# Second-order fluid dynamics for the unitary Fermi gas from kinetic theory

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## Abstract

We compute second order transport coefficients of the dilute Fermi gas at unitarity. The calculation is based on kinetic theory and the Boltzmann equation at second order in the Knudsen expansion. The second order transport coefficients describe the shear stress relaxation time, non-linear terms in the strain-stress relation, and non-linear couplings between vorticity and strain. An exact calculation in the dilute limit gives $\tau_R = \eta/P$, where $\tau_R$ is the shear stress relaxation time, $\eta$ is the shear viscosity, and $P$ is pressure. This relation is identical to the result obtained using the Bhatnagar-Gross-Krook (BGK) approximation to the collision term, but other transport coefficients are sensitive to the exact collision integral.

# I. INTRODUCTION

The dilute Fermi gas at unitarity provides a new paradigm for transport properties of strongly correlated quantum fluids [1–3]. Over the last several years there have been several experimental [4–10] and theoretical [11–23] studies devoted to the shear viscosity of the unitary Fermi gas. It was found that the system behaves as a nearly perfect fluid, which means that the ratio of entropy density to shear viscosity is close to the quantum bound $\eta/s = \hbar/(4\pi k_B)$ [24]. This bound was derived using the holographic duality between string theory in ten space-time dimensions, and field theory in four or fewer dimensions.

The main difficulty in providing more accurate determinations of the temperature and density dependence of $\eta/s$ is that experiments mainly constrain the average value of the viscosity of a harmonically trapped gas cloud. For the range of temperatures probed in experiments the center of the cloud behaves hydrodynamically, but in the dilute corona the mean free path is large and hydrodynamics is not applicable, see Sect. II. The theoretical challenge is to describe the transition between the hydrodynamic regime and the ballistic corona. A possible approach to this problem is to take into account a non-zero dissipative relaxation time [15, 25]. The relaxation time describes the time scale for dissipative stresses to approach the value given by the Navier-Stokes theory. In the dilute corona the relaxation time $\tau_R \sim l_{mfp}$ is large. As a consequence dissipative stresses remain small, and the rate of dissipation smoothly approaches the ballistic limit.

In a systematic approach to fluid dynamics the relaxation time appears at second in the gradient expansion of the stress tensor [26]. The corresponding kinetic coefficients can be determined using kinetic theory at second order in the Knudsen expansion. A simple estimate for the viscous relaxation time can be obtained using the the Bhatnagar-Gross-Krook (BGK) approximation to the collision integral [27]. In this approximation, all departures from equilibrium are assumed to relax at the same rate $1/\tau_0$. Not surprisingly, one finds that the viscous relaxation time $\tau_R$ is given by $\tau_R = \tau_0$. Using the result for the shear viscosity in the BGK theory, $\eta = P\tau_0$, we can write $\tau_R = \eta/P$. The ideal gas law $P \sim nT$ together with the fact that the viscosity of a dilute gas is independent of density implies that the relaxation time is inversely proportional to density.

The BGK estimate $\tau_R = \eta/P$ has been rederived many times in the literature, see [15, 26, 28], but it is not known how reliable this approximation is. Our goal in this paper

is to provide an exact calculation of $\tau_R$ in the dilute limit. The calculation is based on kinetic theory and the Boltzmann equation at leading order in the fugacity of the gas. We also determine other second order transport coefficient related to the shear stress, in particular the leading non-linear terms in strain-stress relation, and the mixing between strain and vorticity. For simplicity we do not compute second order transport coefficients related to heat flow. In the cold atom experiments the cloud is initially isothermal, and temperature gradients are only generated by viscous heating. As a result, the effect of thermal conductivity on the motion of the gas is already a second order effect, and higher order corrections are expected to be very small.

## II. SCALES AND EXPANSION PARAMETERS

Dilute atomic gases are characterized by the condition $k_F r \ll 1$, where $r$ is the range of the interaction, and $k_F$ is the Fermi momentum. The Fermi momentum is defined in terms of the density, $n = \nu k_F^3/(6\pi^2)$, where $\nu$ is the number of species. We are mostly interested in the case $\nu = 2$, which describes an unpolarized two-component gas. The unitary Fermi gas corresponds to the limit $k_F a \to \infty$, where $a$ is the $s$-wave scattering length. This implies that the gas is dilute, but very strongly correlated.

The density of the gas determines a temperature scale, $k_B T_F = k_F^2/(2m)$ [44]. For $T < T_F$ the gas is a strongly correlated quantum fluid, and the only reliable theoretical approach to equilibrium and non-equilibrium properties of the unitary Fermi gas is the quantum Monte Carlo (QMC) method. In the case of thermodynamic properties, QMC computations are very successful [29], but calculations of non-equilibrium properties are more challenging [30, 31]. The unitary gas has a second order phase transition to a superfluid phase. The most precise determination of the critical temperature comes from experiments with trapped atomic gases, which give $T_c/T_F = 0.167(13)$ [32].

At high temperature, $T > T_F$, the thermal de Broglie wave length $\lambda_{dB} = [2\pi/(mT)]^{1/2}$ of the particles is small, and the quantum diluteness of the gas $n\lambda_{dB}^3$ can be used as an expansion parameter. In the case of thermodynamic properties, this is the familiar virial expansion. Note that the fugacity of the gas is given by $z = e^{\beta\mu} \simeq (n\lambda_{dB}^3)/\nu$, and the virial expansion is equivalent to an expansion in powers of $z$. At leading order the equation of state is that of an ideal gas, $P = nT$, and the first non-trivial correction is governed by the

second virial coefficient $b_2$. In the weak coupling limit $b_2 \sim a/\lambda$, but in the $a \to \infty$ limit $b_2$ is a pure number. Experimental data show that at unitarity the viral expansion describes the equation of state for $z \lesssim 1$ [33].

In the high temperature limit transport properties of the unitary gas can be understood in terms of kinetic theory and the Boltzmann equation. Kinetic theory is based on the existence of well-defined quasi-particles, and requires that the quasi-particle width $\Gamma$ is small compared to the mean quasi-particle energy. In the high temperature limit $\Gamma \sim zT \ll E \sim T$ [34]. There are no complete, rigorous, calculations beyond leading order in $z$ in the literature, and as result the regime of validity of kinetic theory is difficult to establish. Experiments and QMC calculations are consistent with the idea that the range of convergence is similar to that of the virial expansion, $z \lesssim 1$ [8, 31].

Experiments involve a larger number of atoms, typically on the order of a few times $10^6$. This implies that the quasi-particle distribution function varies smoothly over the size $L$ of the trap. The microscopic scale is given by the mean free path $l_{mfp}$, and the expansion parameter is the Knudsen number $Kn = l_{mfp}/L$. The mean free path is given by $l_{mfp} = 1/(n\sigma)$, where $n$ is the density and $\sigma$ is the two-body cross section. In the unitary gas $\sigma = 4\pi/q^2$, where $q$ is the momentum transfer. A simple estimate of the Knudsen number can be obtained by averaging $\sigma$ over a thermal distribution. The Knudsen number at the center of the trap is [2]

$$Kn(0) = \frac{3\pi^{1/2}}{4(3\lambda N)^{1/3}} \left( \frac{T}{T_F^{trap}} \right)^2 , \tag{1}$$

where $\lambda$ is the aspect ratio of the trap, and $T_F^{trap}$ is the Fermi temperature of a non-interacting Fermi gas at the center of the trap. For the parameters probed in experiments $Kn(0) \ll 1$ corresponds to $T \lesssim 5T_F$. Since the mean cross section is only a function of temperature the mean free path in the trap scales as $l_{mfp} \sim n^{-1}$. This implies that

$$Kn(x) \simeq Kn(0) \exp \left( \frac{m}{2T} \sum_i \omega_i^2 x_i^2 \right) , \tag{2}$$

where $\omega_i$ $(i = 1, 2, 3)$ are the trapping frequencies. Near the edge of the trap the Knudsen number becomes large, and the expansion in gradients of the distribution function breaks down. Equ. (2) implies that the relevant scale is close to mean square cloud radius.

The estimates in equ. (1,2) refer to a static trapped Fermi gas. Conformal invariance implies that, up to dissipative effects, an expanding gas cloud evolves by a time dependent

scale transformation. This means that in a co-moving frame dimensionless ratios such as the Knudsen number are independent of time. In particular, if $Kn(0) \ll 1$ initially, then the Knudsen expansion remains valid throughout the evolution of the cloud.

In the regime $Kn \ll 1$ kinetic theory is equivalent to fluid dynamics. Fluid dynamics is based on the assumption of local thermodynamic equilibrium, and on the fact that thermodynamic variables vary smoothly over the extent of the system. Gradients of thermodynamic variables determine dissipative corrections to the equations of ideal fluid dynamics. The expansion parameter that controls the validity of fluid dynamics is the ratio of dissipative to ideal contributions in the energy and momentum currents. The main parameter is the inverse Reynolds number

$$Re^{-1} = \frac{\eta}{\rho L u} , \tag{3}$$

which measures the ratio of dissipative and ideal contributions to the stress tensor. Here, $\eta$ is the shear viscosity, $\rho$ is the mass density, and $u$ is the velocity of the fluid. In the kinetic regime $Kn \simeq Re^{-1}$, and the gradient expansion in kinetic theory is equivalent to the expansion in gradients of thermodynamic variables [3]. The fluid dynamical description is valuable because the gradient expansion remains valid even in the regime $T \lesssim T_F$ where the quasi-particle width is comparable to the quasi-particle energy and kinetic theory is not applicable.

In the case of the experiments described in [4, 9, 10] fluid dynamics is applicable at the center of the trap, but kinetic theory is not. The mean free path is comparable to the inter-particle spacing, and $\eta/s \ll 1$. Fluid dynamics breaks down in the dilute part of the cloud, but in this regime kinetic theory is reliable. In order to study the transition between fluid dynamics and kinetic theory we will compute the leading second order gradient corrections to the Navier-Stokes equation. These terms can used to quantify the breakdown of the Navier-Stokes theory. In Sect. IX we will show that one can resum second order gradient corrections, and achieve a smooth transition to the kinetic regime.

## III.  GRADIENT EXPANSION

We determine second order transport coefficients by matching the conserved currents in hydrodynamics to the currents in kinetic theory. In hydrodynamics the particle current is $\vec{j} = n\vec{u}$. This relation defines the fluid velocity $\vec{u}$, and does not receive dissipative corrections.

The stress tensor is

$$\Pi_{ij} = \rho u_i u_j + P\delta_{ij} + \delta\Pi_{ij} , \tag{4}$$

where $P$ is the pressure and $\delta\Pi_{ij}$ is the dissipative part of the stress tensor. At first order in the gradient expansion $\delta\Pi_{ij}$ can be written as $\delta\Pi_{ij} = -\eta\sigma_{ij} - \zeta\delta_{ij}\langle\sigma\rangle$ with

$$\sigma_{ij} = \nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\langle\sigma\rangle , \qquad \langle\sigma\rangle = \vec{\nabla}\cdot\vec{u} , \tag{5}$$

where $\eta$ is the shear viscosity and $\zeta$ is the bulk viscosity. In a scale invariant fluid $\zeta = 0$ [35]. The general structure of dissipative corrections in a scale invariant fluid up to second order in the gradient expansion was studied in [26]. The result is

$$\begin{aligned} \delta\Pi_{ij} &= -\eta\sigma_{ij} + \eta\tau_R\left[\dot{\sigma}_{ij} + u^k\nabla_k\sigma_{ij} + \frac{2}{3}\langle\sigma\rangle\sigma_{ij}\right] + \lambda_1\sigma_{\langle i}{}^k\sigma_{j\rangle k} + \lambda_2\sigma_{\langle i}{}^k\Omega_{j\rangle k} \\ &+ \lambda_3\Omega_{\langle i}{}^k\Omega_{j\rangle k} + \gamma_1\nabla_{\langle i}T\nabla_{j\rangle}T + \gamma_2\nabla_{\langle i}P\nabla_{j\rangle}P + \gamma_3\nabla_{\langle i}T\nabla_{j\rangle}P \\ &+ \gamma_4\nabla_{\langle i}\nabla_{j\rangle}T + \gamma_5\nabla_{\langle i}\nabla_{j\rangle}P . \end{aligned} \tag{6}$$

Here, $\mathcal{O}_{\langle ij\rangle} = \frac{1}{2}(\mathcal{O}_{ij} + \mathcal{O}_{ji} - \frac{2}{3}\delta_{ij}\mathcal{O}^k{}_k)$ denotes the symmetric traceless part of a tensor $\mathcal{O}_{ij}$, and $\Omega_{ij} = \nabla_i u_j - \nabla_j u_i$ is the vorticity tensor. In this work we focus terms related to gradients of the velocity field and determine $\tau_R$ and $\lambda_i$. We will discuss the physical significance of these parameters in Sec. IX.

## IV.   KINETIC THEORY AND THE CHAPMAN-ENSKOG METHOD

In kinetic theory the conserved currents are expressed in terms of quasi-particle distribution functions $f_p(\vec{x}, t)$. The mass current is

$$\vec{j} = \int d\Gamma_p\, m\vec{v} f_p(\vec{x}, t) , \tag{7}$$

where $d\Gamma_p = (d^3 p)/(2\pi)^3$, $m$ is the mass of the particles, $\vec{v} = \vec{\nabla}_p E_p$ is the quasi-particle velocity, and $E_p$ is the quasi-particle energy. We will compute transport coefficients at leading order in the fugacity $z = \exp(\mu/T)$. For this purpose we can approximate $E_p$ by the energy of a free fermion, $E_p = p^2/(2m)$. The stress tensor is given by

$$\Pi_{ij} = \int d\Gamma_p\, v_i p_j f_p(\vec{x}, t) . \tag{8}$$

The distribution function is determined by the Boltzmann equation

$$\left(\partial_t + \vec{v}\cdot\vec{\nabla}_x - \vec{F}\cdot\vec{\nabla}_p\right) f_p(\vec{x}, t) = C[f_p] , \tag{9}$$

where $\vec{F} = -\vec{\nabla}_x E_p$. In the dilute limit $E_p$ is not a function of $\vec{x}$ and we can set $\vec{F} = 0$. We can write the Boltzmann equation as $\mathcal{D}f_p = C[f_p]$, where we have defined

$$\mathcal{D} = \partial_t + \vec{v} \cdot \vec{\nabla}_x \, . \tag{10}$$

At leading order in the fugacity the collision term is dominated by two-body collisions and the effects of quantum statistics can be neglected. We have

$$C[f_1] = - \prod_{i=2,3,4} \left( \int d\Gamma_i \right) w(1,2;3,4) \left( f_1 f_2 - f_3 f_4 \right) , \tag{11}$$

where $f_i = f_{p_i}$ and $w(1,2;3,4)$ is the transition probability for $\vec{p}_1 + \vec{p}_2 \to \vec{p}_3 + \vec{p}_4$. In the dilute Fermi gas the scattering amplitude is dominated by s-wave scattering and

$$w(1,2;3,4) = (2\pi)^4 \left( \sum_i E_i \right) \delta \left( \sum_i \vec{p}_i \right) |\mathcal{A}|^2 , \quad |\mathcal{A}|^2 = \frac{16\pi^2}{m^2} \frac{a^2}{q^2 a^2 + 1} , \tag{12}$$

where $a$ is the s-wave scattering length and $2\vec{q} = \vec{p}_2 - \vec{p}_1$. The collision term vanishes in local thermal equilibrium, corresponding to the distribution function

$$f_p^0 = \exp\left( \frac{\mu}{T} \right) \exp\left( -\frac{m\vec{c}^2}{2T} \right) , \tag{13}$$

where $\vec{c} = \vec{v} - \vec{u}$, and the thermodynamic variables $\mu, T$ and $\vec{u}$ are functions of $\vec{x}$ and $t$. We will solve the Boltzmann by expanding the distribution function $f_p$ around the local equilibrium distribution,

$$f_p = f_p^0 + f_p^1 + f_p^2 + \ldots = f_p^0 \left( 1 + \frac{\psi_p^1}{T} + \frac{\psi_p^2}{T} + \ldots \right) . \tag{14}$$

Inserting this ansatz into the Boltzmann equation gives

$$\mathcal{D}f_p^0 + \mathcal{D}f_p^1 + \ldots = \frac{f_p^0}{T} \left( C_L^1 \left[ \psi_p^1 \right] + C_L^2 \left[ \psi_p^1 \right] + C_L^1 \left[ \psi_p^2 \right] + \ldots \right) , \tag{15}$$

where we have linearized the collision term,

$$C_L^1 [\psi_1] = - \int \left( \prod_{i=2,3,4} d\Gamma_i \right) w(1,2;3,4) \, f_2^0 \left( \psi_1 + \psi_2 - \psi_3 - \psi_4 \right) , \tag{16}$$

$$C_L^2 [\psi_1] = - \int \left( \prod_{i=2,3,4} d\Gamma_i \right) w(1,2;3,4) \frac{f_2^0}{T} \left( \psi_1 \psi_2 - \psi_3 \psi_4 \right) . \tag{17}$$

The left hand side of equ. (15) represents an expansion in gradients of the thermodynamic variables, and the right hand side is an expansion in powers of the inverse mean free path $l_{mfp}^{-1}$.

The dimensionless expansion parameter is the Knudsen number $Kn = l_{mfp}/L$, where $L$ is the characteristic length scale associated with the spatial dependence of the thermodynamic variables.

We will solve the Boltzmann equation to second order in the Knudsen number. At first order we have

$$\psi_p^1 = \left(C_L^1\right)^{-1} X_p^0, \quad X_p^0 \equiv \frac{T}{f_p^0} \left(\mathcal{D} f_p^0\right). \tag{18}$$

Given the function $\psi_p^1$ the dissipative contribution to the stress tensor at first order in the gradient expansion is determined by

$$\delta\Pi_{ij}^1 = \frac{\nu m}{T} \langle v_i v_j | \psi_p^1 \rangle, \tag{19}$$

where $\nu = 2$ is the spin degeneracy and we have used $p_i = m v_i$. We have defined the inner product

$$\langle \psi_p | \chi_p \rangle = \int d\Gamma_p \, f_p^0 \, \psi_p \chi_p. \tag{20}$$

The symmetries of the collision term imply that $C_L^1$ is a hermitean operator with respect to this inner product. In a scale invariant fluid $\delta\Pi_{ij}$ is traceless and we can replace $v_{ij} \equiv v_i v_j$ by $\bar{v}_{ij} \equiv v_{ij} - \frac{\delta_{ij}}{3} v^2$. If scale invariance is broken then $\langle \bar{v}_{ij} | \psi_p^1 \rangle$ projects out the traceless part of the stress tensor.

At next order in the Knudsen expansion the solution of the Boltzmann equation is

$$\psi_p^2 = \left(C_L^1\right)^{-1} \left\{ \left( X_p^0 - C_L^1 \left[\psi_p^1\right] \right) + X_p^1 - C_L^2 \left[\psi_p^1\right] \right\}, \quad X_p^1 \equiv \frac{T}{f_p^0} \left(\mathcal{D} f_p^1\right). \tag{21}$$

The second order contribution to $\delta\Pi_{ij}$ is determined by the matrix element $\langle v_i p_j | \psi_p^2 \rangle$. Because $C_L^1$ is hermitean we can write

$$\delta\Pi_{ij}^2 = \frac{\nu m}{T} \langle \left(C_L^1\right)^{-1} \bar{v}_{ij} | \Delta X_p^0 + X_p^1 - C_L^2 \left[\psi_p^1\right] \rangle, \tag{22}$$

where $\Delta X_p^0 = X_p^0 - C_L^1 \left[\psi_p^1\right]$. Equ. (18) implies that $\Delta X_p^0 = 0$ at first order in the gradient expansion, but in general $\Delta X_p^0$ is non-vanishing at $O(\nabla^2)$. The advantage of letting $\left(C_L^1\right)^{-1}$ act to the left is that $C_L$ is an integral transformation which is not easy to invert. Indeed, solving equ. (21) using the exact collision operator is quite involved. However, computing $\left(C_L^1\right)^{-1} \bar{v}_{ij}$ is essentially equivalent to computing $\left(C_L^1\right)^{-1} X_p^0$ and requires no extra effort once $\psi_p^1$ is determined.

The calculation of $\delta\Pi_{ij}^2$ involves the following steps: 1) Solve the first order equation for $\psi_p^1$. The solution to this problem is well known [13]. 2) Compute the second order streaming

terms $\Delta X_p^0$ and $X_p^1$. Some of the required calculations can be found [26]. 3) Determine the second order collision term $C_L^2 \left[ \psi_p^1 \right]$. 4) Compute the matrix element in equ. (22). We will go through these steps in the following sections.

## V. FIRST ORDER SOLUTION

We begin by computing the first order streaming term $X_p^0 = \frac{T}{f_p^0} (\mathcal{D} f_p^0)$. Using equ. (13) we find

$$
\begin{aligned}
X_p^0 = \frac{m}{2} \Bigg\{ & 2 \frac{T}{m} \mathcal{D}_u \alpha + 2 c^i \left[ \mathcal{D}_u u_i + \frac{T}{m} \nabla_i \alpha \right] \\
& + c^i c^j \left[ \sigma_{ij} + \delta_{ij} \left( \mathcal{D}_u \log(T) + \frac{2}{3} \langle \sigma \rangle \right) \right] + c^2 c^k \nabla_k \log(T) \Bigg\},
\end{aligned}
\tag{23}
$$

where $\mathcal{D}_u = \partial_0 + \vec{u} \cdot \vec{\nabla}$ is the comoving derivative relative to the fluid velocity $\vec{u}$, and we have defined $\alpha = \mu/T$. This equation can be simplified using the equations of fluid dynamics. Since equ. (23) is first order in gradients we can neglect gradient terms in the hydrodynamic equations. To leading order in the fugacity we can also use the equation of state of a free gas, $P = nT$. The continuity equation, the Euler equation, and the equation of energy conservation are

$$
\mathcal{D}_u \alpha = 0 , \quad \mathcal{D}_u u_i + \frac{T}{m} \nabla_i \alpha = -\frac{5T}{2m} \nabla_i \log(T) , \quad \mathcal{D}_u \log(T) + \frac{2}{3} \langle \sigma \rangle = 0 ,
\tag{24}
$$

which leads to

$$
X_p^0 = \frac{m}{2} \left\{ c^i c^j \sigma_{ij} + c^k \left[ \frac{5T}{m} - c^2 \right] q_k \right\},
\tag{25}
$$

where we have defined $q_k = -\nabla_k \log(T)$. Note that equ. (25) is orthogonal to the zero modes of the collision operator, $\langle X_p^0 | \phi^{0,k} \rangle$ with $\phi^{0,k} = \{1, \vec{c}, c^2\}$ ($k = 1, 2, 3$). These zero modes are associated with conservation of particle number, momentum, and energy.

As explained in the introduction we will focus on the case of no heat flow, $q_k = 0$. In order to solve the Boltzmann equation $|X_p^0\rangle = C_L^1 |\psi_p^1\rangle$ we make an ansatz for $\psi_p^1$,

$$
\psi_p^1 = \sum_{k}^{N-1} a_k S_k (x_c) \, \vec{c}^{ij} \sigma_{ij} , \quad x_c = \frac{mc^2}{2T} ,
\tag{26}
$$

where $S_k(x)$ is a complete set of functions. In practice we choose $S_k(x) = L_k^{5/2}(x)$, where $L_k^{5/2}$ is a generalized Laguerre (Sonine) polynomial. This choice is convenient because of the

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orthogonality relation $\langle S_k\,\bar{c}^{ij}|S_l\,\bar{c}_{ij}\rangle \sim \delta_{kl}$. We determine the coefficients $a_k$ by computing moments of the Boltzmann equation

$$\langle S_k\,\bar{c}^{ij}|(X_p^0)_{ij}\rangle = \langle S_k\,\bar{c}^{ij}|C_L^1|(\psi_p^1)_{ij}\rangle , \quad (k = 0,\ldots,N-1) , \tag{27}$$

where we have defined $X_p^0 = (X_p^0)^{ij}\sigma_{ij}$ and $\psi_p^1 = (\psi_p^1)^{ij}\sigma_{ij}$. From equ. (25) we get $(X_p^0)^{ij} = \frac{m}{2}\,\bar{c}^{ij}$. As a first approximation we can set $N = 1$, so that $\psi_p^1 = a_0\,\bar{c}^{ij}\sigma_{ij}$ with

$$a_0 = \frac{m}{2}\,\frac{\langle\bar{c}^{kl}|\bar{c}_{kl}\rangle}{\langle\bar{c}^{ij}|C_L^1|\bar{c}_{ij}\rangle} . \tag{28}$$

The matrix element of the collision operator is

$$\langle\bar{c}_{ij}|C_L^1|\bar{c}^{ij}\rangle = -\int\left(\prod_{i=1}^{4} d\Gamma_i\right) w(1,2;3,4)\,f_1^0 f_2^0\,(\bar{c}_1)_{ij}\left(\bar{c}_1^{ij}+\bar{c}_2^{ij}-\bar{c}_3^{ij}-\bar{c}_4^{ij}\right) . \tag{29}$$

At unitarity this integral can be computed analytically, see Sec. VII. We find

$$a_0 \equiv \bar{a}_0\,\frac{m}{zT} = -\frac{15\pi}{32\sqrt{2}}\,\frac{m}{zT} , \tag{30}$$

and the shear viscosity is

$$\eta = \frac{15}{32\sqrt{\pi}}\,(mT)^{3/2} . \tag{31}$$

Using $(C_L^1)^{-1}|(X_p^0)_{ij}\rangle = |(\psi_p^1)_{ij}\rangle$ together with $(X_p^0)_{ij} = \frac{m}{2}\bar{c}_{ij}$ we observe that this result determines the quantity $(C_L^1)^{-1}|\bar{v}_{ij}\rangle$ which enters in equ. (22). We find

$$\left(C_L^1\right)^{-1}\left|\bar{v}_{ij}\right\rangle = \left|\bar{v}_{ij}\right\rangle\frac{2}{zT}\,\bar{a}_0 , \tag{32}$$

which is correct up to higher order terms in the Sonine polynomial expansion. The next-to-leading order correction is determined in App. A.


## VI. SECOND ORDER STREAMING TERMS

Once $f_p^1 = f_p^0\psi_p^1/T$ is determined we can compute the second order streaming term $X_p^1 = (T/f_P^0)(\mathcal{D}f_p^1)$. The Boltzmann equation implies that the sum $X_p^1 + \Delta X_p^0$ must be orthogonal to the zero modes of the collision operator, but the two terms do not satisfy the orthogonality constraints individually. We can decompose $X_p^1 = (X_p^1)_{orth} + (X_p^1)_{ct}$, where $(X_p^1)_{orth}$ is orthogonal to the zero modes, and $(X_p^1)_{ct}$ is a "counterterm" that will have to

cancel against contributions contained in $\Delta X_P^0$. We find

$$
\left(X_p^1\right)_{orth} = \frac{m\bar{a}_0}{zT} \left\{ \frac{m}{2T}\left(c^i c^j c^k c^l - \frac{2}{15}\delta^{ik}\delta^{jl}c^4\right)\sigma_{ij}\sigma_{kl} \right. \tag{33}
$$

$$
+ \left(c^i c^j - \frac{1}{3}\delta^{ij}c^2\right)\left[\left(\mathcal{D}_u + \frac{2}{3}\langle\sigma\rangle\right)\sigma_{ij} - \sigma_{ik}\sigma_j{}^k - \sigma_{ik}\Omega_j{}^k\right]
$$

$$
\left. + \left(c^i c^j c^k - \frac{3}{5}\delta^{(ij}c^{k)}c^2\right)\nabla_k\sigma_{ij}\right\} \,,
$$

where $\Omega_{ij} = \nabla_i u_j - \nabla_j u_i$ is the vorticity tensor, and $A_{(ijk)}$ is symmetrized in all tensor indices. We have dropped two-derivative terms proportional to gradients of $T$ and $\alpha$. The counterterms are

$$
\left(X_p^1\right)_{ct} = \frac{m\bar{a}_0}{zT} \left\{ \left(\frac{m}{15T}c^4 - \frac{1}{3}c^2\right)\sigma^2 + \frac{3}{5}\delta^{(ij}c^{k)}c^2\,\nabla_k\sigma_{ij}\right\} \,, \tag{34}
$$

where $\sigma^2 = \sigma^{ij}\sigma_{ij}$. The second streaming term, $\Delta X_p^0$, can be determined as in Sect. V, but at second order in the gradient expansion we have to use the Navier-Stokes equation rather than the Euler equation. We have

$$
\mathcal{D}_u\alpha = -\frac{\eta}{2P}\,\sigma^2 \,, \tag{35}
$$

$$
\mathcal{D}_u u_i = -\frac{T}{m}\left(\nabla_i\alpha + \frac{5}{2}\nabla_i\log(T)\right) - \frac{1}{\rho}\nabla^k\left(\eta\sigma_{ki}\right) \,, \tag{36}
$$

$$
\mathcal{D}_u\log(T) = -\frac{2}{3}\langle\sigma\rangle + \frac{\eta}{3P}\,\sigma^2 \,, \tag{37}
$$

where we have neglected second order terms that involve gradients of the temperature. In order to be consistent with the first order calculation we use the result for $\eta$ found in Sect. V. We can write

$$
\eta = -\frac{\sqrt{2}}{\pi^{3/2}}\,\bar{a}_0(mT)^{3/2} \,, \tag{38}
$$

as well as $P = nT$ and $\rho = mn$. Here, $n = \nu z\lambda^{-3}$ is the density and $\lambda = [(2\pi)/(mT)]^{1/2}$ is the thermal de Broglie wave length. Combining equ. (23) with the Navier-Stokes equation (35-37) we find

$$
\Delta X_p^0 = -\frac{m\bar{a}_0}{zT}\left\{\left(\frac{1}{3}c^2 - \frac{T}{m}\right)\sigma^2 + \frac{2T}{m}\,c^i\,\nabla_k\sigma_{ki}\right\} \,. \tag{39}
$$

We observe that $\Delta X_p^0$ is a sum of terms that are proportional to zero modes of the collision operator, but that it does not cancel against $(X_p^1)_{ct}$. In particular, $(X_p^1)_{ct}$ contains terms of order $c^4$ and $c^3$, whereas $\Delta X_p^1$ is a second order polynomial in $c$. We can, however, write any

polynomial in $c$ as the sum of a polynomial orthogonal to the zero modes, and a polynomial of lower order in $c$. In particular, we can write

$$c^4 = \chi_4 + \left(c^4 - \chi_4\right) , \qquad\qquad \chi_4 = c^4 - 10\frac{T}{m}c^2 + 15\left(\frac{T}{m}\right)^2 , \qquad (40)$$

$$c_i c^2 = \chi_{3,i} + \left(c_i c^2 - \chi_{3,i}\right) , \qquad\qquad \chi_{3,i} = c_i\left(c^2 - 5\frac{T}{m}\right) , \qquad (41)$$

where $\langle \chi_4 | \phi^{0,k} \rangle = \langle \chi_{3,i} | \phi^{0,k} \rangle = 0$ with $\phi^{0,k} = \{1, c_i, c^2\}$ $(k = 1, 2, 3)$. Using equ. (40,41) we can write $(X_p^1)_{ct} = (X_p^1)_{ct,orth} + (X_p^1)_{ct,zm}$ where $(X_p^1)_{ct,orth}$ is orthogonal to the zero modes. We find $\Delta X_p^0 + (X_p^1)_{ct,zm} = 0$, and the second order streaming term is orthogonal to the zero modes. The complete streaming term at second order in the gradient expansion is

$$X_p^1 + \Delta X_p^0 = (X_p^1)_{orth} + (X_p^1)_{ct,orth} , \qquad (42)$$

where $(X_p^1)_{orth}$ is given in equ. (33) and

$$\left(X_p^1\right)_{ct,orth} = \frac{m\bar{a}_0}{zT}\left\{\frac{m}{15T}\left(c^4 - \frac{10T}{m}c^2 + \frac{15T^2}{m^2}\right)\sigma^2 + \frac{2}{5}\left(c^2 - \frac{5T}{m}\right)c^i\nabla_j\sigma_{ij}\right\} . \qquad (43)$$

In order to determine the transport coefficients we need to compute scalar products with $|c_{ij}\rangle$. Because of the orthogonality properties of $\chi_4$ and $\chi_{3,i}$ we find

$$\left\langle c_{ij}\right| \left(X_p^1\right)_{ct,orth} \left.\right\rangle = \frac{1}{3}\left\langle c^2\right| \left(X_p^1\right)_{ct,orth} \left.\right\rangle = 0 , \qquad (44)$$

and only $(X_p^1)_{orth}$ contributes to $\delta\Pi_{ij}^2$.


## VII.   SECOND ORDER COLLISION TERM

The second order collision operator is

$$
\begin{aligned}
C_L^2\left[\psi_1^1\right] &= -\int d\Gamma_{234}\, w(1,2;3,4)\, f_2^0\, \left(\psi_1^1\psi_2^1 - \psi_3^1\psi_4^1\right) \\
&= -\frac{1}{T}\left(\frac{\bar{a}_0 m}{zT}\right)^2 \sigma_{ij}\sigma_{kl}\int d\Gamma_{234}\, w(1,2;3,4)\, f_2^0\, \left(\bar{c}_1^{ij}\bar{c}_2^{kl} - \bar{c}_3^{ij}\bar{c}_4^{kl}\right) , \qquad (45)
\end{aligned}
$$

where $d\Gamma_{234} = d\Gamma_2\, d\Gamma_3\, d\Gamma_4$ and we have used $\psi_p^1 = \bar{a}_0 m(zT)^{-1}\bar{c}^{ij}\sigma_{ij}$. In order to determine the stress tensor we need

$$\left\langle \bar{c}_1^{ab}\right| C_L^2\left[\psi_1^1\right] \left.\right\rangle \equiv \left(\frac{\bar{a}_0 m}{zT}\right)^2 (C_L^2)^{abijkl}\sigma_{ij}\sigma_{kl} . \qquad (46)$$

$(C_L^2)_{abijkl}$ is a rank 6 tensor which is symmetric and traceless in $(ab)$, $(ij)$ and $(kl)$, and symmetric under the exchange $(ij) \leftrightarrow (kl)$. These symmetries completely fix the tensor structure and we find

$$(C_L^2)_{abijkl} \, \sigma^{ij} \sigma^{kl} = \frac{12}{35} \, \mathcal{C}_L^2 \, \sigma_{\langle a}{}^c \sigma_{b \rangle c} \, , \tag{47}$$

where we have defined the scalar integral

$$\mathcal{C}_L^2 \equiv (C_L^2)_{ab}{}^a{}_c{}^{cb} = -\int d\Gamma_{1234} \, w(1,2;3,4) \, f_1^0 f_2^0 \, (\bar{c}_1)_{ab} \left[ (\bar{c}_1)^a{}_c (\bar{c}_2)^{cb} - (\bar{c}_3)^a{}_c (\bar{c}_4)^{cb} \right] \, . \tag{48}$$

The scalar collision integral can be computed in analogy with the first order collision integral. We introduce center-of-mass and relative momenta

$$m \, \vec{c}_{1,2} = \frac{\vec{P}}{2} \pm \vec{q}, \quad m \, \vec{c}_{3,4} = \frac{\vec{P}}{2} \pm \vec{q}' \, , \tag{49}$$

and write the phase space measure as

$$\int d\Gamma_{1234} \, (2\pi)^4 \delta^3 \left( \sum_i \vec{p}_i \right) \delta \left( \sum_i E_i \right)$$
$$= \frac{2}{(2\pi)^6} \int P^2 dP \int q^2 dq \, \frac{qm}{2} \int d\cos\theta_q \int d\cos\theta_{q'} \int d\phi_{q'} \, , \tag{50}$$

where we have chosen a coordinate system in which $\vec{P} = P\hat{z}$, so that $\hat{P} \cdot \hat{q} = \cos\theta_q$. We also have $\hat{P} \cdot \hat{q}' = \cos\theta_{q'}$ and $\hat{q} \cdot \hat{q}' = \cos\theta_q \cos\theta_{q'} + \sin\theta_q \sin\theta_{q'} \cos\phi_{q'}$. Neither the product of distribution functions, $f_1^0 f_2^0$, nor the scattering amplitude, $|\mathcal{A}|^2$, depend on the angles $\theta_q, \theta_{q'}$ and $\phi_{q'}$. The angular integral can be performed by symmetrizing the integrand

$$\int d\cos\theta_q \int d\cos\theta_{q'} \int d\phi_{q'} \, \frac{1}{8} \left[ (\bar{c}_1)_{ab} + (\bar{c}_2)_{ab} - (\bar{c}_3)_{ab} - (\bar{c}_4)_{ab} \right]$$
$$\times \left[ (\bar{c}_1)^a{}_c (\bar{c}_2)^{cb} - (\bar{c}_3)^a{}_c (\bar{c}_4)^{cb} + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \right] = \frac{4\pi}{27} \frac{q^4}{m^6} \left( 12q^2 - 7P^2 \right) \, . \tag{51}$$

The integrals over $P$ and $q$ factorize and

$$\mathcal{C}_L^2 = -\frac{2}{(2\pi)^6} \int P^2 dP \int q^2 dq \, f_1^0 f_2^0 \, \frac{mq}{2} \left[ \frac{4\pi}{27} \frac{q^4}{m^6} \left( 12q^2 - 7P^2 \right) \right] \frac{16\pi^2}{m^2 q^2}$$
$$= \frac{4 T^{11/2}}{9\pi^{5/2} m^{3/2}} \, , \tag{52}$$

where we have taken the unitary limit, $a \to \infty$. This result determines the matrix element of the second order collision term,

$$\left\langle (\bar{c}_1)_{ij} \left| C_L^2 \left[ \psi_1^1 \right] \right\rangle = \frac{16 \, \bar{a}_0^2 \, m^{1/2} T^{5/2}}{105\pi^{5/2}} \, \sigma_{\langle i}{}^k \sigma_{j \rangle k} \, . \tag{53}$$

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# VIII. STRESS TENSOR AND SECOND ORDER TRANSPORT COEFFICIENTS

We now have all the ingredients in place to compute the dissipative stress tensor at second order in the gradient expansion. We start from equ. (22) and use equ. (32) to compute $(C_L^1)^{-1}\bar{c}_{ij}$. We find

$$\delta\Pi_{ij}^2 = \frac{2\nu m \bar{a}_0}{zT^2}\left(\langle \bar{c}_{ij}|(X_p^1)_{orth}\rangle - \langle \bar{c}_{ij}|C_L^2\left[\psi_p^1\right]\rangle\right). \tag{54}$$

The projection of the collision term on $|\bar{c}_{ij}\rangle$ is given in equ. (53). The projection of the streaming term is

$$\langle \bar{c}_{ij}|(X_p^1)_{orth}\rangle = \frac{\bar{a}_0 m^{1/2}T^{5/2}}{\sqrt{2}\pi^{3/2}}\left[\left(\mathcal{D}_u + \frac{2}{3}\langle\sigma\rangle\right)\sigma_{ij} + \sigma_{\langle i}{}^k\sigma_{j\rangle k} - \sigma_{\langle i}{}^k\Omega_{j\rangle k}\right]. \tag{55}$$

We can read off the transport coefficients by matching equ. (54) to the general result in conformal fluid dynamics, equ. (6). We use equ. (38) to relate $\bar{a}_0$ to the shear viscosity. We find

$$\tau_R = \frac{\eta}{P}, \quad \lambda_1 = \frac{15\eta^2}{14P}, \quad \lambda_2 = -\frac{\eta^2}{P}, \quad \lambda_3 = 0. \tag{56}$$

This result is exact at leading order in the fugacity and Sonine polynomial expansion. It is straightforward to compute higher order terms in the Sonine polynomial expansion. For the shear viscosity this is done in App. A. This solution can be inserted into the second order streaming and collision terms in order to compute the next order correction to $\tau_R$ and $\lambda_i$. We can see that the relation $\tau_R = \eta/P$ is not modified. The coefficients $\lambda_i$ receive corrections that are parametrically of the same magnitude as the corrections to $\eta$, which is less than 2%. Higher order corrections in the fugacity expansion are more difficult to compute. These corrections include higher order corrections to the equation of state and the quasi-particle properties, quantum effects, and three body collisions. Estimates of these effects can be obtained from the $T$-matrix calculation described in [19] and the molecular dynamics simulation in [36]. Both calculations show that corrections to the dilute limit become large for $T \lesssim T_F$, and that these effects tend to increase the shear viscosity.

Equ. (56) can be compared to the result of the relaxation time (BGK) approximation [26]. In this case we replace the full collision operator by $C[f^0 + \delta f] \simeq -\delta f/\tau_0$. This is a very crude approximation, but one that has been successfully applied in many areas of kinetic theory. Of course, one can always choose $\tau_0$ to obtain the correct shear viscosity, but the error in other transport properties is not necessarily small. We find, however, that $\tau_R$

and $\lambda_{2,3}$ agree with the BGK approximation, and that the correction in $\lambda_1$, the factor $15/14$ in equ. (56), is close to one.

The reason that $\lambda_1$ is modified is easily traced to the fact that $\psi_p^1 \sim c_{ij}\sigma^{ij}$, so that the non-linear collision term generates terms proportional to $\sigma_i{}^k\sigma_{jk}$. The fact that numerically this correction is small is essentially an accident, which depends on the structure of the collision cross section. The result that $\tau_R$ and $\lambda_2$ are not modified is somewhat harder to understand. The two main reasons are that the structure of $\psi_p^1$ is correctly reproduced by the BGK approximation, and that the second order streaming term $\mathcal{D}f_p^1$ is constrained by scale invariance. Indeed, the BGK approximation leads to the correct relaxation time $\tau_R$ in units of $\eta/P$ provided the collision time scales as $\tau_0 \sim T^{-1}h(\alpha)$ for any function $h$.

In this work we have not studied higher order corrections to heat flow. In this case the BGK approximation is expected to be less useful. If the collision time $\tau_0$ is fixed using the shear viscosity, then the thermal conductivity is too small by a factor $2/3$ [26, 37]. In addition to that, non-linearities in the collision term will give corrections to $q_i q_j$ terms in the stress tensor.

## IX. DISCUSSION

The main result of our study is equ. (56), which provides the transport coefficients related to terms of order $O(\nabla^2 u)$ in the stress tensor of a unitary Fermi gas. The results are exact at leading order in the fugacity $z$. In order to study the physical significance of second order terms we note that it is possible to rewrite the equations of fluid dynamics as the Navier-Stokes equation coupled to a relaxation equation for the dissipative stresses $\pi_{ij} \equiv \delta\Pi_{ij}$. For this purpose we use the first order relation $\pi_{ij} = -\eta\sigma_{ij}$ and write equ. (6) as [26]

$$\pi_{ij} = -\eta\sigma_{ij} - \tau_R\left[\dot{\pi}_{ij} + u^k\nabla_k\pi_{ij} + \frac{5}{3}\langle\sigma\rangle\pi_{ij}\right] + \frac{\lambda_1}{\eta^2}\pi_{\langle i}{}^k\pi_{j\rangle k} - \frac{\lambda_2}{\eta}\pi_{\langle i}{}^k\Omega_{j\rangle k} + \lambda_3\Omega_{\langle i}{}^k\Omega_{j\rangle k}, \quad (57)$$

where we have dropped terms of order $O(\nabla^2 T)$. Equation (57) is easiest to solve in systems in which the time dependence is harmonic, and non-linear terms in the velocity are small, $(\nabla u)^2 \ll \nabla\dot{u}$. In this case the relaxation time equation is solved by $\pi_{ij} = -\eta(\omega)\sigma_{ij}$, where $\eta(\omega) = \eta/(1 - i\omega\tau_R)$ is an effective, frequency dependent, viscosity.

The two conditions stated above are satisfied in the case of collective modes of a trapped Fermi gas [15, 25]. We consider the damping of the transverse breathing mode [5]. In order

to study the sensitivity of the damping rate to the values of the transport coefficients we write $\eta = c_\eta(mT)^{3/2}$ and $\tau_R = c_\tau\eta/P$. In kinetic theory $c_\eta = 15/(32\sqrt{\pi})$ and $c_\tau = 1$, see equ. (31) and (56). The damping rate is determined by the spatial integral over the frequency dependent shear viscosity. We find [25]

$$\Gamma = -\frac{c_\eta\omega_\perp}{(3\lambda N)^{1/3}}\left(\frac{E_F}{E_0}\right)\left(\frac{T}{\bar{\omega}}\right)^3 \text{Li}_{-3/2}\left(-\frac{3N^2\bar{\omega}^2}{80c_\eta^2c_\tau^2\pi^3T^4}\right) , \tag{58}$$

where $\omega_\perp$ is the transverse trap frequency, $\bar{\omega}$ is the geometric mean of the trapping frequencies, and $\lambda = \omega_z/\omega_\perp$. $Li$ is a polylogarithm, $N$ is the total number of particles and $E_0/E_F$ is the total energy per particle in units of the Fermi energy. At low temperature the damping rate scales as $\Gamma \sim c_\eta T^3 \log(c_\eta c_\tau T^2)$. In this regime fluid dynamics is valid over most of the cloud, and there is only a weak, logarithmic, dependence on the second order coefficient $c_\tau$. In the high temperature limit we find $\Gamma \sim 1/(c_\eta c_\tau^2 T)$. In this case the dependence on the second order coefficient $c_\tau$ is more important than the dependence on $c_\eta$ and the gradient expansion is not valid. However, the result agrees with the prediction of the Boltzmann equation for a trapped gas [15]. This implies that equ. (58) smoothly interpolates between second order fluid dynamics and kinetic theory. In particular, we can view the result $\eta(\omega) = \eta/(1 - i\omega\tau_R) \simeq \eta + i\omega\eta\tau_R$ as a resummation of the second order term that builds in the correct extrapolation to the limit $\tau_R \to \infty$.

Equation (58) was compared to data in [15, 25], and it was found that the agreement with experiment in the regime $0.3 \lesssim T/T_F \lesssim 1$ is quite good. In the original studies equ. (58) was derived using the BGK model, which is not a systematic approximation. What we have shown in the present work is that the same result can be derived from a reliable calculation based on kinetic theory and the fugacity expansion.

The role of $\lambda_1$ and $\lambda_2$ can be studied by considering the hydrodynamic expansion of a Fermi gas after release from a harmonic trap. The initial state is in hydrostatic equilibrium in an axisymmetric harmonic potential with $\omega_z \ll \omega_z$. Hydrodynamic expansion converts the asymmetry of the potential into differential acceleration and leads to transverse flow. Shear viscosity counteracts this effect and suppresses transverse expansion. We can obtain a qualitative understanding of the effects of dissipative terms by computing the stress tensor for the velocity field that solves the Euler equation for an expanding gas cloud [25, 38]. The velocity field is of the form $u_i(x, t) = \alpha_i(t)x_i$ (no sum over $i$), which is analogous to Hubble expansion in cosmology and to Bjorken expansion in relativistic heavy ion physics. In the

case of a strongly deformed trap $\alpha \equiv \alpha_\perp \gg \alpha_z$. The solution to the Euler equation can be written as $\alpha(t) = \dot{b}(t)/b(t)$, where $b(t)$ is the transverse scale factor of the expansion. At early time $b(t) \simeq 1 + \frac{1}{2}\omega_\perp^2 t^2$, and at late time $b(t) \simeq \sqrt{\frac{3}{2}}\omega_\perp t$. The strain tensor $\sigma_{ij}$ is diagonal, $\sigma_{ij} = \frac{2}{3}\mathrm{diag}(\alpha, \alpha, -2\alpha)$.

1. At first order in the gradient expansion we compare the dissipative stresses $\delta\Pi_{ij} = -\eta\sigma_{ij} = -\frac{2\eta}{3}\mathrm{diag}(\alpha, \alpha, -2\alpha)$ to the ideal stresses $\Pi_{ij} = P\delta_{ij} + \rho u_i u_j$. We observe, as expected, that dissipative effects tend to suppress transverse expansion and accelerate longitudinal expansion.

2. The coefficient $\lambda_1$ determines non-linearities in the stress-strain relation. In the case of anisotropic expansion we find $\delta\Pi_{ij}^2 = \frac{4\lambda_1}{9}\mathrm{diag}(-\alpha^2, -\alpha^2, 2\alpha^2)$ and $\lambda_1 > 0$ implies that viscous stresses are increased by second order effects.

3. The transport coefficient $\lambda_2$ only plays a role in rotating systems. An example is the expansion from a rotating trap studied in [39]. The initial state supports a velocity field of the form $\vec{u} = (\beta z, 0, \beta x)$, which carries non-zero angular momentum but no vorticity. The first order stress tensor is $\delta\Pi_{ij}^1 = -2\beta\eta(\delta_{xi}\delta_{zj} + \delta_{xj}\delta_{zi})$. The main effect of viscosity is to convert a fraction of the initial irrotational flow to rigid rotation, and generate non-zero vorticity [25]. At second order in the gradient expansion vorticity couples to transverse expansion and the angular momentum carried by the irrotational flow. This leads to two effects, an enhancement of transverse flow in-plane versus out of the rotation plane, and a further enhancement of rigid rotation.

## X. FINAL REMARKS

In this paper we have computed second order transport coefficients for a dilute Fermi gas. Second order transport properties were first considered by Burnett, who computed $\psi_p^2$ and $\delta\Pi_{ij}^2$ for Maxwell molecules, which are classical particles subject to a repulsive $1/r^5$ force [40]. The calculation presented in this work is substantially simpler than Burnett's. Part of the simplification is due to a more compact notation. We also avoid explicitly calculating $\psi_p^2$, and we focus on a simpler interaction, albeit one that can be realized experimentally. Finally, exact scale invariance reduces the number of independent kinetic coefficients.

Second order kinetic coefficients have also been computed for a relativistic quark gluon plasma [41]. The general structure of the result is very similar to the non-relativistic case. In particular, in the case of a quark gluon plasma one finds $\tau_R \simeq 3\eta/(2P)$, and $\lambda_1 > 0$, $\lambda_2 < 0$ as well as $\lambda_3 = 0$. All these results refer to the weak coupling, kinetic, limit. Second order transport coefficients of a relativistic scale invariant plasma have been computed in the strong coupling limit using the AdS/CFT correspondence [42]. In this case one finds $\tau_R = (1 - \log(2)/2)\eta/P$ and, again, $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_3 = 0$.

Assessing the full impact of $\tau_R$, $\lambda_1$ and $\lambda_2$ on the non-equilibrium evolution of expanding Fermi gas clouds will require numerical simulation similar to those reported in [43]. This work is in progress.

**Appendix A: The shear viscosity at next-to-leading order in the Sonine polynomial expansion**

It is straightforward, if somewhat tedious, to go beyond leading order in the Sonine polynomial expansion. At next-to-leading order we write

$$\left(\psi_p^1\right)_{ij} = \left(a_0 + a_1 S_1\left(x_c\right) + \ldots\right)\bar{c}_{ij}, \tag{A1}$$

where $S_1(x) = \frac{7}{2} - x$ and $x_c = mc^2/(2T)$. We define the matrix elements

$$(C_L^1)_{IJ} = \left\langle S_I \bar{c}^{ij} \middle| C_L^1 \middle| S_J \bar{c}_{ij} \right\rangle, \tag{A2}$$

as well as the normalization constants

$$\mathcal{N}_I = \frac{m}{2}\left\langle S_I \bar{c}^{ij} \middle| S_I \bar{c}_{ij} \right\rangle. \tag{A3}$$

The expansion coefficients $a_i$ are determined by equ. (27). If we truncate the expansion at $N = 1$ we get

$$a_0^{(1)} = \mathcal{N}_0 \frac{(C_L^1)_{11}}{(C_L^1)_{00}(C_L^1)_{11} - (C_L^1)_{01}^2}, \tag{A4}$$

$$a_1^{(1)} = \mathcal{N}_0 \frac{-(C_L^1)_{01}}{(C_L^1)_{00}(C_L^1)_{11} - (C_L^1)_{01}^2}. \tag{A5}$$

This should be compared to the $N = 0$ solution $a_0^{(0)} = \mathcal{N}_0/(C_L^1)_{00}$. The matrix elements $(C_L^1)_{IJ}$ can be computed using the methods described in Sect. VII. Because of the orthogonality relation $\langle S_k\, \bar{c}_{ij}|S_l\, \bar{c}_{ij}\rangle \sim \delta_{kl}$ the shear viscosity is determined by $a_0^{(N)}$. For $N = 1$ we obtain [14]

$$\eta^{(1)} = \eta^{(0)} \frac{(C_L^1)_{00}(C_L^1)_{11}}{(C_L^1)_{00}(C_L^1)_{11} - (C_L^1)_{01}^2} = \eta^{(0)} \frac{193}{190}\,, \tag{A6}$$

which is a 2% correction. Note that the Sonine polynomial expansion is variational. In particular, the shear viscosity computed from the exact solution of the Boltzmann equation is larger or equal to the $N$'th order approximant. Also note that the $N = 1$ correction to the distribution function is somewhat larger than the correction to the shear viscosity. We find $a_1^{(1)}/a_0^{(1)} = -12/193 \simeq -0.06$. The sign of $a_1^{(1)}/a_0^{(1)}$ implies that particle are pushed out to slightly larger momenta compared to the $N = 0$ approximation $\psi_p^1 \sim c^{ij}\sigma_{ij}$.

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[44] Note that in the following we will set the Boltzmann constant $k_B$ and Planck's constant $\hbar$ equal to one.