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# Waste Not, Want Not: Heisenberg-Limited Metrology With Information Recycling

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Information recycling has been shown to improve the sensitivity of atom interferometers by exploiting atom-light entanglement. In this paper, we apply information recycling to an interferometer where the input quantum state has been partially transferred from some donor system. We demonstrate that when the quantum state of this donor system is from a particular class of number-correlated Heisenberg-limited states, information recycling yields a Heisenberg-limited phase measurement. Crucially, this result holds *irrespective* of the fraction of the quantum state transferred to the interferometer input and also for a general class of number-conserving quantum-state-transfer processes, including ones that destroy the first-order phase coherence between the branches of the interferometer. This result could have significant applications in Heisenberg-limited atom interferometry, where the quantum state is transferred from a Heisenberg-limited photon source, and in optical interferometry where the loss can be monitored.

When performing an interferometric measurement with a limited number of particles,  $N$ , there can be significant benefit to using a nonclassical input state to improve the phase sensitivity beyond the standard quantum noise limit (QNL) (shot-noise limit) of  $\Delta\phi \sim 1/\sqrt{N}$  [1, 2]. The ultimate limit to sensitivity is the Heisenberg limit  $\Delta\phi \sim 1/N$  [3, 4]. In particular, a Mach-Zehnder (MZ) interferometer can achieve Heisenberg-limited phase sensitivity if the input state has perfect number correlations between the two interferometer modes [5, 6]. An example is the two-mode squeezed vacuum state [7], which is routinely generated in quantum optics laboratories [2].

There exist metrological devices, however, where Heisenberg-limited input states are difficult to generate, such as inertial sensors based on atom interferometry. In such cases, Heisenberg-limited interferometry might still be possible provided a Heisenberg-limited state from a *donor system* (e.g., two-mode squeezed optical vacuum) can be mapped to this *acceptor system*. This possibility was demonstrated theoretically in [8], where quantum state transfer (QST) between squeezed light and atoms was shown to enhance the sensitivity of atom interferometry well below the QNL. Similar results are also possible in other contexts, as proposals exist for achieving QST between donor photons and a range of acceptor systems, including atomic motional states [9], room-temperature and laser cooled atomic vapours [10], Bose-Einstein condensates of dilute atomic vapors [11–15], ions [16], solid state systems [10], and mechanical oscillators [17].

Unfortunately, in practice any QST process is imperfect, and even a small degree of imperfection results in a large degradation of the acceptor system’s phase sensitivity from the Heisenberg limit [8, 18]. It was first shown in [19] that atom-light entanglement can be used to enhance the sensitivity of atom interferometry by applying the technique of information recycling. Furthermore, [8, 20] revealed that if this atom-light entanglement takes the form of a QST process, then in very specific situations, information recycling can help atom interferometers achieve Heisenberg-limited sensitivities. Here

we explicitly prove a generalized version of this result and identify the precise (but still very general) conditions under which it holds. That is, we show that if the donor source displays perfect number correlations, then the acceptor particles give Heisenberg-limited sensitivity *regardless of the QST efficiency* when used in a Mach-Zehnder (MZ) interferometer, provided information recycling is applied. This is true regardless of the physical mechanism for QST, provided that the QST process is number conserving.

*Number-correlated MZ interferometer.* To determine the best phase sensitivity possible for a given interferometry scheme, we appeal to the quantum Fisher information. As discussed in [6, 21], the quantum Fisher information  $\mathcal{F}$  places an absolute lower bound on the phase sensitivity,  $\Delta\phi \geq 1/\sqrt{\mathcal{F}}$ , called the quantum Cramér-Rao bound (QCRB), which applies regardless of the choice of measurement and phase estimation procedure; the bound depends only on the input state.

It is known [6, 21] that when a pure state is used as the input to a lossless MZ interferometer (i.e., beamsplitter-mirror-beamsplitter configuration), the quantum Fisher information for estimating a differential phase shift is given by  $\mathcal{F} = 4(\langle \hat{L}_y^2 \rangle - \langle \hat{L}_y \rangle^2)$ , where  $\hat{L}_k \equiv \frac{1}{2} \mathbf{b}^\dagger \sigma_k \mathbf{b}$  defines pseudo-spin operators,  $\mathbf{b} = (\hat{b}_1, \hat{b}_2)^T$ ,  $\hat{b}_j$  are the usual bosonic annihilation operators for the two modes, and  $\sigma_k$  are the set of Pauli spin matrices.

Consider now a two-mode state that displays perfect number correlations between the two input modes,

$$|\Psi_b\rangle = \sum_{N=0}^{\infty} c_N |N, N\rangle. \quad (1)$$

When used as the input to a MZ interferometer, the quantum Fisher information is given by

$$\mathcal{F}_b = \frac{V(\hat{N}_t) + N_t(N_t + 2)}{2}, \quad (2)$$

where  $\hat{N}_t = \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2$  is the operator for the total number of particles,  $N_t = \langle \hat{N}_t \rangle$  is its expectation value,

and  $V(\hat{X})$  denotes the variance of  $\hat{X}$ . For the twin-Fock state  $|\Psi_{\text{TF}}\rangle = |N/2, N/2\rangle$ , the variance is zero, so  $\mathcal{F}_b = N_t(N_t + 2)/2$ . Two-mode squeezed vacuum [22–24],

$$|\Psi_{\text{sq}}(r)\rangle = \text{sech } |r| \sum_{N=0}^{\infty} (-e^{-i\theta} \tanh |r|)^N |N, N\rangle, \quad (3)$$

with  $r = |r|e^{i\theta}$ , has variance  $V(\hat{N}_t) = N_t(N_t + 2)$  and thus  $\mathcal{F}_b = N_t(N_t + 2)$ .

For a particular choice of measurement signal,  $\hat{\mathcal{S}}$ , the phase uncertainty is given by  $\Delta\phi = \sqrt{V(\hat{\mathcal{S}})/|\partial_\phi\langle\hat{\mathcal{S}}\rangle|}$ . Input states of the form (1) have no mean field, so the resulting interferometer runs on what would conventionally be called noise; more precisely, they rely on second-order coherence [25] between the branches of the MZ interferometer, in contrast to the first-order coherence that is required for conventional interferometry. The signal choice  $\hat{\mathcal{S}} = \hat{L}_z^2$  is optimal at the operating points  $\phi = 0, \pi$ , giving a phase uncertainty [26]

$$\Delta\phi = \sqrt{\frac{2}{V(\hat{N}_t) + N_t(N_t + 2)}} \quad (4)$$

for sensing small changes away from the operating point. This signal choice thus achieves the QCRB.

Since the MZ interferometer does not require first-order coherence between the branches, the phase uncertainty (4) is achieved by any input (mixed) state of the form [26]

$$\hat{\rho}_b = \sum_{M,N=0}^{\infty} \rho_{MN} |M, M\rangle\langle N, N|, \quad (5)$$

not just by the pure states (1), for which  $\rho_{MN} = c_M c_N^*$ . We define  $p_N \equiv \rho_{NN}$ . When  $\rho_{MN}$  is diagonal, i.e.,  $\rho_{MN} = p_N \delta_{MN}$ , the number correlations between the input branches are purely classical.

*Donor-enhanced MZ interferometer.* Now suppose we want to map the Heisenberg-limited state  $\hat{\rho}_b$  from this donor system to some two-mode acceptor system. This scenario is depicted in Fig. 1. At  $t = t_0$ , the quantum state of the system is prepared such that the state of the donor system is  $\hat{\rho}_b$ , while the two modes of the acceptor system (annihilation operators  $\hat{a}_1$  and  $\hat{a}_2$ ) are unoccupied, giving a total state

$$\hat{\rho}(t_0) = \sum_{M,N=0}^{\infty} \rho_{MN} |M, 0, M, 0\rangle\langle N, 0, N, 0|. \quad (6)$$

A QST process is implemented such that at  $t = t_1$ , some or all of the particles are transferred from mode 1(2) of our donor system to mode 1(2) of our acceptor system. The acceptor particles are then used as the input to a MZ interferometer.

A perfect QST process performs the map  $|N, 0\rangle \rightarrow |0, N\rangle$  in each branch of the interferometer, and consequently the MZ interferometer composed of the two acceptor modes is Heisenberg limited. In practice, however,

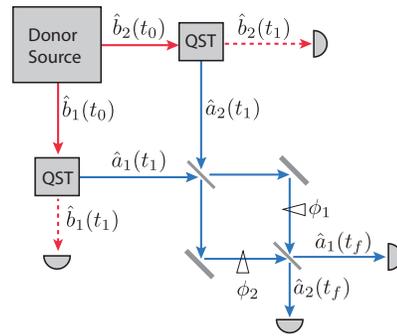


FIG. 1. Schematic of a donor-enhanced MZ interferometer. Initially, the two-mode donor system (annihilation operators  $\hat{b}_1$  and  $\hat{b}_2$ ) is prepared in the state  $\hat{\rho}_b$ ; both modes of the acceptor system (annihilation operators  $\hat{a}_1$  and  $\hat{a}_2$ ) are initially in vacuum, so we do not depict their inputs to the QST processes in the diagram. Each mode of the donor system undergoes some QST process, transferring part or all of its quantum state to the corresponding mode of the acceptor system at time  $t_1$ . The two modes of the acceptor system then form the inputs to a conventional MZ interferometer, which is sensitive to the differential phase shift  $\phi = \phi_1 - \phi_2$ . Information recycling is implemented by detecting the number of particles in all four output modes.

the QST process is imperfect. Some particles remain in the donor modes at time  $t_1$ , and this results in a loss of correlations when considering only the acceptor modes. As was shown in [6, 8], even a small loss of correlations can severely degrade sensitivity. Fortunately, we can reduce this degradation by monitoring those donor particles still remaining after the QST process and incorporating this information as part of our phase-estimation procedure. This technique of *information recycling* has been shown to enhance the sensitivity within specific atom interferometric schemes reliant on two-photon Raman transitions [8, 19]. The surprising result we show here is that a Heisenberg-limited donor source coupled with information recycling yields Heisenberg-limited interferometry with the acceptor modes *irrespective of the QST efficiency* or the physical mechanism of the QST process.

To show this, we now consider the state after incomplete QST. Without specifying the physical mechanism of the QST process, we apply the following physically motivated constraints:

1. The QST process occurs in two *independent branches*; i.e., donor mode  $\hat{b}_1$  ( $\hat{b}_2$ ) can only exchange particles with acceptor mode  $\hat{a}_1$  ( $\hat{a}_2$ ), and neither branch is affected by the other.
2. Each branch of the QST process *conserves particle number*; i.e.,  $\hat{b}_j^\dagger \hat{b}_j + \hat{a}_j^\dagger \hat{a}_j$  is a conserved quantity for  $j = 1, 2$ .
3. The QST process is *symmetric* with respect to the exchange  $\hat{b}_1 \leftrightarrow \hat{b}_2$  and  $\hat{a}_1 \leftrightarrow \hat{a}_2$ ; i.e., the two inde-

pendent branches of the QST process are identical.

Although a *beam splitter* transformation, and (in the low depletion regime), the atom-light QST process from [8] satisfy these requirements, these conditions are also satisfied by a broad class of QST processes, both unitary and nonunitary. For example, they allow very complicated QST processes where the QST Hamiltonian contains higher-order couplings; heuristically, this might result in a QST efficiency that depends on the number of particles in the donor mode. Furthermore, the constraints allow for situations where the QST process is mediated by some other set of modes  $\hat{c}_k$  (e.g., a reservoir), which might be depleted and thus reduce the QST efficiency as more particles are transferred, as seen in [8]. A somewhat fanciful, but certainly not the most general Hamiltonian that satisfies the constraints of such a QST process is

$$\hat{H} = \sum_{i=1,2} \sum_{n,m,l} h_{qpk}^{nml} \left[ (\hat{a}_i^\dagger \hat{a}_i)^n (\hat{b}_i^\dagger \hat{b}_i)^m (\hat{a}_i^\dagger \hat{b}_i)^l (\hat{c}_{i,k}^\dagger)^q \hat{c}_{i,k}^p + h.c. \right]. \quad (7)$$

A general QST process that satisfies the conditions 1–3 performs the following map in each branch:

$$|M, 0\rangle \langle N, 0| \rightarrow \sum_{m=0}^M \sum_{n=0}^N A_{Mm, Nn} |M - m, m\rangle \langle N - n, n|. \quad (8)$$

There are no constraints on  $A_{Mm, Nn}$  other than the usual physical constraints of normalization and complete positivity.  $P_{n|N} \equiv A_{Nn, Nn}$  is the conditional probability that there are  $n$  particles in an acceptor mode, given  $N$  particles initially in the corresponding donor mode.

Under the QST map (8), the state  $\hat{\rho}(t_0)$  of Eq. (6) is mapped to the (generally mixed) state

$$\begin{aligned} \hat{\rho}(t_1) = & \sum_{M, N=0}^{\infty} \rho_{MN} \sum_{\substack{m_1, n_1 \\ m_2, n_2}} A_{Mm_1, Nn_1} A_{Mm_2, Nn_2} \\ & \times |M - m_1, m_1, M - m_2, m_2\rangle \langle N - n_1, n_1, N - n_2, n_2|. \end{aligned} \quad (9)$$

Notice that we only require that number correlations between the branches be maintained; dephasing within or between the branches is perfectly acceptable.

Introducing the pseudo-spin operators for the acceptor modes,  $\hat{J}_k \equiv \frac{1}{2} \mathbf{a}^\dagger \sigma_k \mathbf{a}$ , where  $\mathbf{a} = (\hat{a}_1, \hat{a}_2)^T$ , the unitary operator for the MZ interferometer performs the following transformations:  $\hat{J}_z(t_f) = U^\dagger \hat{J}_z(t_1) U = \hat{J}_z(t_1) \cos \phi - \hat{J}_x(t_1) \sin \phi$ , and  $\hat{L}_z(t_f) = U^\dagger \hat{L}_z(t_1) U = \hat{L}_z(t_1)$ , since only the acceptor particles take part in the interferometric process. As in [8], we estimate the phase by measuring the number of particles at the four output ports (see Fig. 1) and constructing the signal  $\hat{S} = [\hat{J}_z(t_f) + \hat{L}_z(t_f)]^2$ . Although only  $\hat{J}_z$  contains phase

information, the noise in  $\hat{J}_z$  is anticorrelated with  $\hat{L}_z$ , so measuring both quantities allows us to correct for this noise and therefore improve sensitivity.

To evaluate the phase sensitivity, we need the first and second moments of  $\hat{S}$  in the state (9). Since the QST process and the angular-momentum operators preserve total particle number, there is no interference between sectors with different numbers of particles; the desired moments are averages over  $p_N = \rho_{NN}$ . The anticorrelation of  $\hat{J}_z$  and  $\hat{L}_z$ , expressed by  $\hat{J}_z \hat{\rho}(t_1) = -\hat{L}_z \hat{\rho}(t_1)$ , allows us to convert  $\hat{L}_z$  in these moments to  $\hat{J}_z$ . The anticorrelation implies that  $\hat{\rho}(t_1)$  is invariant under rotations about the  $z$  axis; in particular, a rotation by  $\pi$ , which takes  $\hat{J}_x$  to  $-\hat{J}_x$ , implies that all terms with an odd number of  $\hat{J}_x$  operators have vanishing expectation value. At the most sensitive operating point,  $\phi = 0$ , the phase sensitivity is [26]

$$\Delta\phi = \frac{\sqrt{V(\hat{S})}}{|\partial_\phi \langle \hat{S} \rangle|} = \frac{1}{2 \langle \hat{J}_x^2 \rangle^{1/2}} = \sqrt{\frac{1}{2 \langle \hat{N}_1 \hat{N}_2 \rangle + N_a}}, \quad (10)$$

where  $\hat{N}_j = \hat{a}_j^\dagger(t_1) \hat{a}_j(t_1)$ , and  $N_a = \langle \hat{N}_1 + \hat{N}_2 \rangle$  is the average number of acceptor particles and thus the number of particles that take part in the interferometric process.

We can put a lower bound on  $\langle \hat{N}_1 \hat{N}_2 \rangle$  by noting that a state of the form (9) gives

$$\langle \hat{N}_1 \hat{N}_2 \rangle = \sum_{N=0}^{\infty} p_N \langle \hat{N}_1 \rangle_N \langle \hat{N}_2 \rangle_N = \sum_{N=0}^{\infty} p_N \langle \hat{N}_1 \rangle_N^2. \quad (11)$$

Here  $\langle \hat{N}_j \rangle_N = \sum_{n_j=0}^N n_j P_{n_j|N}$  is the *conditional expectation value* of the number of particles in acceptor mode  $j$ , given  $N$  initial particles in donor mode  $j$ . That the conditional probabilities are the same in the two branches ensures that  $\langle \hat{N}_1 \rangle_N = \langle \hat{N}_2 \rangle_N$ . Convexity implies that

$$\langle \hat{N}_1 \hat{N}_2 \rangle \geq \left( \sum_{N=0}^{\infty} p_N \langle \hat{N}_1 \rangle_N \right)^2 = \langle \hat{N}_1 \rangle^2 = \frac{1}{4} N_a^2, \quad (12)$$

which gives an upper bound on the phase sensitivity of any QST process applied to the initial state  $|\Psi(t_0)\rangle$ ,

$$\Delta\phi \leq \sqrt{\frac{2}{N_a(N_a + 2)}} \simeq \frac{\sqrt{2}}{N_a}. \quad (13)$$

The important feature of this result is that the Heisenberg limit is recovered, with respect to the number of particles,  $N_a$ , taking part in the interferometer, rather than the total number of particles  $N_t$ . Although the absolute sensitivity is less than with perfect QST, this is purely due to loss of particles, rather than to loss of correlations. We stress that this is not the *true* Heisenberg limit, in the sense that we have used  $N_t \geq N_a$  particles to make the measurement, but only  $N_a$  of them have passed through the interferometer. Without the application of information recycling, however, the sensitivity is significantly worse than  $1/N_a$  [26].

For the specific case when the donor source is a twin-Fock state,  $|\Psi_b\rangle = |\Psi_{\text{TF}}\rangle$ , we get  $\langle \hat{N}_1 \hat{N}_2 \rangle = \langle \hat{N}_1 \rangle \langle \hat{N}_2 \rangle$ , which gives a phase sensitivity that saturates the bound (13) and is entirely independent of the QST efficiency or even the form of the number-conserving QST interaction. For other initial states, there might be a weak dependence on the QST process (as seen for the beamsplitting case below); nevertheless the phase sensitivity is guaranteed to be at least as good as that given by the twin-Fock state. To be more quantitative about the performance of states other than  $|\Psi_{\text{TF}}\rangle$ , we need to specify a particular Hamiltonian governing the QST process.

*Beamsplitter QST process.* We now consider the simplest possible QST process, a beamsplitter. The Hamiltonian describing this process,  $\hat{H} \propto \sum_{j=1,2} (\hat{a}_j \hat{b}_j^\dagger + \hat{a}_j^\dagger \hat{b}_j)$ , leads to the unitary transformation

$$\hat{a}_j(t_1) = \sqrt{1 - \mathcal{Q}} \hat{a}_j(t_0) - i\sqrt{\mathcal{Q}} \hat{b}_j(t_0), \quad (14a)$$

$$\hat{b}_j(t_1) = \sqrt{1 - \mathcal{Q}} \hat{b}_j(t_0) - i\sqrt{\mathcal{Q}} \hat{a}_j(t_0). \quad (14b)$$

Here  $\mathcal{Q}$  is the *QST efficiency*, i.e., the fraction of donor particles mapped to the acceptor modes.

The transformation (14) allows us to evaluate Eq. (10) explicitly to determine the precise dependence on the QST efficiency. With the initial state (6), we get  $\langle \hat{N}_1 \hat{N}_2 \rangle = (\mathcal{Q}^2 V(\hat{N}_t) + \langle N_a \rangle^2)/4$ , and the phase sensitivity in the presence of information recycling is

$$\Delta\phi = \sqrt{\frac{2}{\mathcal{Q}^2 V(\hat{N}_t) + N_a(N_a + 2)}}. \quad (15)$$

For a twin-Fock input, which has  $V(\hat{N}_t) = 0$ , the phase sensitivity does not depend on  $\mathcal{Q}$  and is given by the bound in Eq. (13). When the donor state is two-mode squeezed vacuum,  $|\Psi_b\rangle = |\Psi_{\text{sq}}\rangle$ , we find that  $\Delta\phi = 1/\sqrt{N_a(N_a + 1 + \mathcal{Q})}$ , which has only a weak dependence on  $\mathcal{Q}$ . Indeed, it is clear that to leading order in the total number of acceptor particles,  $N_a = \mathcal{Q}N_t$ , the sensitivity (15) has Heisenberg scaling for any donor input state (5), *regardless of the QST efficiency*  $\mathcal{Q}$ . This gives a clear illustration of the power of information recycling as a tool to enable quantum metrology.

It is instructive to compute the quantum Fisher information  $\mathcal{F}_a$  for the donor-acceptor interferometer. With the pure initial state (1) and a beamsplitter QST process, the state remains pure, and the quantum Fisher information is simply  $\mathcal{F}_a = 4[\langle \hat{J}_y(t_1)^2 \rangle - \langle \hat{J}_y(t_1) \rangle^2]$ . The transformations (14) allow us to compute these expectations with respect to the initial state. Since the acceptor modes are initially vacuum, we obtain

$$\mathcal{F}_a = \mathcal{Q}^2 \mathcal{F}_b + (1 - \mathcal{Q}) N_a = \frac{\mathcal{Q}^2 V(\hat{N}_t) + N_a(N_a + 2)}{2}. \quad (16)$$

Comparing with the sensitivity (15), it is clear that our information-recycled signal achieves the best possible Heisenberg scaling, i.e., by saturating the QCRB.

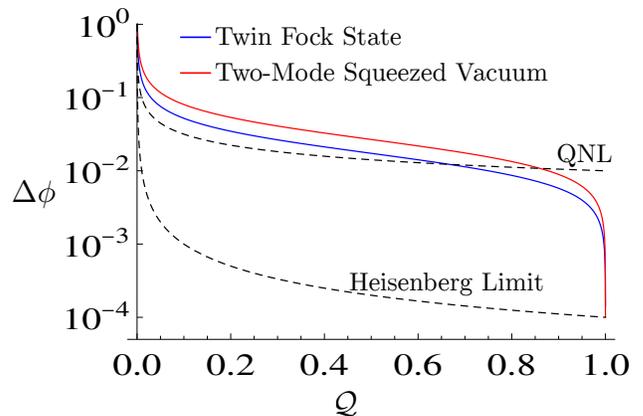


FIG. 2. Examples of the QST-dependence of the phase sensitivity at the optimal operating point without information recycling (i.e.,  $\hat{S} = \hat{J}_z^2$ ), assuming a beamsplitter QST process, for initial donor states  $|\Psi_{\text{TF}}\rangle$  (solid blue) and  $|\Psi_{\text{sq}}\rangle$  (solid red) and using  $N_t = 10^4$ . The upper and lower dashed lines mark the standard QNL,  $\Delta\phi = 1/\sqrt{N_a}$ , and the Heisenberg limit,  $\Delta\phi = 1/N_a$ . Heisenberg scaling is rapidly lost for small departures from perfect QST. In contrast, the sensitivity (15) with information recycling has Heisenberg scaling  $\propto 1/N_a$  for all  $\mathcal{Q}$ . The analytic expressions for the sensitivity  $\Delta\phi$ , as a function of  $\phi$  and at the optimal operating point, are in the Supplemental Material [26].

In contrast to these results, when information recycling is not applied, the beamsplitter QST process acts as a linear loss mechanism and Heisenberg scaling is lost (see Fig. 2). This loss of Heisenberg scaling occurs for relatively small deviations of  $\mathcal{Q}$  from perfect QST and affects any initial state of the form (5) [26] (see also [27, 28]).

*Applications.* Donor-enhanced interferometry with information recycling requires the following: (i) a correlated source of donor particles, (ii) partial QST between the donor particles and some acceptor system that operates in two independent and symmetric branches, and (iii) the ability to detect both donor and acceptor particles. It might be particularly useful in situations where there are abundant donor particles and a limited number of acceptor particles [such as QST from photons (donor) to atoms (acceptor) for the purposes of atom interferometry], since the QST efficiency becomes irrelevant once  $N_a$  equals the total number of available acceptor particles. In addition to Heisenberg-limited atom interferometry, another potential application for this scheme is optical interferometry which requires coupling into optical fibers before an interferometer (Fig. 3). Here, coupling between the freely propagating modes (donor system) and the fiber modes (acceptor system) represents the QST process. Typically there will be some scattering into other modes, which is a source of inefficient QST. Information recycling could be implemented by detecting the scattered photons. Since our scheme only requires photon counting, rather than homodyne detection, in-

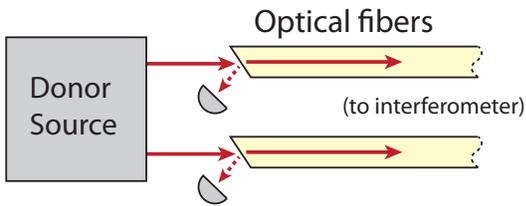


FIG. 3. Free-space photons from a donor source are coupled into optical fibers, which form the arms of a MZ interferometer. By measuring scattered photons, information recycling can be used to counteract the deleterious effects of inefficient coupling between the donor photon source and the optical fibers.

formation recycling could still be implemented even if the scattering is incoherent, and into a range of spatial modes.

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