Medium effects and the shear viscosity of the dilute Fermi gas away from the conformal limit

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M. Bluhm and T. Schäfer[1]

[1]*Department of Physics, North Carolina State University,*

Raleigh, North Carolina 27695, USA

Abstract

We study the shear viscosity of a dilute Fermi gas as a function of the scattering length in the vicinity of the unitarity limit. The calculation is based on kinetic theory, which provides a systematic approach to transport properties in the limit in which the fugacity $z = n\lambda^3/2$ is small. Here, n is the density of the gas and λ is the thermal wave length of the fermions. At leading order in the fugacity expansion the shear viscosity is independent of density, and the minimum shear viscosity is achieved at unitarity. At the next order medium effects modify the scattering amplitude as well as the quasi-particle energy and velocity. We show that these effects shift the minimum of the shear viscosity to the Bose-Einstein condensation (BEC) side of the resonance, in agreement with the result of recent experiments.

I. INTRODUCTION

Cold fermionic gases provide a unique arena for the study of strongly correlated matter. In these systems the s-wave interaction between atoms can be controlled by altering an external magnetic field. The dimensionless parameter that governs the interaction is $k_F a$, where a is the s-wave scattering length and k_F is the magnitude of the Fermi momentum. In a homogeneous Fermi gas the latter is related to the particle density by $n = k_F^3/(3\pi^2)$. The Fermi momentum also defines a temperature scale, the Fermi temperature by $T_F = k_F^2/(2m)$. Tuning the system into a Feshbach resonant state corresponds to the limit $k_F a \to \infty$. In this regime the interaction cross section is limited only by unitarity. Equilibrium and non-equilibrium properties of the dilute Fermi gas at unitarity have been investigated in a number of experiments, for example [1–4].

In the unitarity limit the dilute Fermi gas is a scale and conformally invariant non-relativistic many-body system [5]. At high temperature T it behaves like a weakly interacting gas [6], but in the low T limit it is a strongly correlated quantum fluid, which shares many interesting properties with other strongly interacting systems. An important example is nearly perfect fluidity, which has also been observed in the relativistic quark-gluon plasma [7, 8].

Detuning the system away from unitarity, scale invariance is lost. At low T and positive a a Bose-Einstein condensate (BEC) of strongly bound diatomic molecules is formed [9–11]. On the atomic side of the resonance, i.e. for negative a, a Bardeen-Cooper-Schrieffer (BCS) superfluid state is realized at low T. The crossover between the BEC and BCS regimes is known to be smooth [12, 13]. As the temperature is increased the superfluid Fermi gas undergoes a phase transition to a normal fluid. On the BEC side this is the Einstein transition, which occurs at a critical temperature $T_c \sim T_F$. In the BCS regime the critical temperature is exponentially small compared to T_F. The maximum transition temperature T_c/T_F is achieved slightly on the BEC side of the BEC-BCS crossover.

It is natural to ask how transport properties of the fluid change along the BEC-BCS crossover. The bulk viscosity, for example, vanishes at unitarity but is expected to be non-zero on either side of the Feshbach resonance. In kinetic theory bulk viscosity is thought to arise from scale invariance breaking encoded in the density dependence of the effective fermion mass [14]. The shear viscosity, on the other hand, is expected to be large in the

BCS and BEC limits and become minimal close to unitarity. Indeed, kinetic theory in the high temperature limit predicts that the shear viscosity has a minimum exactly at unitarity [15]. Experimental measurements of the shear viscosity at unitarity and at temperatures close to the phase transition have been reported in [16–18]. These experiments obtain values for η/s, the ratio of shear viscosity to entropy density, that are only a few times larger than the conjectured universal bound $\eta \geq \hbar s/(4\pi k_B)$ [19]. Recent measurements [20] indicate, however, that the kinematic viscosity η/n is minimized on the molecular side of the BEC-BCS crossover.

In this work we study the dependence of the shear viscosity on $k_F a$ in kinetic theory. Kinetic theory can be viewed as a systematic expansion in the fugacity $z = n\lambda^3/2$ of the gas. Here, $n/2$ is the density per spin state and $\lambda = [(2\pi\hbar)/(mT)]^{1/2}$ is the thermal de Broglie wave length. At leading order in the fugacity the shear viscosity is independent of z and has a minimum at $k_F a \to \infty$ [15]. This minimum at unitarity is a simple consequence of the maximum in the vacuum cross section at resonance. We will show that including medium effects in the scattering amplitude shifts the minimum away from unitarity. The physical reason for this behavior is related to Pauli blocking in the in-medium scattering amplitude, which is more efficient on the BCS side. Formally, the minimum in the shear viscosity arises from the competition between $(\lambda/a)^2$ and $z(\lambda/a)$ corrections to η. We will show that it is possible to compute all one- and two-body effects of order $\mathcal{O}(z(\lambda/a))$. In addition to in-medium corrections to the scattering amplitude, these terms arise from medium corrections to the quasi-particle energy and velocity.

Kinetic theory based on atomic degrees of freedom is reliable in the limits of high temperature, $T \gg T_F$, or weak coupling, $k_F|a| \ll 1$. Previous investigations have indicated that at unitarity kinetic theory is applicable at temperatures as low as $T/T_F \simeq 0.4$ [15, 21, 22]. This condition is satisfied for part of the data reported in [20]. Early studies of medium effects were reported in [23, 24]. Medium effects are also included in the T-matrix approaches of Enss et al. [25] and Levin et al. [26, 27].

This paper is structured as follows: in Sect. II we introduce a quasi-particle description for the dilute Fermi gas near unitarity. In Sect. III A we discuss the kinetic theory calculation of the shear viscosity. A simple model based on medium-corrections to the cross section is described in Sect. III B, and a systematic expansion in powers of z and (λ/a) is given in Sect. III C. We conclude in Sect. IV, and relegate details of the expansion to two appendices.

II. QUASI-PARTICLE DESCRIPTION

In this section we introduce a quasi-particle model for the dilute Fermi gas near unitarity. The effective Lagrangian for non-relativistic spin $1/2$ fermions interacting via a short range potential is

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi - \frac{C_0}{2} \left(\psi^\dagger \psi \right)^2 , \tag{1}$$

where the coupling C_0 is determined by the s-wave scattering length a. In the weak coupling limit we find $C_0 = 4\pi a/m$. In the high temperature limit thermodynamic properties of the gas can be computed as a systematic expansion in the fugacity z, cf. [6]. This is the well known virial expansion. The pressure is given by

$$P = \frac{\nu T}{\lambda^3} \left(z + b_2 z^2 + \ldots \right) , \tag{2}$$

with $\nu = 2$ for two spin degrees of freedom. The second virial coefficient is obtained by summing the two-particle interaction to all orders. Near unitarity we get

$$b_2 = -\frac{1}{4\sqrt{2}} + \frac{1}{\sqrt{2}} \left(1 + \frac{\sqrt{2}\,\lambda}{\pi\,a} + \ldots \right) , \tag{3}$$

which is valid on either side of the resonance. The temperature dependence of $b_2(T)$ is a measure for the scale invariance breaking. Given $P(\mu, T)$ we can compute other thermodynamic properties. The particle density $n = (\partial P)/(\partial \mu)_T$ is given by

$$n = \frac{\nu}{\lambda^3} \left(z + 2b_2 z^2 + \ldots \right) , \tag{4}$$

and the entropy density $s = (\partial P)/(\partial T)_\mu$ is

$$s = \frac{5}{2} \frac{\nu}{\lambda^3} \left(z \left[1 - \frac{2}{5} \frac{\mu}{T} \right] + b_2 z^2 \left[1 - \frac{4}{5} \frac{\mu}{T} \right] + \frac{2}{5} T b_2' z^2 + \ldots \right) . \tag{5}$$

We can construct a quasi-particle model consistent with these results by computing the fermion self-energy at order z. This corresponds to summing the two-body interaction with a fermion in the heat bath to all orders. The fermion dispersion relation is given by $E_p = E_p^0 + \Delta E_p$ with $E_p^0 = p^2/(2m)$ and $\Delta E_p = \mathrm{Re}\,\Sigma(p)$, where m is the mass parameter and p is the magnitude of the momentum. The real part of the on-shell fermion self-energy near unitarity reads

$$\mathrm{Re}\,\Sigma(p) = -\frac{8T}{\sqrt{\pi}} \frac{1}{p} F_D \left(\frac{p}{\sqrt{2mT}} \right) \frac{z}{a} , \tag{6}$$

4

where $F_D(\tilde{p})$ is Dawson's integral and $\tilde{p} = p/\sqrt{2mT}$. The momentum-dependence in ΔE_p modifies the quasi-particle velocity as $\vec{v}_p = \vec{\nabla}_p E_p = \vec{v}_p^0 + \Delta\vec{v}_p$, where $\vec{v}_p^0 = \vec{p}/m$ is the velocity of a free particle and

$$\Delta\vec{v}_p = \frac{\vec{p}}{m}\mathcal{G}(\tilde{p})\frac{z}{a}, \qquad \mathcal{G}(\tilde{p}) = \frac{2}{\pi}\frac{\lambda}{\tilde{p}^3}\left(F_D(\tilde{p})[1 + 2\tilde{p}^2] - \tilde{p}\right). \tag{7}$$

As a consistency check we can verify that the (z/a)-dependence of the quasi-particle properties is compatible with the equation of state controlled by the second virial coefficient. In kinetic theory the enthalpy $\mathcal{E} + P$ can be written as [14]

$$\mathcal{E} + P = \nu\int d\Gamma_p\left(\frac{1}{3}\vec{p}\cdot\vec{v}_p + E_p\right)f_p, \tag{8}$$

where $d\Gamma_p = d^3p/(2\pi)^3$ and $f_p(\vec{x}, t)$ is the quasi-particle distribution function. From Eq. (8), we can compute in equilibrium the $\mathcal{O}(z(\lambda/a))$-shift in the enthalpy due to the change in the quasi-particle energy and velocity discussed above. We get

$$\Delta(\mathcal{E} + P) = \frac{2\nu}{3}\int d\Gamma_p\frac{p^2}{2m}\left(\frac{\Delta v_p}{v_p^0}\right)\hat{f}_p^0 + \nu\int d\Gamma_p\,\Delta E_p\left(1 - \frac{5}{3}\frac{p^2}{2mT}\right)\hat{f}_p^0, \tag{9}$$

where $\hat{f}_p^0 = z\,e^{-E_p^0/T}$ is the equilibrium distribution function for the non-interacting system. Using Eqs. (6) and (7) we find

$$\Delta(\mathcal{E} + P) = \frac{2}{\pi}\frac{\lambda}{a}z^2\frac{\nu T}{\lambda^3}. \tag{10}$$

This result can be compared to the virial expansion. We use the thermodynamic identity $\mathcal{E} + P = \mu n + sT$ with n and s given above and determine $\mathcal{O}(\lambda/a)$ corrections to the enthalpy from the second virial coefficient given in Eq. (3). The result agrees with Eq. (10).

III. SHEAR VISCOSITY FROM KINETIC THEORY

A. Chapman Enskog expansion

We compute the shear viscosity by matching the expression for the dissipative contribution to the stress tensor in fluid dynamics to the result in kinetic theory. In fluid dynamics we write

$$\delta\Pi^{ij} = -\eta\sigma^{ij} - \zeta\delta^{ij}\nabla_k u_k, \qquad \sigma_{ij} = \nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\nabla_k u_k, \tag{11}$$

where $\vec{u}$ is the fluid velocity, η is the shear viscosity, and ζ is the bulk viscosity. In kinetic theory $\delta\Pi_{ij}$ is expressed in terms of the non-equilibrium part $\delta f_p = f_p - f_p^0$ of the distribution function. We have

$$\delta\Pi^{ij} = \nu \int d\Gamma_p \, p^i v_p^j \, \delta f_p \,. \tag{12}$$

In the classical limit it is convenient to define an off-equilibrium function ψ_p by $f_p = f_p^0(1 - \psi_p/T)$ where $f_p^0 = z \, e^{-(E_p - \vec{p}\cdot\vec{u})/T}$. The function ψ_p is determined by the Boltzmann equation

$$\mathcal{D} f_p \equiv \left(\frac{\partial}{\partial t} + \vec{v}_p \cdot \vec{\nabla}_x + \vec{F} \cdot \vec{\nabla}_p \right) f_p = \mathcal{C} \,. \tag{13}$$

In Eq. (13), $\mathcal{D} f_p$ denotes the streaming term in which $\vec{F} = -\vec{\nabla}_x E_p$ is the force acting on the quasi-particles between collisions, and $\mathcal{C}$ is the collision operator term. In the high temperature limit the collision term is dominated by two-body scatterings. Linearizing in the off-equilibrium function ψ_p we get [28]

$$\mathcal{C} = \frac{f_{p_1}^0}{T} \int \prod_{i=2}^{4} d\Gamma_{p_i} \, f_{p_2}^0 w(1,2;3,4) \left(\psi_{p_1} + \psi_{p_2} - \psi_{p_3} - \psi_{p_4} \right) , \tag{14}$$

where $w(1,2;3,4) = (2\pi)^4 \, \delta^{(3)}(\vec{p_1} + \vec{p_2} - \vec{p_3} - \vec{p_4}) \, \delta(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4})|\mathcal{A}|^2$ is the transition rate and $|\mathcal{A}|^2$ is the absolute square of the scattering amplitude.

In order to determine the shear viscosity we expand ψ_p in gradients of the thermodynamic variables. This is known as the Chapman-Enskog expansion. At linear order in gradients of $\vec{u}$ we can write $\psi_p = \chi^{ij}(p) \, \sigma_{ij}$, where we have dropped terms that contribute to the bulk viscosity [14]. In the local rest-frame of the fluid we find

$$\frac{T}{f_{p_1}^0}(\mathcal{D} f_{p_1}^0) = \frac{1}{2} v_{p_1}^i p_1^j \sigma_{ij} = \frac{1}{2m} p_1^i p_1^j \left(1 + \mathcal{G}(\tilde{p}_1)\frac{z}{a} \right) \sigma_{ij} = \mathcal{C}[\chi^{ij}(p_1)] \, \sigma_{ij} \,, \tag{15}$$

where, using Eq. (14),

$$\mathcal{C}[\chi^{ij}(p_1)] \, \sigma_{ij} = \int \prod_{i=2}^{4} d\Gamma_{p_i} f_{p_2}^0 w(1,2;3,4) \left(\chi^{ij}(p_1) + \chi^{ij}(p_2) - \chi^{ij}(p_3) - \chi^{ij}(p_4) \right) \sigma_{ij} \,. \tag{16}$$

The linearized Boltzmann equation in the shear channel can be written as

$$\frac{1}{2m} \left\langle \chi^{ij}(p_1) \middle| (p_1)_{ij} \left(1 + \mathcal{G}(\tilde{p}_1)\frac{z}{a} \right) \right\rangle = \left\langle \chi^{ij}(p_1) \middle| \mathcal{C}\left[\chi_{ij}(p_1) \right] \right\rangle , \tag{17}$$

where $p_{ij} = p_i p_j - \frac{1}{3}\delta_{ij} \, p^2$ with $p^{ij}\sigma_{ij} = p^i p^j \sigma_{ij}$. We have also defined an inner product on the space of linearized distribution functions

$$\langle a(p)|b(p) \rangle = \int d\Gamma_p \, f_p^0 \, a(p)b(p) \,. \tag{18}$$

The shear part of the stress tensor can be written as

$$\delta\Pi^{ij} = -\frac{\nu}{mT} \int d\Gamma_{p_1} f^0_{p_1} \chi(p_1) \left(1 + \mathcal{G}(\tilde{p}_1)\frac{z}{a}\right) p^i_1 p^j_1 p^{kl}_1 \sigma_{kl} \,, \tag{19}$$

where we have defined $\chi^{ij}(p) = p^{ij}\chi(p)$. Using $p^{ij}p_{ij} = p^i p^j p_{ij} = \frac{2}{3}p^4$ and Eq. (17) we finally obtain

$$\eta = \frac{\nu}{10\, m^2 T} \frac{\left\langle \chi^{ij}(p_1) \middle| (p_1)_{ij} \left(1 + \mathcal{G}(\tilde{p}_1)\frac{z}{a}\right) \right\rangle^2}{\left\langle \chi^{ij}(p_1) \middle| \mathcal{C}\left[\chi_{ij}(p_1)\right] \right\rangle} \,. \tag{20}$$

This expression determines η for the off-equilibrium function $\chi(p)$ that solves the linearized Boltzmann equation.

By writing η in the specific form given in Eq. (20) one obtains a lower bound on the shear viscosity for a given trial function $\chi(p)$. The actual result for η can then be found by maximizing Eq. (20) over all trial functions. In practice, $\chi(p)$ is expanded in a series of generalized Laguerre polynomials which is known to converge rapidly. Truncating this series at leading order, i.e. using $\chi(p) = 1$ as a trial function, was shown to provide already an excellent approximation, accurate to better than 2%, for the shear viscosity at unitarity [24]. We will therefore use this truncation in the following.

B.   In-medium cross section

Medium effects influence the shear viscosity in a variety of ways. The medium modification of the quasi-particle velocity impacts the streaming term in Eq. (15), and the stress tensor in Eq. (19). Both of these contribute to the numerator in Eq. (20). The matrix element of the collision operator in the denominator is affected by medium modifications of the quasi-particle energy that enters into the distribution functions and the transition rate $w(1, 2; 3, 4)$. Moreover, medium effects modify also the squared scattering amplitude $|\mathcal{A}|^2$.

In order to explore these effects we begin with a simple model calculation in which we take into account medium effects in the scattering amplitude, and thus in the cross section, only. The absolute square of the vacuum scattering amplitude is

$$|\mathcal{A}|^2 = \frac{16\pi^2}{m^2} \frac{a^2}{a^2 q^2 + 1} \,, \tag{21}$$

where $\vec{q} = (\vec{p}_2 - \vec{p}_1)/2$ is the relative momentum between two scattering particles. In terms of the zero range Lagrangian given in Eq. (1) the amplitude arises from the sum of all two-body scattering diagrams. These diagrams form a geometric series and $\mathcal{A} = C_0/(1 - \Pi_0 C_0)$,

where Π_0 is the two-particle polarization function. In dimensional regularization we find $\Pi_0(q) = -imq/(4\pi)$ and $C_0 = 4\pi a/m$. At leading order in the fugacity medium effects arise from Pauli-blocking of the fermion lines in Π_0. We can write $\Pi = \Pi_0 + \delta\Pi$ with [23, 24]

$$\delta\Pi(P, q) = -\int \frac{d^3k}{(2\pi)^3} \frac{\hat{f}^0_{|\vec{P}/2+\vec{k}|} + \hat{f}^0_{|\vec{P}/2-\vec{k}|}}{(q^2 - k^2)/m + i\epsilon}, \tag{22}$$

which depends on both the relative momentum q and the total momentum $\vec{P} = \vec{p_1} + \vec{p_2}$. The real and imaginary parts of the in-medium polarization function are given by

$$\mathrm{Im}\,\delta\Pi = \frac{z}{\pi}m^2T\frac{e^{-P^2/(8mT)}}{P}e^{-q^2/(2mT)}\sinh(Pq/(2mT)), \tag{23}$$

$$\mathrm{Re}\,\delta\Pi = -\frac{2z}{\pi^2}m^2T\frac{e^{-P^2/(8mT)}}{P}\int_0^\infty dx \frac{xe^{-x^2}\sinh(Px/\sqrt{2mT})}{(q^2/(2mT) - x^2)}, \tag{24}$$

where the integral in Eq. (24) is a Cauchy principle value integral. The full in-medium scattering amplitude squared is

$$|\mathcal{A}|^2 = \frac{16\pi^2}{m^2}\frac{1}{\left(q - \frac{4\pi}{m}\mathrm{Im}\,\delta\Pi\right)^2 + \left(\frac{1}{a} - \frac{4\pi}{m}\mathrm{Re}\,\delta\Pi\right)^2}, \tag{25}$$

which agrees with the "broad-resonance" expression discussed in [23]. Phenomenological consequences of this result were also discussed in [23]. The important observation in our context is that, whereas the squared vacuum amplitude is even in a, the in-medium expression has odd corrections of order $\mathcal{O}(z(a/\lambda))$.

The calculation of the shear viscosity based on the in-medium scattering amplitude is now straightforward. We define the total and relative momenta in the initial and final state as $\vec{P} = \vec{p_1} + \vec{p_2}$ and $\vec{P'} = \vec{p_3} + \vec{p_4}$, as well as $\vec{q} = (\vec{p_1} - \vec{p_2})/2$ and $\vec{q'} = (\vec{p_3} - \vec{p_4})/2$. The vector $\vec{P}$ can be aligned along the z-axis, and the integration over d^3P' is performed by using the condition for total momentum conservation. This leaves three angular integrals, $d\cos\theta_q\,d\cos\theta_{q'}\,d\phi$, where

$$\vec{P}\cdot\vec{q} = Pq\cos\theta_q, \quad \vec{P}\cdot\vec{q'} = Pq'\cos\theta_{q'}, \quad \vec{q}\cdot\vec{q'} = qq'\left[\cos\theta_q\cos\theta_{q'} + \sin\theta_q\sin\theta_{q'}\cos\phi\right]. \tag{26}$$

Finally, the integration over dq' can be performed by making use of the condition for energy conservation $q^2/m - q'^2/m = 0$. Inside the integral the off-equilibrium factor $\chi^{ij}(p_1)\left(\chi^{ij}(p_1) + \chi^{ij}(p_2) - \chi^{ij}(p_3) - \chi^{ij}(p_4)\right)$ can be symmetrized in the in- and out-going momenta. We find

$$\frac{1}{4}\left(\chi^{ij}(p_1) + \chi^{ij}(p_2) - \chi^{ij}(p_3) - \chi^{ij}(p_4)\right)^2 = q^4 + q'^4 - \frac{1}{3}(q^2 - q'^2)^2 - 2q^2q'^2\cos^2\Theta, \tag{27}$$
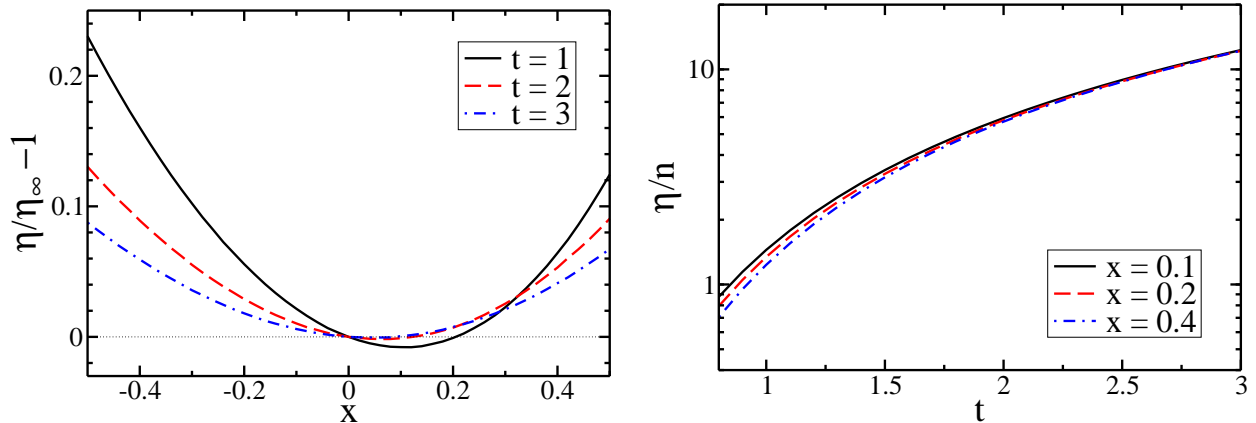
FIG. 1: (Color online) Left panel: Scaled shear viscosity difference from unitarity $(\eta - \eta_\infty)/\eta_\infty$ as a function of $x = 1/(k_F a)$ for different values of $t = T/T_F$. The shear viscosity is computed from the in-medium cross section containing only the influence of $\mathrm{Re}\,\delta\Pi$. Furthermore, medium corrections to the quasi-particle energy and velocity are neglected. Right panel: Shear viscosity η scaled by the particle density n from Eq. (4) as a function of t for different values of x.

where $\vec{q}\cdot\vec{q}' = qq'\cos\Theta$. The matrix element of the linearized collision operator is then given by

$$\langle \chi^{ij}|\mathcal{C}[\chi_{ij}]\rangle = z^2 \frac{m}{6\pi^5} \int_0^\infty dP \int_0^\infty dq\, P^2 q^7 e^{-P^2/(4mT)} e^{-q^2/(mT)} |\mathcal{A}|^2 \,. \qquad (28)$$

The remaining integrations can be performed numerically. The integral in the numerator of Eq. (20) is $\langle \chi^{ij}(p)|p_{ij}\rangle = 5z(mT)^{7/2}/\sqrt{2\pi^3}$.

Numerical results are shown in Fig. 1. In the left panel we focus on the difference $\eta - \eta_\infty$, where η_∞ is the shear viscosity at unitarity. In this difference medium corrections that are independent of a cancel. We plot the dimensionless quantity $\eta/\eta_\infty - 1$ as a function of $1/(k_F a)$ for different values of T/T_F. In the right panel of Fig. 1 we show the behavior of η/n as a function of T/T_F for three different values of $1/(k_F a)$ on the BEC side.

Our results can be compared to recent measurements reported by the North Carolina State University group [20]. Elliott et al. studied dissipative corrections to the expansion of a dilute Fermi gas for several different values of $1/(k_F a)$ near unitarity. The results are reported in terms of a trap averaged kinematic shear viscosity $\langle \eta/n \rangle$. It is difficult to compare these results directly to our calculations for a homogeneous gas because performing the trap average involves a poorly constrained cutoff on the spatial integral over the shear viscosity. However, the results of Elliott et al. are quite remarkable, even on a qualitative

level. They find that $\langle \eta/n \rangle$ has a minimum on the BEC-side of the resonance. As the temperature of the cloud increases, this minimum is shifted toward the unitarity limit. At a given value of $k_F a$, $\langle \eta/n \rangle$ increases with temperature on the BEC side, and decreases on the BCS side. This behavior is consistent with a $(\lambda/a)^2$ dependence that dominates at high temperature, and competes with a $z(\lambda/a)$ contribution that becomes more important as the temperature is lowered. On a more quantitative level, Elliott et al. study the expanding gas at a cloud energy $\tilde{E}/E_F = 1$ and find a minimum of $\langle \eta/n \rangle$ at $1/(k_F a) \simeq 0.18$. Here, $\tilde{E}$ is a virial theorem based measure of the cloud energy. For exactly harmonic traps $\tilde{E} = E_{tot}/N$, where E_{tot} is the total energy (internal plus potential) of the cloud. In the high temperature limit we expect $E_{tot} = 3NT$.

The results shown in Fig. 1 are consistent with these findings. The figure in the left panel demonstrates that we observe a minimum of $\eta/\eta_\infty - 1$ on the BEC side. The minimum shifts toward unitarity with increasing temperature. Moreover, on the BCS side the scaled difference decreases with increasing temperature, while on the BEC side, close to unitarity, it increases with increasing T. The figure in the right panel shows that η/n is independent of $1/(k_F a)$ for large T/T_F, and that the sensitivity to $1/(k_F a)$ grows with decreasing temperature. On the BEC side η/n drops with increasing $1/(k_F a)$ even beyond the point where a minimum was observed in the left panel. This is a consequence of the decrease of n with increasing $1/(k_F a)$ as predicted by Eq. (4).

C.   Expansion in $z(\lambda/a)$

As discussed above, in-medium effects influence the shear viscosity in several ways. In addition to the effects of the in-medium scattering amplitude, the medium modification of the quasi-particle energy affects the energy conserving delta function and the final state momenta. With the interaction included, energy conservation implies $q^2 - q'^2 = m\mathcal{F}(P, q^2, q'^2, \theta_q, \theta_{q'})$ with

$$
\begin{aligned}
\mathcal{F} = {} & \Delta E_p(P, q'^2, \cos\theta_{q'}) + \Delta E_p(P, q'^2, -\cos\theta_{q'}) \\
& - \Delta E_p(P, q^2, \cos\theta_q) - \Delta E_p(P, q^2, -\cos\theta_q),
\end{aligned}
\tag{29}
$$

see Appendix A for details. We can solve this equation for q'^2 order-by-order in the fugacity and perform the integration over dq' in the matrix element of the collision operator up to
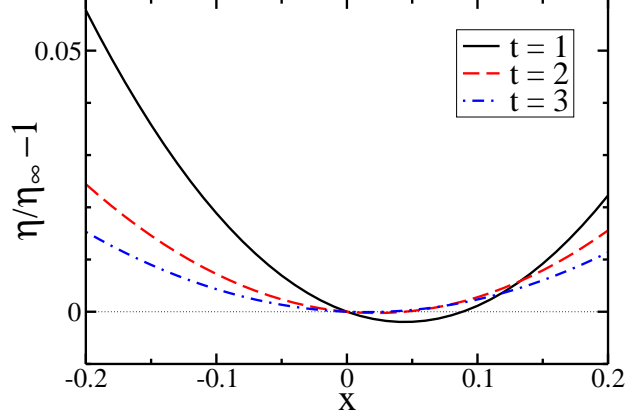
FIG. 2: (Color online) Scaled shear viscosity difference $(\eta - \eta_\infty)/\eta_\infty$ as a function of $x = 1/(k_F a)$ for different values of $t = T/T_F$. This figure shows the result of a systematic expansion to order $\mathcal{O}((\lambda/a)^2)$ and $\mathcal{O}(z(\lambda/a))$, see Eq. (32).

order $\mathcal{O}(z)$. This amounts to

$$
\langle \chi^{ij} | \mathcal{C}[\chi_{ij}] \rangle = \frac{2}{(2\pi)^6} \int_0^\infty dP \int_0^\infty dq \int_{-1}^1 d\cos\theta_q \int_{-1}^1 d\cos\theta_{q'} \int_0^{2\pi} d\phi \, P^2 q^2 \, f_{p_1}^0 f_{p_2}^0 \, |\mathcal{A}|^2
$$
$$
\times \left\{ mq^5(1 - \cos^2\Theta) \left(1 - m \frac{\partial \mathcal{F}}{\partial q'^2} \bigg|_{q'^2 = q^2} \right) + \frac{3}{2} mq^3 (1 - \cos^2\Theta) \, \Delta(q'^2) \right\}, \quad (30)
$$

where $f_p^0 = z e^{-E_p/T}$ contains medium effects through ΔE_p and $\Delta(q'^2) = -m\mathcal{F}|_{q'^2 = q^2}$. Medium corrections to the numerator of Eq. (20) arise from modifications of the quasi-particle energy and velocity. We find

$$
\langle \chi^{ij}(p) | p_{ij} (1 + \mathcal{G}(\tilde{p})z/a) \rangle = \frac{z(2mT)^{7/2}}{3\pi^2} \int_0^\infty dy \, y^6 e^{-y^2} e^{-\operatorname{Re}\Sigma(\sqrt{2mT}y)/T} \left\{ 1 + \mathcal{G}(y) \frac{z}{a} \right\}. \quad (31)
$$

Equations (30) and (31) contain all the terms needed to compute the dependence of η on (λ/a) at leading order in z. Close to unitarity we can expand

$$
\frac{\eta - \eta_\infty}{\eta_\infty} = c_0 \left(\frac{\lambda}{a} \right)^2 + c_1 z \left(\frac{\lambda}{a} \right) + \dots . \quad (32)
$$

Details of the calculation of c_0 and c_1 are described in Appendix B. We find

$$
c_0 = \frac{1}{4\pi} \simeq 0.07958, \qquad c_1 \simeq -0.03325, \quad (33)
$$

where the value of c_1 is the result of a numerical calculation. The final result for c_1 involves subtle cancellations between several effects. We showed in the previous section that the

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in-medium cross section alone leads to a minimum of the shear viscosity on the BEC side, corresponding to a negative contribution to c_1. In contrast, corrections to the quasi-particle velocity give a positive contribution to c_1. This effect is largely cancelled by corrections to the quasi-particle energy, see Appendix B. The final result of Eq. (32) is shown in Fig. 2. We observe that the complete result to $\mathcal{O}((\lambda/a)^2)$ and $\mathcal{O}(z(\lambda/a))$ is remarkably similar to Fig. 1, which only includes the in-medium cross section.

IV.   CONCLUSIONS AND OUTLOOK

In summary, we studied the influence of in-medium effects on the scattering length dependence of the shear viscosity in the dilute Fermi gas near unitarity. To zeroth order in the fugacity, η only depends on $(\lambda/a)^2$, and the minimum occurs at unitarity. Medium effects give, however, corrections of order $\mathcal{O}(z(\lambda/a))$, and the minimum of $(\eta - \eta_\infty)/\eta_\infty$ shifts to the BEC side. The main effect that causes this behavior is Pauli blocking in the in-medium cross section. Our results are in qualitative agreement with the experimental observations recently reported in [20]. More detailed comparisons will require an improved understanding of how to average the shear viscosity over the trap.

In our calculation we focused on one- and two-body effects, and truncated the systematic expansion in z and (λ/a) at order $\mathcal{O}((\lambda/a)^2)$ and $\mathcal{O}(z(\lambda/a))$. As demonstrated in Sect. III B, a simple model involving only in-medium scattering can be studied at any value of z and (λ/a), although the applicability of kinetic theory becomes questionable for $z \gtrsim 1$ or $(\lambda/a) \gtrsim 1$. Note that the former condition arises from the applicability of kinetic theory in a homogeneous system, whereas the latter condition arises in the context of applying kinetic theory to a finite system, in which the mean free path must be small compared to the system size. We have not considered three-body collisions, which are expected to contribute to the shear viscosity at $\mathcal{O}(z)$. Vacuum terms in the three-body amplitude can contain terms of $\mathcal{O}(1/(qa))$, which would contribute to the shear viscosity at $\mathcal{O}(z(\lambda/a))$. There is no experimental evidence for three-body effects in transport coefficients, but a theoretical estimate is certainly desirable.

The results presented in this work rely on kinetic theory and are limited to temperatures significantly above the phase transition. Near T_c quantum statistics and pseudogap effects are likely to be important. These effects can be incorporated into kinetic theory, but a

diagrammatic framework along the lines of [25–27] is likely to be more reliable.

Acknowledgements

Appendix A: Condition for energy conservation

The matrix element of the collision operator in Eq. (20) can be written as

$$\langle \chi^{ij}|\mathcal{C}[\chi_{ij}]\rangle = \frac{2}{(2\pi)^6}\int_0^\infty dP \int_0^\infty dq \int_{-1}^1 d\cos\theta_q \int_{-1}^1 d\cos\theta_{q'} \int_0^{2\pi} d\phi\, P^2 q^2\, f^0_{p_1} f^0_{p_2}\, |\mathcal{A}|^2$$

$$\times \int_0^\infty dq' q'^2 \left[q^4 + q'^4 - 2q^2 q'^2 \cos^2\Theta - \frac{1}{3}(q^2 - q'^2)^2 \right] \delta\left(q^2/m - q'^2/m - \mathcal{F}\right). \quad \text{(A.1)}$$

We change variables from dq' to dq'^2 and use the energy conserving delta function to evaluate the q'^2-integral up to leading order in z. For this purpose we solve the condition $q^2/m - q'^2/m = \mathcal{F}(P, q^2, q'^2, \theta_q, \theta_{q'})$ for q'^2 order-by-order in the fugacity. Since $m\mathcal{F}$ is of order $\mathcal{O}(z(\lambda/a))$, we can replace q'^2 in $\mathcal{F}$ by the solution at $\mathcal{O}(z^0)$, i.e. $q'^2 = q^2$, and find $q'^2 = q^2\left(1 + \Delta(q'^2)/q^2\right)$ with the $z(\lambda/a)$-correction

$$\Delta(q'^2) = -m\mathcal{F}|_{q'^2=q^2} = m\bigg(\Delta E_p(P, q^2, \cos\theta_q) + \Delta E_p(P, q^2, -\cos\theta_q)$$

$$-\Delta E_p(P, q^2, \cos\theta_{q'}) - \Delta E_p(P, q^2, -\cos\theta_{q'})\bigg), \quad \text{(A.2)}$$

where

$$\Delta E_p(\alpha, \beta, \gamma) = -\frac{8T}{\sqrt{\pi}} \frac{1}{p(\alpha, \beta, \gamma)} F_D\left(\frac{p(\alpha, \beta, \gamma)}{\sqrt{2mT}}\right) \frac{z}{a}, \quad \text{(A.3)}$$

$$p(\alpha, \beta, \gamma) = \sqrt{\frac{\alpha^2}{4} + \beta + \alpha\gamma\sqrt{\beta}} \quad \text{(A.4)}$$

such that $\Delta(q'^2) = 0$ for $q^2 = 0$. With this, the integral over dq'^2 can be evaluated. We find

$$\frac{1}{2}\int_0^\infty dq'^2 \sqrt{q'^2}\left[q^4 + q'^4 - 2q^2 q'^2 \cos^2\Theta - \frac{1}{3}(q^2 - q'^2)^2 \right] \delta\left(q^2/m - q'^2/m - \mathcal{F}\right)$$

$$= mq^5(1 - \cos^2\Theta) - m^2 q^5(1 - \cos^2\Theta)\left.\frac{\partial\mathcal{F}}{\partial q'^2}\right|_{q'^2=q^2} + \frac{3}{2}mq^3(1 - \cos^2\Theta)\Delta(q'^2) + \dots, \quad \text{(A.5)}$$

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where the first term is of $\mathcal{O}(z^0)$ and the second and third term are of $\mathcal{O}(z)$. Since $\partial\mathcal{F}/\partial q'^2$ is already of $\mathcal{O}(z)$, it suffices to evaluate this factor at $q'^2 = q^2$.

Appendix B: Systematic expansion

Ignoring all medium effects in Eq. (20), in particular using Eq. (21) for the squared scattering amplitude, one obtains for the shear viscosity

$$\eta_0 = \frac{15}{2^8\sqrt{\pi}}(mT)^{3/2}\mathcal{J}^{-1} \tag{B.1}$$

with

$$\mathcal{J} = \int_0^\infty d\tilde{q}\, \frac{\tilde{q}^5 e^{-2\tilde{q}^2}}{1 + 1/(2mTa^2\tilde{q}^2)} = \frac{1}{8}\left(1 - \frac{1}{2mTa^2} + \dots\right), \tag{B.2}$$

where $\tilde{q} = q/\sqrt{2mT}$. In-medium corrections alter this result as $\eta = \eta_0 + \Delta\eta$. Since η in Eq. (20) is of the form $\eta = A/B$, one can determine $\Delta\eta$ to leading order in the deviations from $\eta_0 = A_0/B_0$ as $\Delta\eta = \Delta A/B_0 - \eta_0\Delta B/B_0$. At unitarity, one finds from Eq. (20) to leading order in z

$$\eta_\infty = \frac{15\pi}{8\sqrt{2}}\frac{1}{\lambda^3}\left(1 - \frac{2^5\sqrt{2}}{\sqrt{\pi}}z\int_0^\infty d\tilde{P}\int_0^\infty d\tilde{q}\, \tilde{P}\tilde{q}^4\, e^{-3\tilde{P}^2/4}e^{-3\tilde{q}^2}\sinh(\tilde{P}\tilde{q})\right). \tag{B.3}$$

The second term is associated with $\mathrm{Im}\,\delta\Pi$ in $|\mathcal{A}|^2$ and leads to in-medium corrections that cancel in the difference $\eta - \eta_\infty$.

The term ΔA arises from the change in the quasi-particle velocity as well as the energy which enters the distribution function f_p^0 inside the numerator-integral. We find to leading order

$$\frac{(\Delta A/B_0)}{\eta_\infty} = \frac{8}{3\pi}z\left(\frac{\lambda}{a}\right), \tag{B.4}$$

which gives a non-vanishing contribution to the scaled difference $(\eta - \eta_\infty)/\eta_\infty$. This tends to decrease $(\eta - \eta_\infty)/\eta_\infty$ for $a < 0$ and to increase it for $a > 0$, i.e. to shift the minimum to the atomic side of the resonance.

The term ΔB is associated with the collision operator. The real part of $\delta\Pi$ gives a contribution of $\mathcal{O}(z(\lambda/a))$ which is positive for $a < 0$ and negative for $a > 0$. The imaginary part of $\delta\Pi$ contributes at $\mathcal{O}(z(\lambda/a)^2)$ in the scaled difference, which we have neglected throughout. The distribution functions $f_{p_1}^0 f_{p_2}^0$ in Eq. (30) give an order $\mathcal{O}(z(\lambda/a))$ term which is positive for $a < 0$ and negative for $a > 0$. Finally, the medium corrections to the

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energy conserving delta function give rise to two terms, cf. Eq. (A.5), which are both of $\mathcal{O}(z(\lambda/a))$ and decrease (increase) the scaled difference for $a < 0$ ($a > 0$).

With this, the scaled difference can be expanded systematically in powers of z and (λ/a) as

$$\frac{\eta - \eta_\infty}{\eta_\infty} = c_0 \left(\frac{\lambda}{a}\right)^2 + c_1 z \left(\frac{\lambda}{a}\right) + \dots \tag{B.5}$$

with $c_0 = 1/(4\pi)$ and

$$c_1 = \frac{8}{3\pi} + \frac{32\sqrt{2}}{\pi^2}\mathcal{I}_1 - \frac{16\sqrt{2}}{\pi^{3/2}}\mathcal{I}_2 - \frac{9}{\sqrt{2}\pi^{5/2}}\mathcal{I}_3 - \frac{4\sqrt{2}}{\pi^{3/2}}\mathcal{I}_4 \,. \tag{B.6}$$

In Eq. (B.6), the integrals are given by

$$\mathcal{I}_1 = \int_0^\infty d\tilde{P} \int_0^\infty d\tilde{q} \, \tilde{P}\tilde{q}^3 \, e^{-3\tilde{P}^2/4} e^{-2\tilde{q}^2} \int_0^\infty dx \, \frac{xe^{-x^2}\sinh(\tilde{P}x)}{(\tilde{q}^2 - x^2)} \,, \tag{B.7}$$

$$\mathcal{I}_2 = \int_0^\infty d\tilde{P} \int_0^\infty d\tilde{q} \int_0^\pi d\theta_q \, \tilde{P}^2\tilde{q}^5 \sin\theta_q \, e^{-\tilde{P}^2/2} e^{-2\tilde{q}^2}$$
$$\times \left[\frac{F_D(p(\tilde{P}, \tilde{q}^2, \cos\theta_q))}{p(\tilde{P}, \tilde{q}^2, \cos\theta_q)} + \frac{F_D(p(\tilde{P}, \tilde{q}^2, -\cos\theta_q))}{p(\tilde{P}, \tilde{q}^2, -\cos\theta_q)}\right] \,, \tag{B.8}$$

$$\mathcal{I}_3 = \int_0^\infty d\tilde{P} \int_0^\infty d\tilde{q} \int_0^\pi d\theta_q \int_0^\pi d\theta_{q'} \int_0^{2\pi} d\phi \, \tilde{P}^2\tilde{q}^3 \sin\theta_q \sin\theta_{q'} (1 - \cos^2\Theta)$$
$$\times e^{-\tilde{P}^2/2} e^{-2\tilde{q}^2} \Delta C(\tilde{P}, \tilde{q}, \theta_q, \theta_{q'}) \,, \tag{B.9}$$

$$\mathcal{I}_4 = \int_0^\infty d\tilde{P} \int_0^\infty d\tilde{q} \int_0^\pi d\theta_{q'} \, \tilde{P}^2\tilde{q}^4 \sin\theta_{q'} \, e^{-\tilde{P}^2/2} e^{-2\tilde{q}^2} C'(\tilde{P}, \tilde{q}, \theta_{q'}) \tag{B.10}$$

with $p(\alpha, \beta, \gamma)$ defined in Eq. (A.4) and

$$\Delta C(\tilde{P}, \tilde{q}, \theta_q, \theta_{q'}) = \frac{F_D(p(\tilde{P}, \tilde{q}^2, \cos\theta_{q'}))}{p(\tilde{P}, \tilde{q}^2, \cos\theta_{q'})} + \frac{F_D(p(\tilde{P}, \tilde{q}^2, -\cos\theta_{q'}))}{p(\tilde{P}, \tilde{q}^2, -\cos\theta_{q'})}$$
$$- \frac{F_D(p(\tilde{P}, \tilde{q}^2, \cos\theta_q))}{p(\tilde{P}, \tilde{q}^2, \cos\theta_q)} - \frac{F_D(p(\tilde{P}, \tilde{q}^2, -\cos\theta_q))}{p(\tilde{P}, \tilde{q}^2, -\cos\theta_q)} \,, \tag{B.11}$$

$$C'(\tilde{P}, \tilde{q}, \theta_{q'}) = \frac{p(\tilde{P}, \tilde{q}^2, \cos\theta_{q'}) - [1 + 2(p(\tilde{P}, \tilde{q}^2, \cos\theta_{q'}))^2]F_D(p(\tilde{P}, \tilde{q}^2, \cos\theta_{q'}))}{2(p(\tilde{P}, \tilde{q}^2, \cos\theta_{q'}))^3}$$
$$\times \left(2\tilde{q} + \tilde{P}\cos\theta_{q'}\right)$$
$$+ \frac{p(\tilde{P}, \tilde{q}^2, -\cos\theta_{q'}) - [1 + 2(p(\tilde{P}, \tilde{q}^2, -\cos\theta_{q'}))^2]F_D(p(\tilde{P}, \tilde{q}^2, -\cos\theta_{q'}))}{2(p(\tilde{P}, \tilde{q}^2, -\cos\theta_{q'}))^3}$$
$$\times \left(2\tilde{q} - \tilde{P}\cos\theta_{q'}\right) \,. \tag{B.12}$$

By evaluating the above integrals numerically, we find $\mathcal{I}_1 \approx -0.02194$, $\mathcal{I}_2 \approx 0.23899$, $\mathcal{I}_3 \approx -0.00059$ and $\mathcal{I}_4 \approx -0.18650$. In c_1, the first and the third term are both large compared to

the others, but of opposite sign. It is interesting to note that these together with the fourth and fifth term basically cancel each other, leaving the $z(\lambda/a)$-dependence of $(\eta - \eta_\infty)/\eta_\infty$ determined by the second term in c_1 which is related to $\operatorname{Re}\delta\Pi$ in the in-medium cross section.

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