Rényi generalizations of quantum information measures

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Quantum information measures such as the entropy and the mutual information find applications in physics, e.g., as correlation measures. Generalizing such measures based on the Rényi entropies is expected to enhance their scope in applications. We prescribe Rényi generalizations for any quantum information measure which consists of a linear combination of von Neumann entropies with coefficients chosen from the set $\{-1, 0, 1\}$. As examples, we describe Rényi generalizations of the conditional quantum mutual information, some quantum multipartite information measures, and the topological entanglement entropy. Among these, we discuss the various properties of the Rényi conditional quantum mutual information and sketch some potential applications. We conjecture that the proposed Rényi conditional quantum mutual informations are monotone increasing in the Rényi parameter, and we have proofs of this conjecture for some special cases.

I. INTRODUCTION

Quantum information theory shares large common grounds with a multitude of areas in physics. The von Neumann entropy, which is a central measure of information, for instance, is widely used under the name of entanglement entropy to study entanglement in ground states of quantum many body systems and lattice systems [1, 2], relativistic quantum field theory [3, 4], and the holographic theory of black holes [5, 6]. The entanglement entropy is calculated as the entropy of the reduced state of a subsystem which consists of any bounded region in a large system, where the overall system is assumed to be in a pure state. The entanglement entropy captures the entanglement shared between the subsystem and the rest of the system across the boundary. A large body of work in the literature has focused on various questions related to the entanglement entropy, such as its scaling with respect to the size of the subsystem. The area law establishes that this scaling depends only on the area of the boundary [7–12].

Recently, the class of α-Rényi entropies [13] has been considered in the above contexts [14–16]. The α-Rényi quantum entropy is a generalization of the von Neumann entropy. For a state ρ_A on a quantum system A, the α-Rényi entropy is defined as

$$H_\alpha(A)_\rho \equiv [1 - \alpha]^{-1} \log \text{Tr}\{\rho_A^\alpha\}, \qquad (1)$$

where the parameter $\alpha \in (0, 1) \cup (1, \infty)$. (It is traditionally defined for $\alpha \in \{0, 1, \infty\}$ in the limit as α approaches 0, 1, and ∞, respectively). The α-Rényi entropy is non-negative, additive on tensor-product states, and converges to the von Neumann entropy $H(A)_\rho \equiv$

$-\text{Tr}\{\rho_A \log \rho_A\}$ in the limit as the Rényi parameter α tends to one. The α-Rényi entanglement entropies characterize the entanglement spectra of condensed matter systems, akin to moments of a probability distribution [17, 18]. In Gaussian quantum information theory, the $\alpha = 2$ Rényi entropy is useful in studying Gaussian entanglement and other more general quantum correlations [19]. In quantum thermodynamics, α-Rényi entropies represent the derivative of the free energy with respect to temperature [20] and are relevant for the work value of information [21]. In quantum information theory, the von Neumann entropy captures the amount of quantum information contained in an ensemble of a large number of independent and identically distributed (i.i.d.) copies of the state [22], while the α-Rényi entropies (for values of α other than one) are relevant in scenarios beyond the i.i.d. setting [23]. For example, the α-Rényi entropies are useful for characterizing information processing tasks in regimes of a single or finite number of resource utilizations [24–27] and for establishing strong converse theorems [28–35]. Given the rich variety of applications of the Rényi entropies, there has been a substantial effort towards obtaining Rényi generalizations of other information measures, such as the conditional quantum entropy (CQE), or the quantum mutual information (QMI) [26, 29, 30, 32–34, 36, 37]. Recently, the authors of the present paper have contributed to this effort by proposing Rényi generalizations of the conditional quantum mutual information (CQMI) [38].

In this paper, we consider more generally a whole class of quantum information measures—we prescribe Rényi generalizations for any quantum information measure which is equal to a linear combination of von Neumann entropies with coefficients chosen from the set $\{-1, 0, 1\}$.

This criterion is met by many useful measures including the CQE, the QMI, and the CQMI, which are defined respectively as

$$H(A|B)_\rho \equiv H(AB)_\rho - H(B)_\rho, \qquad (2)$$

$$I(A;B)_\rho \equiv H(A)_\rho + H(B)_\rho - H(AB)_\rho, \qquad (3)$$

$$I(A;B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho$$
$$- H(ABC)_\rho, \qquad (4)$$

where ρ is taken to be a bipartite state ρ_{AB} in (2) and (3), and a tripartite state ρ_{ABC} in (4). As examples, we describe Rényi CQMIs, some quantum multipartite information measures and topological entanglement entropy (TEE) obtained based on the prescription. In particular, we focus on the Rényi CQMIs; we discuss the desired properties of the von Neumann CQMI that they retain, and some applications. We conjecture that the proposed Rényi CQMIs are monotone increasing in the Rényi parameter, with established proofs for some special cases. The solution of this conjecture would imply a characterization of states with CQMI nearly equal to zero, which could be helpful for solving some open problems in quantum information theory [39, 40] and condensed matter physics [41, 42].

The paper is organized as follows. In Section II, we set the stage by describing the earlier approaches to Rényi generalization of quantum information measures and their shortcomings. In Section III, we present the new prescription for Rényi generalization of quantum information measures which are equal to a linear combination of von Neumann entropies with coefficients chosen from the set $\{-1, 0, 1\}$. In Section IV, we describe examples, namely, Rényi CQMI, some Rényi multipartite information measures and Rényi TEE. Finally, we state our conclusions in Section V.

II. BACKGROUND

Suppose that we would like to establish a Rényi generalization of the following linear combination of entropies:

$$\sum_{S \subseteq \{A_1, \ldots, A_l\}} a_S H(S)_\rho, \qquad (5)$$

where $\rho_{A_1 \ldots A_l}$ is a density operator on l systems, the coefficients $a_S \in \{-1, 0, 1\}$, and the sum runs over all subsets of the systems $A_1, \ldots, A_l$. A first approach one might consider is simply to replace the linear combination of von Neumann entropies with the corresponding linear combination of α-Rényi entropies:

$$\sum_{S \subseteq \{A_1, \ldots, A_l\}} a_S H_\alpha(S)_\rho. \qquad (6)$$

However, the work of [43] establishes that there are no universal constraints on such a quantity. For example, consider the following quantity obtained by replacing the von Neumann entropies in (4) with α-Rényi entropies, namely

$$I'_\alpha(A;B|C)_\rho \equiv H_\alpha(AC)_\rho + H_\alpha(BC)_\rho - H_\alpha(C)_\rho$$
$$- H_\alpha(ABC)_\rho. \qquad (7)$$

For $\alpha \in (0,1) \cup (1,\infty)$, this quantity does not generally satisfy non-negativity [43], while the von Neumann CQMI is known to be non-negative, the latter being a result of the strong subadditivity inequality [44]. Since strong subadditivity is consistently useful in applications and often regarded as a "law of quantum information theory," the work in [43] suggests that the Rényi generalization in (6) is perhaps not the appropriate one to be using in applications [45].

On the other hand, one can write a quantum information measure in terms of the relative entropy, and subsequently replace the relative entropy with a Rényi relative entropy [46], [36], [33] in order to obtain a Rényi generalization of the measure. For density operators ρ and σ, the relative entropy is defined as

$$D(\rho \| \sigma) \equiv \text{Tr}\{\rho \log \rho\} - \text{Tr}\{\rho \log \sigma\} \qquad (8)$$

if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$, and it is equal to $+\infty$ otherwise. The Rényi relative entropy between density operators ρ and σ is defined for $\alpha \in [0,1) \cup (1,\infty)$ as [46]

$$D_\alpha(\rho \| \sigma) \equiv \frac{1}{\alpha - 1} \log \text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\} \qquad (9)$$

if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ or $(\alpha \in [0,1)$ and $\rho \not\perp \sigma)$. It is equal to $+\infty$ otherwise (these support conditions were established in [47]). The von Neumann entropy of a state ρ_A on system A can itself be written in terms of the relative entropy as $-D(\rho_A \| I_A)$, where I_A is the identity operator. The CQE and the QMI can also be written in terms of the relative entropy as

$$H(A|B)_\rho = -\min_{\sigma_B} D(\rho_{AB} \| I_A \otimes \sigma_B), \qquad (10)$$

$$I(A;B)_\rho = \min_{\sigma_B} D(\rho_{AB} \| \rho_A \otimes \sigma_B), \qquad (11)$$

respectively, where ρ_A is the reduced density operator $\text{Tr}_B\{\rho_{AB}\}$ and σ_B is any density operator on the Hilbert space $\mathcal{H}_B$ of system B. (The unique optimum σ_B in the above expressions turns out to be the reduced density operator ρ_B.) Therefore, one can obtain Rényi generalizations of the above quantities by using the Rényi relative entropy in place of the relative entropy. Rényi generalizations of quantum information measures obtained via the above procedure converge to the corresponding von Neumann entropy based quantities in the limit as α tends to one. They also retain most of the desired properties of the original quantities. For example, a Rényi QMI obtained from (11), just like the original von Neumann entropy based quantity, is non-negative and non-increasing under the action of local completely positive and trace preserving (CPTP) maps for $\alpha \in [0,1) \cup (1,2]$. This is because the Rényi relative entropy for $\alpha \in [0,1) \cup (1,2]$,

just like the relative entropy, is non-negative and non-increasing under the action of any CPTP map, in the sense that

$$D_\alpha(\rho\|\sigma) \geq D_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \qquad (12)$$

for a quantum map $\mathcal{N}$ [46].

One could also use the sandwiched Rényi relative entropy [36], [33]—a new variant of the Rényi relative entropy—instead of the Rényi relative entropy. The sandwiched Rényi relative entropy has found a number of applications in quantum information theory recently in the context of strong converse theorems [33], [27], [34], [35]. It is defined for $\alpha \in (0,1) \cup (1,\infty)$ as

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1} \log\left[\text{Tr}\left\{\left(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right)^\alpha\right\}\right] \qquad (13)$$

if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ or ($\alpha \in (0,1)$ and $\rho \not\perp \sigma$). It is equal to $+\infty$ otherwise. The sandwiched Rényi relative entropy is non-negative and non-increasing under the action of any CPTP map for $\alpha \in [1/2,1) \cup (1,\infty)$ [48]. Sandwiched Rényi generalizations of quantum information measures as discussed above thus also satisfy the above properties.

In order to write an information quantity in terms of a relative entropy, the key task is to identify the second argument for the relative entropy. This task, however, can be nontrivial in some cases. For example, it is not obvious as to what the second argument should be for the CQMI. Taking a cue from the QMI of (11), in which the second argument (when suitably normalized) has vanishing QMI, one may try to write the CQMI as an optimized relative entropy with respect to the set of quantum Markov states [49], which are defined as those tripartite states which have zero CQMI. However, it has been shown that such a quantity is not equal to the CQMI, and can in general be arbitrarily large compared to the latter [50]. Therefore, the relative entropy distance to the set of quantum Markov states does not lead to a good definition for Rényi CQMI.

III. PRESCRIPTION FOR RÉNYI GENERALIZATION

Having discussed the general approach towards obtaining Rényi generalizations of quantum information measures of the form given in (5), and the hurdles faced, we now give our prescription for a Rényi generalization. It is also based on the relative entropy and its variants.

In the case that $a_{A_1\ldots A_l}$ is nonzero, without loss of generality, we can set $a_{A_1\ldots A_l} = -1$ (otherwise, factor out -1 to make this the case). Then, we can rewrite the quantity in (5) in terms of the relative entropy as follows:

$$D\left(\rho_{A_1\ldots A_l}\left\|\exp\left\{\sum_{S\subseteq A'} a_S \log\rho_S\right\}\right.\right), \qquad (14)$$

where $A' = \{A_1,\ldots,A_l\}\backslash A_1\cdots A_l$. On the other hand, if $a_{A_1\ldots A_l} = 0$, i.e., if all the marginal entropies in the sum are on a number of systems that is strictly smaller than the number of systems over which the state ρ is defined (as is the case with $H(AB)+H(BC)+H(AC)$, for example), we can take a purification of the original state and call this purification the state $\rho_{A_1\ldots A_l}$. This state is now a pure state on a number of systems strictly larger than the number of systems involved in all the marginal entropies. We then add the entropy $H(A_1\ldots A_l)_\rho = 0$ to the sum of entropies and apply the above recipe (so we resolve the issue with this example by purifying to a system R, setting the sum formula to be $H(ABCR) + H(AB) + H(BC) + H(AC)$, and proceeding with the above recipe).

We then appeal to a multipartite generalization of the Lie-Trotter product formula [51] to rewrite the second argument in (14) as

$$\lim_{\alpha\to 1}\left[\bigcirc_{S\subseteq A'}\Theta_{\rho_S^{a_S(1-\alpha)/2}}\left(I_{A_1\ldots A_l}\right)\right]^{1/(1-\alpha)}, \qquad (15)$$

where the map

$$\Theta_{\rho_S^{a_S(1-\alpha)/2}}(X) \equiv \rho_S^{a_S(1-\alpha)/2} X \rho_S^{a_S(1-\alpha)/2} \qquad (16)$$

and the composition $\bigcirc$ of maps $\Theta_{\rho_S^{a_S(1-\alpha)/2}}$ for all subsets S can proceed in any order, and $I_{A_1\ldots A_l}$ is the identity operator on the support of the state $\rho_{A_1\ldots A_l}$. Finally, we obtain a Rényi generalization of the linear combination in (5) as

$$D_\alpha\left(\rho_{A_1\ldots A_l}\left\|\left[\bigcirc_{S\subseteq A'}\Theta_{\rho_S^{a_S(1-\alpha)/2}}\left(I_{A_1\ldots A_l}\right)\right]^{1/(1-\alpha)}\right.\right), \qquad (17)$$

where D_α is the Rényi relative entropy, and we have used (15) in (14) and promoted the parameter α in (15) to take the role of the Rényi parameter.

A similar Rényi generalization can also be obtained using the sandwiched Rényi relative entropy as

$$\widetilde{D}_\alpha\left(\rho_{A_1\ldots A_l}\left\|\left[\bigcirc_{S\subseteq A'}\Theta_{\rho_S^{a_S(1-\alpha)/(2\alpha)}}\left(I_{A_1\ldots A_l}\right)\right]^{\alpha/(1-\alpha)}\right.\right). \qquad (18)$$

Note that there further exist a number of different possible variants of the above Rényi generalizations since different choice of orderings of the maps $\Theta_{\rho_S^{a_S(1-\alpha)/2}}$ are possible. Moreover, we could also consider arbitrary density operators on the appropriate subsystems for the maps Θ instead of the reduced density operators (marginals) of $\rho_{A_1\ldots A_l}$, and then optimize over these operators (under the assumption that the support of $\rho_{A_1\ldots A_l}$ is contained in the intersection of the supports of these operators).

For any quantum information measure, it is possible to prove that these different Rényi generalizations converge to the original von Neumann entropy based quantity in

(5) in the limit as $\alpha \to 1$. Also, consider that we can write the linear combination in (5) as

$$\sum_{S \subseteq \{A_1, \dots, A_l\}} a_S H(S)_\rho$$

$$= -\operatorname{Tr}\left\{\rho_{A_1 \dots A_l}\left[\sum_{S \subseteq \{A_1, \dots, A_l\}} a_S \log \rho_S\right]\right\}. \quad (19)$$

The *information second moment* corresponding to this combination of entropies is then

$$V(\rho_{A_1 \dots A_l}, \{a_S\})$$

$$\equiv \operatorname{Tr}\left\{\rho_{A_1 \dots A_l}\left[\sum_{S \subseteq \{A_1, \dots, A_l\}} a_S \log \rho_S\right]^2\right\}. \quad (20)$$

It can be shown that the Rényi generalization in (17) has the following Taylor expansion about $\gamma = 0$, where $\gamma = \alpha - 1$:

$$\frac{1}{\gamma}\log\left[\operatorname{Tr}\{\rho_{A_1 \dots A_l}\} + \gamma \sum_{S \subseteq \{A_1, \dots, A_l\}} a_S H(S)_\rho\right.$$

$$\left. + \frac{\gamma^2}{2}V(\rho_{A_1 \dots A_l}, \{a_S\}) + O(\gamma^3)\right], \quad (21)$$

thus recovering the information second moment as the second order term in the Taylor expansion. (See [38, Appendix E.1], for example, which shows the Taylor expansion in a neighborhood of $\gamma = 0$ for the Rényi CQMI.) Note that the Rényi generalization in (6) does not recover the information second moment in a Taylor expansion. Furthermore, we leave it as an open question to determine whether the following statement is generally true: *if a von Neumann entropy based measure is non-negative and non-increasing under the action of local CPTP maps, then its Rényi generalizations of the above type are also non-negative and non-increasing under local CPTP maps.*

IV. EXAMPLES OF RÉNYI QUANTUM INFORMATION MEASURES

A. Rényi conditional quantum mutual information

The CQMI quantifies how much correlation exists in a tripartite state between two parties from the perspective of the third. A compelling operational interpretation of the quantity has been given in terms of the quantum state redistribution protocol [52, 53]: given a four-party pure state ψ_{ADBC}, with a sender possessing systems D and B and a receiver possessing system C, the optimal rate of quantum communication necessary to transfer the system B to the receiver is given by $\frac{1}{2}I(A;B|C)_\psi$. As mentioned before, due to strong subadditivity [44], the CQMI is non-negative. Additionally, also due to strong

subadditivity, the CQMI can never increase under local CPTP maps performed on the systems A or B [54], so that $I(A;B|C)_\rho$ is a sensible measure of the correlations present between systems A and B, from the perspective of C. That is, the following inequality holds

$$I(A;B|C)_\rho \geq I(A';B'|C)_\omega, \quad (22)$$

where $\omega_{A'B'C} \equiv (\mathcal{N}_{A \to A'} \otimes \mathcal{M}_{B \to B'})(\rho_{ABC})$ with $\mathcal{N}_{A \to A'}$ and $\mathcal{M}_{B \to B'}$ arbitrary local CPTP maps performed on the systems A and B, respectively. One other property of the CQMI is that it obeys a duality relation [52, 53]. That is, for a four-party pure state ψ_{ABCD}, the following equality holds

$$I(A;B|C)_\psi = I(A;B|D)_\psi. \quad (23)$$

The CQMI finds numerous applications. In quantum many body physics, the quantum Markov states (tripartite states which have zero CQMI) can be readily used to study gapped systems in two spatial dimensions [42]. The CQMI can also be used to study the mediation of quantum correlations in condensed matter systems [55]. In entanglement theory, the CQMI underlies the squashed entanglement [54, 56–58], which is the only measure of entanglement known to satisfy all axioms for an entanglement measure. The squashed entanglement of a state is an upper bound on the amount of entanglement that can be extracted from the state by an entanglement distillation protocol [54]. Furthermore, the squashed entanglement of a channel is known to be an upper bound on the quantum communication capacity of any channel assisted by unlimited forward and backward classical communication [59]. The CQMI also underlies quantum discord [60]. Quantum discord is a measure of quantum correlations different from entanglement [61] and has been studied quite extensively [62].

Using the prescribed formula of (17), a Rényi CQMI can be defined as

$$I_\alpha(A;B|C)_\rho \equiv \min_{\sigma_{BC}} D_\alpha(\rho_{ABC}\|Y(\alpha))$$

$$= \min_{\sigma_{BC}} \frac{1}{\alpha - 1}\log \operatorname{Tr}\left\{\rho_{ABC}^\alpha Y(\alpha)^{1-\alpha}\right\}, \quad (24)$$

where D_α is the Rényi relative entropy of order α given in (9), and

$$Y(\alpha) \equiv \left[\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\sigma_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right]^{1/(1-\alpha)}. \quad (25)$$

The choice of minimization over the BC subsystem enables the duality property of (23) to hold for the resulting quantity [38, Theorem 32]. Using a Sibson-like identity [63], the unique optimum state σ_{BC} can be determined explicitly, and the above Rényi CQMI can be rewritten as

$$I_\alpha(A;B|C)_\rho = \frac{\alpha}{\alpha - 1}\log \operatorname{Tr}\{Z(\alpha)\}, \quad (26)$$

Formula	CQMI in (4)	Rényi CQMI in (7)	Rényi CQMI in (26)	Rényi CQMI in (28)
Non-negative	✓	✗	✓	✓
Monotone under local op.'s on A	✓	✗	?	?
Monotone under local op.'s on B	✓	✗	✓	✓
Duality	✓	✓	✓	✓
Converges to (4) as $\alpha \to 1$	N/A	✓	✓	?
Monotone in α	N/A	✗	?	?

TABLE I. Rényi generalizations of the conditional quantum mutual information (CQMI). The Rényi generalizations prescribed in this work are applicable to the CQMI. The leftmost column of the table lists some desired properties of a Rényi CQMI. These properties are satisfied by the original von Neumann CQMI $I(A;B|C)_\rho$ in (4) as shown in Column 2. The Rényi CQMI in (7) obtained by simply replacing the linear sum of von Neumann entropies with the corresponding linear sum of Rényi entropies, in Column 3, is compared with the Rényi generalizations obtained through the formula prescribed in this work, in Columns 4 and 5. The question marks indicate open questions, with numerical evidence supporting a positive answer. The quantity in Column 3 does not retain many of the desired properties. On the contrary, the quantities in Columns 4 and 5 retain some of these desired properties. The table suggests that the latter are more useful Rényi generalizations of the CQMI.

where $Z(\alpha)$ is

$$\left(\rho_C^{(\alpha-1)/2} \operatorname{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_{ABC}^{\alpha} \rho_{AC}^{(1-\alpha)/2} \right\} \rho_C^{(\alpha-1)/2} \right)^{1/\alpha} \tag{27}$$

[38, Proposition 8]. The above quantity converges to the von Neumann CQMI in the limit as $\alpha \to 1$ for any tripartite state ρ_{ABC} [38, Theorem 11]. It is non-negative and non-increasing under the action of local CPTP maps on the B system in the range of $\alpha \in [0,1) \cup (1,2]$ [38, Corollaries 16 and 15]; it is an open question to determine whether *the quantity is also non-increasing under local CPTP maps on the A system* (numerical work has supported a positive answer). We can also define a sandwiched Rényi conditional mutual information $\widetilde{I}_\alpha(A;B|C)_\rho$ using the sandwiched Rényi relative entropy $\widetilde{D}_\alpha$. A particular sandwiched Rényi CQMI that satisfies the desired properties of non-negativity, monotonicity under local CPTP maps on the B system (in the range of $\alpha \in [1/2,1) \cup (1,\infty)$), and duality [38, Section 6] is given by

$$\tilde{I}_\alpha(A;B|C)_\rho \equiv \inf_{\sigma_{BC}} \sup_{\omega_C} \tilde{D}_\alpha \left(\rho_{ABC} \| Y'(\alpha) \right), \tag{28}$$

where $\tilde{D}_\alpha$ is the sandwiched Rényi relative entropy of order α given in (13), and

$$Y'(\alpha) \equiv \left[\rho_{AC}^{(1-\alpha)/(2\alpha)} \omega_C^{(\alpha-1)/(2\alpha)} \sigma_{BC}^{(1-\alpha)/\alpha} \right. \\ \left. \times \omega_C^{(\alpha-1)/(2\alpha)} \rho_{AC}^{(1-\alpha)/(2\alpha)} \right]^{\alpha/(1-\alpha)}. \tag{29}$$

Table I summarizes the various properties of the above-described Rényi and sandwiched Rényi CQMIs in comparison with the Rényi CQMI given in (7). This includes the property of monotonicity of these quantities in the Rényi parameter, which we discuss later in Section IV A 3. The comparison elucidates the effectiveness of the prescribed formula. The Rényi CQMIs also converge to the quantity in (4) in the limit as the Rényi parameter tends to one.

1. Rényi squashed entanglement

The squashed entanglement of a bipartite state ρ_{AB} is defined as [54]

$$E_{\text{sq}}(A;B)_\rho \\ \equiv \frac{1}{2} \inf_{\rho_{ABE}} \left\{ I(A;B|E)_\rho : \rho_{AB} = \operatorname{Tr}_E \{\rho_{ABE}\} \right\}. \tag{30}$$

Thus, a straightforward Rényi generalization is as follows:

$$E_{\text{sq}}^\alpha(A;B)_\rho \\ \equiv \frac{1}{2} \inf_{\rho_{ABE}} \left\{ I_\alpha(A;B|E)_\rho : \rho_{AB} = \operatorname{Tr}_E \{\rho_{ABE}\} \right\}. \tag{31}$$

A Rényi squashed entanglement could potentially be used to strengthen the results on distillable entanglement of a state and the quantum communication capacity of a channel assisted by unlimited two-way classical communication by establishing the squashed entanglement as a strong converse rate for these respective tasks. It remains a topic of future research to investigate these applications of the Rényi squashed entanglement in full detail. Also, it is important to establish that the Rényi conditional mutual informations satisfy monotonicity under local quantum operations on both A and B in order for this quantity to be a sensible correlation measure. Assuming the truth of the above statement (with the support of numerical evidence), in a followup work, we have explored the various properties of the Rényi squashed entanglement that would qualify it as an entanglement monotone [64, Section 4].

2. Rényi quantum discord

We obtain a Rényi generalization of the quantum discord [61], since we can write the discord of a bipartite

state ρ_{AB} as [60]

$$I\left(A;B\right) - \sup_{\{\Lambda\}} I\left(X;B\right)$$

$$= \inf_{\{\Lambda\}} I\left(A;B\right) - I\left(X;B\right) \tag{32}$$

$$= \inf_{\{\Lambda\}} I\left(EX;B\right) - I\left(X;B\right) \tag{33}$$

$$= \inf_{\{\Lambda\}} I\left(E;B|X\right), \tag{34}$$

where the optimization is over all POVMs acting on the system A, with classical output X. We can then find an isometric extension of any such measurement, where we label the environment system as E. So we just define a Rényi quantum discord in the following way:

$$\inf_{\{\Lambda\}} I_\alpha\left(E;B|X\right), \tag{35}$$

where I_α could be any Rényi generalization of the conditional mutual information. Such a Rényi discord might potentially bear new insights over the von Neumann discord [64, Section 5].

3. Potential operational characterization of states with small CQMI

The Rényi CQMIs could also be useful in connection with some important open problems in quantum information theory and condensed matter physics. We conjecture that *our Rényi generalizations of the CQMI are monotonically increasing functions of the Rényi parameter.* That is, for $0 \leq \alpha \leq \beta$, we conjecture that

$$I_\alpha(A;B|C)_\rho \leq I_\beta(A;B|C)_\rho, \tag{36}$$

as well as the analogous statement for the sandwiched Rényi CQMI. These conjectures are true in a number of special cases. For example, we it can be shown that these conjectures hold when the Rényi parameter α is in a neighborhood of one, and that (36) is true in the case when $\alpha + \beta = 2$ (or when $1/\alpha + 1/\beta = 2$ for the sandwiched Rényi CQMI) [38, Section 8]. If proven to be correct generally, the above conjecture would establish the truth of an open conjecture from [41] (up to a constant):

$$I(A;B|C)_\rho \geq \widetilde{I}_{1/2}(A;B|C)_\rho \tag{37}$$

$$= -\log F(\rho_{ABC}, \mathcal{R}(\rho_{BC})), \tag{38}$$

where $\mathcal{R}(\cdot) \equiv \rho_{AC}^{1/2} \rho_C^{-1/2} (\cdot) \rho_C^{-1/2} \rho_{AC}^{1/2}$ denotes Petz's recovery map for the partial trace over A [65] and $F(P,Q) \equiv \|\sqrt{P}\sqrt{Q}\|_1^2$ is the quantum fidelity. This would give an operational characterization of quantum states with small CQMI (i.e., states that fulfill strong subadditivity with near equality). This characterization could be useful for understanding topological order in condensed matter physics [41, 42], for solving some open

questions related to squashed entanglement [39], as well as for deriving quantum communication complexity lower bounds [66], as discussed in [40]. After the completion of the present paper, recent progress on establishing (38) has appeared in [67].

B. Rényi multipartite information measures

The conditional multipartite information of a state $\rho_{A_1 \cdots A_l C}$ is defined as

$$I\left(A_1; A_2; \cdots; A_l|C\right)_\rho$$

$$\equiv \left[\sum_{i=1}^l H\left(A_i|C\right)_\rho\right] - H\left(A_1 A_2 \cdots A_l|C\right)_\rho \tag{39}$$

$$= \left(\sum_{i=1}^l \left[H\left(A_i C\right)_\rho - H\left(C\right)_\rho\right]\right)$$

$$- H\left(A_1 A_2 \cdots A_l C\right)_\rho + H\left(C\right)_\rho \tag{40}$$

and has appeared in various contexts (see [68] and references therein). This quantity can be written as a relative entropy as follows:

$$I\left(A_1; A_2; \cdots; A_l|C\right)_\rho$$

$$= D\left(\rho_{A_1 \cdots A_l C} \middle\| \exp\left\{\sum_{i=1}^l \log \rho_{A_i C} - \sum_{i=1}^{l-1} \log \rho_C\right\}\right). \tag{41}$$

Thus, we obtain two natural Rényi generalizations of this measure, defined as

$$I_\alpha\left(A_1; A_2; \cdots; A_l|C\right)_{\rho|\rho}$$

$$\equiv \frac{1}{\alpha - 1} \log \operatorname{Tr}\left\{\rho_{A_1 \cdots A_l C}^\alpha \sigma_{A_1 \cdots A_l C}\left(\alpha\right)\right\}, \tag{42}$$

$$\widetilde{I}_\alpha\left(A_1; A_2; \cdots; A_l|C\right)_{\rho|\rho}$$

$$\equiv \frac{1}{\alpha - 1} \log \operatorname{Tr}\left\{\left(\rho_{A_1 \cdots A_l C}^{1/2} \omega_{A_1 \cdots A_l C}\left(\alpha\right) \rho_{A_1 \cdots A_l C}^{1/2}\right)^\alpha\right\}, \tag{43}$$

where

$$\sigma_{A_1 \cdots A_l C}\left(\alpha\right) \equiv \rho_{A_l C}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \cdots \rho_C^{\frac{\alpha-1}{2}} \rho_{A_2 C}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{A_1 C}^{1-\alpha}$$
$$\times \rho_C^{\frac{\alpha-1}{2}} \rho_{A_2 C}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \cdots \rho_C^{\frac{\alpha-1}{2}} \rho_{A_l C}^{\frac{1-\alpha}{2}}, \tag{44}$$

$$\omega_{A_1 \cdots A_l C}\left(\alpha\right) \equiv \rho_{A_l C}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \cdots \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{A_2 C}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{A_1 C}^{\frac{1-\alpha}{\alpha}}$$
$$\times \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{A_2 C}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \cdots \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{A_l C}^{\frac{1-\alpha}{2\alpha}}. \tag{45}$$

These quantities satisfy monotonicity under local quantum operations on the system A_1 for $\alpha \in [0,1) \cup (1,2]$ and $\alpha \in [1/2,1) \cup (1,\infty)$, respectively, and they reduce to the von Neumann entropy in the limit as $\alpha \to 1$. The proofs for these facts follow along similar lines given in previous sections of this paper. It is an open question to

determine if the monotonicity holds with respect to local operations on the individual systems $A_1, \ldots, A_l$.

For a state $\rho_{A_1 A_1' \cdots A_l A_l'}$, another multipartite information-based quantity that has appeared in various contexts (see [69–71] and references therein) is as follows:

$$I\left(A_1 A_1' : \cdots : A_l A_l'\right) - I\left(A_1' : \cdots : A_l'\right). \quad (46)$$

The measure gives an alternate approach for "quantum conditioning," as discussed in [69]. It can be written as the following relative entropy:

$$D\Bigg(\rho_{A_1 A_1' \cdots A_l A_l'} \Bigg\| \exp\bigg\{ \log \rho_{A_1' \cdots A_l'} + \sum_{i=1}^l \log \rho_{A_i A_i'} - \log \rho_{A_i'} \bigg\}\Bigg). \quad (47)$$

Thus, we obtain two natural Rényi generalizations of this quantity, defined as

$$\frac{1}{\alpha - 1} \log \mathrm{Tr}\left\{\rho_{A_1 \cdots A_l}^\alpha \sigma_{A_1 \cdots A_l}(\alpha)\right\}, \quad (48)$$

$$\frac{1}{\alpha - 1} \log \mathrm{Tr}\left\{\left(\rho_{A_1 \cdots A_l}^{1/2} \omega_{A_1 \cdots A_l}(\alpha) \rho_{A_1 \cdots A_l}^{1/2}\right)^\alpha\right\}, \quad (49)$$

where

$$\sigma_{A_1 \cdots A_l C}(\alpha) \equiv \rho_{A_1' \cdots A_l'}^{\frac{1-\alpha}{2}} \left(\rho_{A_1'}^{\frac{\alpha-1}{2}} \otimes \cdots \otimes \rho_{A_l'}^{\frac{\alpha-1}{2}}\right) \times$$
$$\left(\rho_{A_1 A_1'}^{1-\alpha} \otimes \cdots \otimes \rho_{A_l A_l'}^{1-\alpha}\right) \left(\rho_{A_1'}^{\frac{\alpha-1}{2}} \otimes \cdots \otimes \rho_{A_l'}^{\frac{\alpha-1}{2}}\right) \rho_{A_1' \cdots A_l'}^{\frac{1-\alpha}{2}}, \quad (50)$$

$$\omega_{A_1 \cdots A_l C}(\alpha) \equiv \rho_{A_1' \cdots A_l'}^{\frac{1-\alpha}{2\alpha}} \left(\rho_{A_1'}^{\frac{\alpha-1}{2\alpha}} \otimes \cdots \otimes \rho_{A_l'}^{\frac{\alpha-1}{2\alpha}}\right) \times$$
$$\left(\rho_{A_1 A_1'}^{\frac{1-\alpha}{\alpha}} \otimes \cdots \otimes \rho_{A_l A_l'}^{\frac{1-\alpha}{\alpha}}\right) \left(\rho_{A_1'}^{\frac{\alpha-1}{2\alpha}} \otimes \cdots \otimes \rho_{A_l'}^{\frac{\alpha-1}{2\alpha}}\right) \rho_{A_1' \cdots A_l'}^{\frac{1-\alpha}{2\alpha}}, \quad (51)$$

These quantities satisfy monotonicity under local quantum operations on the individual systems $A_1, \ldots, A_l$ for $\alpha \in [0, 1) \cup (1, 2]$ and $\alpha \in [1/2, 1) \cup (1, \infty)$, respectively, and they reduce to the von Neumann entropy in the limit as $\alpha \to 1$. Proofs for these facts follow along similar lines as the proofs in [38, Sections 5 and 6].

C. Rényi topological entanglement entropy

The topological entanglement entropy of a tripartite quantum state ρ_{ABC} is defined as the following linear combination of marginal entropies [72, 73]:

$$H_{\mathrm{topo}}(\rho_{ABC}) \equiv H(A)_\rho + H(B)_\rho + H(C)_\rho - H(AB)_\rho$$
$$- H(BC)_\rho - H(AC)_\rho + H(ABC)_\rho. \quad (52)$$

This quantity is also known in the classical information theory literature as the interaction information [74]. In condensed matter systems, the topological entanglement entropy measures the long range quantum correlations present in a many-body quantum state—it is helpful in classifying if and the degree to which a system is topologically ordered. We can write $H_{\mathrm{topo}}(\rho_{ABC})$ in terms of the relative entropy as

$$-D\Bigg(\rho_{ABC} \| \exp\big\{ \log \rho_{AB} + \log \rho_{BC} + \log \rho_{AC}$$
$$- \log \rho_A - \log \rho_B - \log \rho_C \big\}\Bigg). \quad (53)$$

So this suggests one Rényi generalization $H_{\mathrm{topo}}^\alpha(\rho_{ABC})$ of the topological entanglement entropy to be

$$-\frac{1}{\alpha - 1} \log \mathrm{Tr}\bigg\{ \rho_{ABC}^\alpha \rho_{AC}^{\frac{1-\alpha}{2}} \rho_{BC}^{\frac{1-\alpha}{2}} \rho_{AB}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_B^{\frac{\alpha-1}{2}} \rho_A^{\alpha-1}$$
$$\times \rho_B^{\frac{\alpha-1}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{AB}^{\frac{1-\alpha}{2}} \rho_{BC}^{\frac{1-\alpha}{2}} \rho_{AC}^{\frac{1-\alpha}{2}} \bigg\}, \quad (54)$$

and a sandwiched Rényi generalization $\widetilde{H}_{\mathrm{topo}}^\alpha(\rho_{ABC})$ is as follows:

$$-\frac{1}{\alpha - 1} \log \mathrm{Tr}\bigg\{ \big(\rho_{ABC}^{1/2} \rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{2\alpha}} \rho_{AB}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_B^{\frac{\alpha-1}{2\alpha}} \rho_A^{\frac{\alpha-1}{\alpha}}$$
$$\times \rho_B^{\frac{\alpha-1}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{AB}^{\frac{1-\alpha}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{2\alpha}} \rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_{ABC}^{1/2}\big)^\alpha \bigg\}. \quad (55)$$

Of course, there are many generalizations depending on the ordering of the operators and whether we use optimizations as discussed before.

In prior work, researchers had suggested that the entire entanglement spectrum of a many-body state could lead to a finer classification of topological order [17]. Following this proposal, [18] considered a Rényi generalization of the topological entanglement entropy along the lines in (6). However, their conclusion was that this particular Rényi generalization does not lead to any further universal information about topological order than that already provided by the von Neumann topological entanglement entropy. A natural question to consider going forward from here is whether the Rényi topological entanglement entropies as defined in (54) and (55) could lead to extra desired universal information about topological order in any setting (that is, whether it would depend on the Rényi parameter α in addition to the "quantum dimension").

V. CONCLUSION

We prescribed Rényi generalizations for quantum information measures which consist of a linear combination of von Neumann entropies with coefficients chosen from the set $\{-1, 0, 1\}$. The said criterion is met by many

useful quantum information measures such as the conditional entropy, the mutual information, the conditional mutual information, the conditional multipartite information, and the topological entanglement entropy. While Rényi generalizations of the conditional entropy and the mutual information were already known, it had been an open problem before this work to obtain valid Rényi generalizations of the remaining quantities listed above, in particular, of the conditional mutual information.

The Rényi generalizations of the prescribed type for the conditional mutual information, $I_\alpha(A; B|C)$ satisfy many of the desired properties of the original von Neumann quantity. They are non-negative, monotone under local operations on either system A or system B (it is left as an open question to prove the property for local operations on the other system, but we have numerical evidence that supports a positive answer), and obey a duality relation. Based on the Rényi conditional mutual informations, we defined Rényi squashed entanglement and Rényi discord and discussed some potential applications of these quantities. We have explored these quantities in much more detail in our followup work [64]. We also posed another open question about the proposed Rényi conditional mutual informations being monotone increasing in the Rényi parameter (we have proofs for this in some special cases and numerical evidence supporting a positive answer in general), and indicated an important implication of the conjecture in characterizing quantum states with small conditional mutual information. Further, we defined a Rényi conditional multipartite information and a Rényi topological entanglement entropy. The Rényi topological entanglement entropies defined in (54) and (55) present potentially interesting, new possibilities in the study of topologically-ordered systems.

In view of the numerous potential uses of Rényi generalizations of quantum information measures, our prescription for Rényi generalization promises new avenues of research in both quantum information theory and other areas of physics.

Note: After the completion of this work, we learned of the recent breakthrough result of [67], in which a variant of the inequality in (38) has been proven. This result has been further strengthened in [75, 76], and its implications to robustness of quantum Markov chains and to other contexts have been further elucidated in [71, 77].

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