need to be combined with Eq. (1) resulting in a system of eight polynomial equations in matrix elements $\boldsymbol{U}_{i,j}$. These equations can be solved analytically using the standard Buchberger's algorithm [12], providing the following 4×4 mode transformation matrix $\boldsymbol{U} = \boldsymbol{A}\cdot\boldsymbol{B}\cdot\boldsymbol{C}$ where,

$$\boldsymbol{A} = e^{ix_2\sigma_z^{(1)}} \oplus e^{ix_1\sigma_x^{(2)}} e^{-ix_2\sigma_z^{(2)}},\ \boldsymbol{C} = \hat{I}^{(1)} \oplus e^{i\frac{\pi}{4}\sigma_y^{(2)}},$$
$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (5)$$

and $\hat{I}^{(1,2)}, \boldsymbol{\sigma}^{(1,2)}_{x,y,z}$ are the 2×2 identity and Pauli matrixes acting on the H and V modes of qubits 1 and 2 respectively. The essential part of the transformation $\boldsymbol{U}$ is the matrix $\boldsymbol{B}$ which performs the following mode operation: $\left(a_H^\dagger\right)_{1,2} \to \left(a_H^\dagger\right)_{1,2}, \left(a_V^\dagger\right)_1 \to -\left(a_V^\dagger\right)_2, \left(a_V^\dagger\right)_2 \to \left(a_V^\dagger\right)_1$. Solutions of the form (5), where x_1 and x_2 are arbitrary real parameters, automatically guarantee fidelity $f(\boldsymbol{U}) = 1$ and yield the maximum possible success probability $s(\boldsymbol{U}) = 1/2$. We remark that the mode transformation is defined in Eq. (1) such that operation $\boldsymbol{A}$ in Eq. (5) precedes operation $\boldsymbol{B}$ and $\boldsymbol{B}$ precedes operation $\boldsymbol{C}$.

The state transform corresponding to the mode operator $e^{ix_1\sigma_x^{(2)}}$, when acting on the states $|H\rangle|+\rangle$ and $|V\rangle|+\rangle$, only adds an overall phase since $\sigma_x|+\rangle = |+\rangle$. The matrix $e^{ix_2\sigma_z^{(1)}} \oplus e^{-ix_2\sigma_z^{(2)}}$ which corresponds to a local two-qubit operator $e^{ix_2\left(\sigma_z^{(1)}-\sigma_z^{(2)}\right)}$, acts as the identity on the space spanned by $|H\rangle|H\rangle$ and $|V\rangle|V\rangle$. Notice also that the space spanned by the states $|H\rangle|V\rangle$ and $|V\rangle|H\rangle$ is mapped outside the computational space by the next operation represented by the matrix $\boldsymbol{B}$. Therefore, parameters

x_1 and x_2 do not affect the state transformation and can be trivially set to zero.

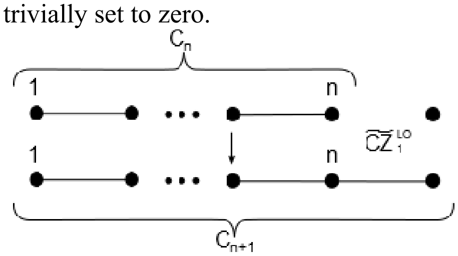

Figure 1. Adding a single qubit to a C_n cluster.

We would like to emphasize that the sequential character of the transformation (which allows for the entangling of n separable qubits as a sequence of four-mode $C\tilde{Z}_1^{LO}$ operations) is important for cluster state generation. First, a sequential approach can reduce the number of physical resources (beam splitters and wave-plates) needed to generate a cluster state. To sequentially generate a linear cluster $|C_n\rangle$ from a product state one only needs $n-1$ polarization-dependent and $n-1$ 50/50 beam splitters used repeatedly. In contrast, a generic $2n$-mode transformation requires $n(2n-1)$ beam splitters [13].

Secondly, unlike a global $2n$-mode operation, a sequential transformation can be implemented simultaneously with one-way computation. If an operation fails before the computation is completed, the remaining unused photonic resources can be saved and reused for another attempt. This can be quantified as follows for construction of a $|C_n\rangle$ cluster out of n qubits. If s is the success probability of a single gate, the probability that a failure first occurs at the m-th application of $C\tilde{Z}_1^{LO}$ (out of a total of $n-1$ applications) is $s^{m-1}(1-s)$, and $n-m-1$ photons are spared. Summing over m and noting that the overall success rate is s^{1-n}, we see that the average number of spared photons per successful computation is $\left[n-(2-s)/(1-s)\right]s^{1-n}+1/(1-s)$.

III. ADDING A BELL PAIR TO A LINEAR CLUSTER

Let us now consider the more complex case $m=2$ when the cluster $|C_2\rangle$ is added to $|C_n\rangle$. We again exploit the notion of hybrid operations and generalize the $C\tilde{Z}_1$ operation to $C\tilde{Z}_2$. The subscript 2 now reflects the fact that two qubits are being added to the cluster. Recall that the state $|C_2\rangle$ is, up to local rotations, equivalent to a Bell state: application of a Hadamard gate to the second qubit in the state $|\Phi^+\rangle$ transforms it into the cluster state $|C_2\rangle \equiv \left(|\Phi^-\rangle+|\Psi^+\rangle\right)/\sqrt{2}$.

The action of $C\tilde{Z}_2$ is analogous to the action of $C\tilde{Z}_1$ given by Eqs. (3a,b), where the state $|+\rangle$ is now replaced by the state $|C_2\rangle$. Similarly, we can define a linear optical transformation $C\tilde{Z}_2$ as follows,

$$\begin{aligned} C\tilde{Z}_2^{LO}|H\rangle|C_2\rangle &= \alpha|H\rangle|C_2\rangle+\beta\left|\Psi^{(\mathrm{out})}_{HH}\right\rangle, \\ C\tilde{Z}_2^{LO}|V\rangle|C_2\rangle &= \alpha|V\rangle|C_2^{\perp}\rangle+\beta\left|\Psi^{(\mathrm{out})}_{HV}\right\rangle, \end{aligned} \qquad (6a,b)$$

where state $|C_2^{\perp}\rangle = CZ|-\rangle|+\rangle$ is orthogonal to the standard cluster state $|C_2\rangle = CZ|+\rangle|+\rangle$.

Unfortunately, a complete analytical optimization of $C\tilde{Z}_2^{LO}$ is not possible due to the algebraic complexity of the problem. However, exploiting the idea of hybrid operations we can find an analytical solution, which reproduces our previous result $|\alpha|^2=1/4$ obtained by numerical optimization [8]. This solution again provides a higher success probability than CZ^{LO} by factor of $9/4$ [8].

To understand the analytic structure of this optimal solution, we start with an assumption that the addition of a Bell pair may be partitioned into two concatenated operations as indicated in Fig. 2: a polarization beam splitter (PBS) gate [14] acting on qubits n and $n+1$ followed by a "stretch" operation acting on qubits n+1 and n+2.

Any size initial cluster state $|C_n\rangle$ has the form $|C_n\rangle = \left(|C_{n-1}\rangle|H\rangle_n+|\tilde{C}_{n-1}\rangle|V\rangle_n\right)/\sqrt{2}$, where we use the notation $|\tilde{C}_{n-1}\rangle = \sigma_z^{(n-1)}|C_{n-1}\rangle$. After the action of a PBS transformation on qubits n and n+1 (see Fig. 2), with success probability 1/2, the state (which is the input state for the stretch gate) can be cast in the following form,

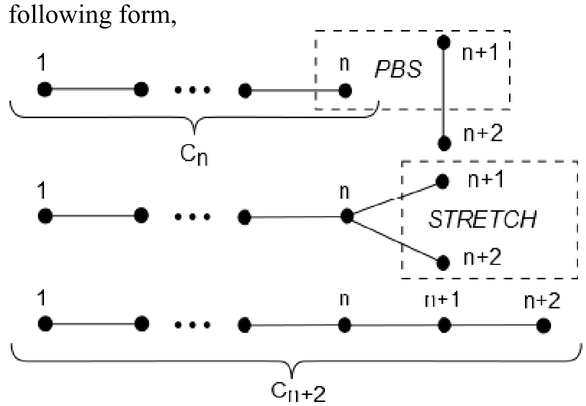

Figure 2. Joining a Bell pair to a C_n cluster.

$$\left|\Psi_{\text{in}}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|C_{n-1}\right\rangle\left|H\right\rangle_n\left|+,+\right\rangle_{n+1,n+2} + \left|\tilde{C}_{n-1}\right\rangle\left|V\right\rangle_n\left|-,-\right\rangle_{n+1,n+2}\right)$$

The final, or "target" state of the stretch gate is, $\left|\Psi_{\text{tar}}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|C_{n-1}\right\rangle\left|H\right\rangle_n\left|C_2\right\rangle_{n+1,n+2} + \left|\tilde{C}_{n-1}\right\rangle\left|V\right\rangle_n \sigma_z^{(n+1)}\left|C_2\right\rangle_{n+1,n+2}\right)$
Interestingly, the stretch gate does not affect the spatial mode *n*; only modes *n*+1 and *n*+2 are involved in the required optical transformations. Therefore, we can consider the action of the stretch gate as a stand-alone hybrid operation on two states:

$$\begin{aligned} &\left|+\right\rangle\left|+\right\rangle \to CZ\left|+\right\rangle\left|+\right\rangle \equiv \left|C_2\right\rangle \\ &\left|-\right\rangle\left|-\right\rangle \to CZ\left|-\right\rangle\left|+\right\rangle \equiv \sigma_z^{(1)}\left|C_2\right\rangle \end{aligned} . \qquad \text{(7a,b)}$$

The four-mode transformation matrix for the stretch gate operation can be defined as $\boldsymbol{U} = \boldsymbol{A}\cdot\boldsymbol{B}\cdot\boldsymbol{C}$,

$$\boldsymbol{B} = \begin{pmatrix} \sin(\pi/8) & 0 & -\cos(\pi/8) & 0 \\ 0 & \cos(\pi/8) & 0 & -\sin(\pi/8) \\ \cos(\pi/8) & 0 & \sin(\pi/8) & 0 \\ 0 & \sin(\pi/8) & 0 & \cos(\pi/8) \end{pmatrix},$$

$$\boldsymbol{A} = e^{i\frac{\pi}{4}\sigma_z^{(1)}} \oplus e^{-i\frac{\pi}{4}\sigma_z^{(2)}} e^{-i\frac{\pi}{4}\sigma_y^{(2)}}, \quad \boldsymbol{C} = e^{i\frac{\pi}{4}\sigma_y^{(1)}} \oplus e^{-i\frac{\pi}{4}\sigma_x^{(2)}} .$$

There is also a more general interpretation of the action of the stretch gate where the gate is considered as a tool for transporting entanglement along a linear cluster (see Fig. 3); the stretch gate implements the operation of moving the central qubit *n* one position while preserving any form of pre-existing entanglement of the central qubit with an arbitrary external system.

IV. ADDING MORE THAN TWO QUBITS

It is also possible to join larger linear clusters using the same approach. For instance, for the addition operations $C_n + C_3 \to C_{n+3}$ and $C_n + C_4 \to C_{n+4}$ we numerically found maximal success rates of 1/4 and 0.153, respectively. Another example is adding a qubit to the *middle* of a linear cluster state (a "grafting" operation) which normally requires three *CZ* gates resulting in a success probability of $1/9^3 \approx 0.00137$.

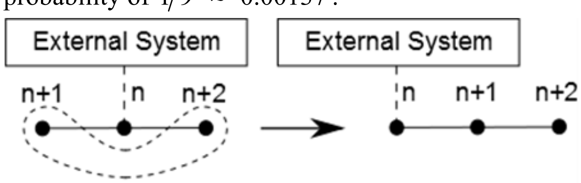

Figure 3. Entanglement swapping by the stretch gate.

When recast in the form of a hybrid operation this gate can be implemented with success probability of ≈ 0.0417 .

One counterintuitive result of our work is that the rate of production of the linear cluster states does not increase when more entanglement resources are invested in the preparation of the initial states. In particular, we have seen that adding a single photonic qubit $\left|C_1\right\rangle$ to a cluster has success probability $1/2$, and thus adding two separable photonic qubits sequentially to a cluster can be implemented with probability $1/4$, which is the same as our optimal success probability for adding a Bell pair $\left|C_2\right\rangle$ to the same initial cluster.

The idea of hybrid operations developed here has applications beyond linear cluster state generation. For instance, in the case of 2D clusters, a weaving operation [15] can be recast in the form of a hybrid gate. 2D clusters can also be formed by a combination of our new transform and standard *CZ* by first creating long linear chains and then using the standard *CZ* gate to create the final cluster state.

V. SUMMARY

We performed an analytical analysis of the problem of photonic cluster-state generation. We suggested a new scheme that provides the most efficient method of cluster state generation and requires no ancilla photons. Our analytical results demonstrate that previous methods of cluster state generation are far from optimal. The success probability of our scheme in comparison with traditional *CZ*-gate-based schemes grows exponentially with the size of the cluster. We expect that future experiments with photonic clusters will exploit this scheme to provide the most efficient realization of linear optics and quantum information technology.

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