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Exploring multipartite quantum correlations with the square of quantum discord

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We explore the quantum correlation distribution in multipartite quantum states based on the square of quantum discord (SQD). For tripartite quantum systems, we derive the necessary and sufficient condition for that the SQD satisfies the monogamy relation. Particularly, we prove for the first time that the SQD is monogamous for three-qubit pure states, based on which a genuine tripartite quantum correlation measure is introduced. In addition, we also address the quantum correlation distributions in four-qubit pure states. As an example, we investigate multipartite quantum correlations in the dynamical evolution of multipartite cavity-reservoir systems.

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I. INTRODUCTION

Beside quantum entanglement, quantum correlation is also a key resource in quantum information processing [1–11]. As a basic tool to characterize the quantum advantage [12], quantum discord (QD) is a prominent bipartite quantum correlation measure [13, 14]. Recently, generalization of the QD to multipartite systems has received much attention [15–19]. However, characterization of quantum correlation structure in multipartite systems is still very challenging. Monogamy relation [20–22] is an important property in multipartite quantum systems. As quantified by the square of concurrences [23], entanglement is monogamous in multiqubit systems [21] *i.e.*,

$$C^2_{A_1|A_2\cdots A_N} \geq C^2_{A_1A_2} + C^2_{A_1A_3} + \cdots + C^2_{A_1A_N}, \tag{1}$$

and this property can be used to construct genuine multipartite entanglement measures [20, 24]. Therefore, it is natural to ask whether or not the quantum correlation is monogamous, especially for the QD.

Prabhu *et al* found that the QD is not monogamous and the monogamy relation

$$D_{A|BC} - D_{A|B} - D_{A|C} \geq 0 \tag{2}$$

is not satisfied even for the three-qubit W state [25]. Giorgi [26] and Fanchini *et al* [27, 28] related the monogamy condition of QD to the entanglement of formation. While Ren and Fan showed that QD is not monogamous under the same measurement party [29]. Recently, Streltsov *et al* further showed that the monogamy relation does not hold in general for quantum correlation measures which are nonzero for separable states [30]. However, these results do not imply that quantum correlation is still not monogamous in a specific case (for example, the geometric measure of discord [31] is monogamous in three-qubit pure states [30]). Since the QD is accepted as a basic tool for quantum correlation, it is desirable to find a kind of monogamous QD even in several qubit systems, which on the one hand gives a clear correlation structure but on the other hand allows the characterization of genuine multipartite quantum correlation.

In this paper, we are motivated by the following two questions: (i) *whether or not* the QD is monogamous in certain form? (ii) *in what degree* the discord is monogamous and can characterize the genuine multipartite quantum correlation? To answer these two questions, we explore the monogamy property of the square of quantum discord (SQD) in multipartite quantum systems. The paper is organized as follows. In Sec. II, we derive the necessary and sufficient condition for that the SQD is monogamous in tripartite quantum states. In three-qubit pure states, we prove that the SQD is monogamous and define a genuine tripartite quantum correlation measure. In Sec. III, we analyze the correlation distribution in multi-qubit pure states and construct multipartite quantum correlation indicators. As an application, we address the dynamics of quantum correlation in multipartite cavity-reservoir systems. Finally, we present discussions and a conclusion in Sec. IV.

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II. MONOGAMY PROPERTY AND CORRELATION MEASURE IN TRIPARTITE QUANTUM STATES

A. Definitions and monogamous condition

In a bipartite quantum system ρ_{AB}, the total correlation can be quantified by quantum mutual information $I_{A:B} = S(A) + S(B) - S(AB)$ with $S(X) = -\mathrm{Tr}\rho_X \log\rho_X$ being von Neumann entropy [13]. While the classical correlation is given by $J_{A:B} = \max_{\{E_j^B\}}[S(A) - \sum_j p_j S(A|E_j^B)]$, in which $\{E_j^B\}$ is a positive operator valued measure (POVM) performed on the subsystem B and $\rho_{A|E_j^B} = \mathrm{Tr}_B(E_j^B \rho_{AB} E_j^{B\dagger})/p_j$ with $p_j = \mathrm{Tr}_{AB}(E_j^B \rho_{AB} E_j^{B\dagger})$ [14]. The QD is used to characterize bipartite quantum correlation, which is defined as the difference between $I_{A:B}$ and $J_{A:B}$, and is expressed as [13]

$$D_{A|B} = S(B) - S(AB) + \min_{\{E_j^B\}} \sum_j p_j S(A|E_j^B), \tag{3}$$

where the minimum runs over all the POVMs, and $D_{A|B}$ is referred to as the discord of system AB with the measurement on subsystem B. The QD can also be written in the form of quantum conditional entropy [7]

$$D_{A|B} = \widetilde{S}(A|B) - S(A|B), \tag{4}$$

where the non-negative quantity $\widetilde{S}(A|B) = \min_{\{E_j^B\}} \sum_j p_j S(A|E_j^B)$ is the measurement-induced quantum conditional entropy and $S(A|B) = S(AB) - S(B)$ is the direct quantum generalization of conditional entropy.

Monogamy relation is an important property in multipartite quantum systems. Coffman *et al* first showed that the monogamy relation of concurrence $\mathcal{C}^2_{A|BC} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{AC} \geq 0$ is satisfied in three-qubit quantum states and the residual entanglement can characterize the genuine tripartite entanglement [20]. It should be noted that, in the monogamy relation, the square of concurrence is monogamous other than the concurrence itself which is not monogamous. Previous studies indicated that the QD is not monogamous even in three-qubit pure states [25–29], which does not imply that the square of QD is not monogamous either.

Here, we explore the monogamy property of SQD in multipartite systems. The SQD can be written as

$$D^2_{A|B} = [\widetilde{S}(A|B) - S(A|B)]^2, \tag{5}$$

which satisfies all the standard requirements for quantum correlation measure [30, 32] and can characterize effectively quantum correlation in bipartite systems. Particularly, in a tripartite pure state $|\psi_{ABC}\rangle$, the measurement-induced quantum conditional entropies are related to the entanglement of formation [23] by the Koashi-Winter formula [33]

$$\widetilde{S}(i|k) = \widetilde{S}(j|k) = E_f(ij), \tag{6}$$

where $\widetilde{S}(i|k)$ and $\widetilde{S}(j|k)$ are the conditional entropies with measurement on the subsystem k, and $E_f(ij) = \min \sum_\epsilon p_\epsilon S(\rho_i^\epsilon)$ is the entanglement of formation in the subsystem ρ_{ij} with the minimum taking over all the pure state decompositions $\{p_\epsilon, \rho_{ij}^\epsilon\}$ and $i \neq j \neq k \in \{A, B, C\}$. Using the formula in Eq. (6), the SQD has the form

$$D^2_{i|k} = [E_f(ij) - S(i|k)]^2, \tag{7}$$

where the measurement is performed on subsystem k, and $i \neq j \neq k \in \{A, B, C\}$. Moreover, in a tripartite pure state $|\psi_{ABC}\rangle$, we have the relation $D^2_{A|BC} = S^2(A) = E_f^2(A|BC)$ in which $E_f(A|BC)$ is the entanglement of formation under the bipartite partition $A|BC$ [13, 14]. Combining this relation with Eq. (7), we can derive the quantum correlation distribution of SQD

$$D^2_{A|BC} - D^2_{A|B} - D^2_{A|C} = T_1 + T_2, \tag{8}$$

where

$$\begin{aligned} T_1 &= E_f^2(A|BC) - E_f^2(AB) - E_f^2(AC), \\ T_2 &= 2S(A|B)[E_f(AC) - E_f(AB) - S(A|B)]. \end{aligned} \tag{9}$$

In the distribution, the first term T_1 is an entanglement distribution relation quantified by the square of entanglement of formation E_f^2 and the second term T_2 is a function of entanglement of formation E_f and conditional entropy $S(A|B)$. According to Eq. (8), the necessary and sufficient condition for the monogamous SQD is

$$T_1 + T_2 \geq 0. \tag{10}$$

B. Monogamy property in three-qubit pure states

We now look into the quantum correlation distribution in two-level (qubit) systems.
Theorem I. In any three-qubit pure state $|\psi_{ABC}\rangle$, the square of quantum discord $D_{A|BC}$ obeys the monogamy relation

$$D^2_{A|BC} - D^2_{A|B} - D^2_{A|C} \geq 0. \tag{11}$$

Proof. The theorem will hold when the monogamy condition in Eq. (10) is satisfied for all three-qubit pure states. In two-qubit quantum states, the entanglement of formation has an analytical expression $E_f(\rho_{ij}) = h[(1+(1-C^2_{ij})^{1/2})/2]$ in which $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy and $C_{ij} = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is the concurrence with the decreasing nonnegative λ_is being the eigenvalues of matrix $\rho_{ij}(\sigma_y \otimes \sigma_y)\rho^*_{ij}(\sigma_y \otimes \sigma_y)$ [23]. As a function of the square of concurrence, the entanglement of formation obeys the following relations

$$\begin{aligned} E^2_f(C^2_{A|BC}) &\geq E^2_f(C^2_{AB} + C^2_{AC}) \\ &\geq E^2_f(C^2_{AB}) + E^2_f(C^2_{AC}), \end{aligned} \tag{12}$$

where the CKW relation $C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}$ [20] and the monotonically increasing property of $E_f(C^2)$ is used in the first equation, and the property that E^2_f is a convex function of C^2 is used in the second equation. According to Eq. (12), we can obtain the first term $T_1 \geq 0$ in the monogamy condition.

For the second term T_2, we first show that $[E_f(AC) - E_f(AB)]$ has the same sign as that of $S(A|B)$. It is straightforward to derive the following relations

$$\begin{aligned} E_f(C^2_{AC}) \geq E_f(C^2_{AB}) &\Rightarrow E_f(C^2_{AB|C}) \geq E_f(C^2_{AC|B}) \\ &\Rightarrow S(C) \geq S(B) \\ &\Rightarrow S(A|B) \geq 0, \end{aligned} \tag{13}$$

where we have used the entanglement distributions $C^2_{AB|C} = C^2_{AC} + C^2_{BC} + \tau_3$ and $C^2_{AC|B} = C^2_{AB} + C^2_{BC} + \tau_3$ with τ_3 being the three-tangle [20], and the monotonically increasing property of $E_f(C^2)$. Similarly, if $E_f(AC) - E_f(AB) \leq 0$, we can obtain the relation $S(A|B) \leq 0$. Therefore $[E_f(AC) - E_f(AB)]$ and $S(A|B)$ have the same sign, and thus the second term in the monogamy condition has the form

$$T_2 = 2|S(A|B)|[|E_f(AC) - E_f(AB)| - |S(A|B)|]. \tag{14}$$

As a result, the nonnegative property of T_2 is equivalent to

$$T'_2 = |E_f(AC) - E_f(AB)| - |S(A|B)| \geq 0, \tag{15}$$

which is proven to be valid as follows.

On one hand, if $E_f(AC) \geq E_f(AB)$, the left hand side of Eq.(15) can be written as

$$T'_2(+) = S(B) - E_f(AB) - S(C) + E_f(AC) \tag{16}$$

where we have used $S(A|B) = S(C) - S(B)$ in tripartite pure states. On the other hand, we have

$$\begin{aligned} & E_f(C^2_{AC}) \geq E_f(C^2_{AB}) \\ \Rightarrow\ & E_f(C^2_{AC} + \Delta) \geq E_f(C^2_{AB} + \Delta) \\ \Rightarrow\ & E_f(C^2_{AC} + \Delta) - E_f(C^2_{AC}) \\ & \leq E_f(C^2_{AB} + \Delta) - E_f(C^2_{AB}), \end{aligned} \tag{17}$$

where Δ is a nonnegative constant. Besides, we have used the monotonic property of $E_f(C^2)$ in the second inequality and the concave property of $E_f(C^2)$ [26] in the third inequality which means that along with the increase of concurrence C^2 the increment of E_f will decrease. When we choose $\Delta = C^2_{BC} + \tau_3$, the entanglement of formation is

$$\begin{aligned} E_f(C^2_{AC} + \Delta) &= E_f(C^2_{AC} + C^2_{BC} + \tau_3) \\ &= E_f(C^2_{C|AB}) \\ &= S(C), \end{aligned} \tag{18}$$

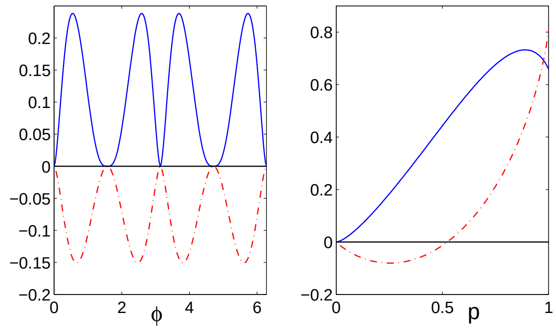

FIG. 1: (Color online) Quantum correlation distribution of SQD (blue solid line) in comparison to that of QD (red dash-dotted line). Left: two distributions for generalized W state in Eq. (22) as a function of parameter ϕ where the parameter θ is set to $\pi/4$; Right: two distributions for the two-parameter state in Eq. (23) as a function of the parameter p where the other parameter is chosen to be $\epsilon = 0.5$.

where the CKW relation has been used. Similarly, the relation $E_f(C_{AB}^2 + \Delta) = S(B)$ can be derived. Substituting the results into Eq. (17), we have the relation

$$S(B) - E_f(AB) \geq S(C) - E_f(AC). \tag{19}$$

Combining Eqs. (19) with (16), we can obtain that $T_2'(+) \geq 0$. In the other case, if $E_f(AC) \leq E_f(AB)$, the left of Eq. (15) becomes

$$T_2'(-) = S(C) - E_f(AC) - S(B) + E_f(AB). \tag{20}$$

Moreover, we have

$$\begin{aligned} & E_f(C_{AC}^2) \leq E_f(C_{AB}^2) \\ \Rightarrow\ & E_f(C_{AC}^2 + \Delta) - E_f(C_{AC}^2) \\ & \geq E_f(C_{AB}^2 + \Delta) - E_f(C_{AB}^2) \\ \Rightarrow\ & S(C) - E_f(AC) \geq S(B) - E_f(AB), \end{aligned} \tag{21}$$

where $\Delta = C_{BC}^2 + \tau_3$ and $E_f(C_{Ak}^2 + \Delta) = S(k)$ with $k \in \{B, C\}$, and the concave property of $E_f(C^2)$ is used. Combining Eqs. (20) with (21), we get $T_2'(-) \geq 0$. Therefore, we have proven that T_2' is nonnegative, namely, T_2 is nonnegative. Due to $T_1 \geq 0$ and $T_2 \geq 0$, the monogamy condition holds, and the proof is completed.

As examples, we consider the quantum correlation distribution of SQD in generalized W state [25]

$$|\psi_W\rangle = \sin\theta\cos\phi|011\rangle + \sin\theta\sin\phi|101\rangle + \cos\theta|110\rangle \tag{22}$$

and the two-parameter state [26]

$$\begin{aligned} |\psi(p,\epsilon)\rangle \ = \ & \sqrt{p\epsilon}|000\rangle + \sqrt{p(1-\epsilon)}|111\rangle \\ & + \sqrt{(1-p)/2}(|101\rangle + |110\rangle). \end{aligned} \tag{23}$$

In Fig.1, we plot the distribution $D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2$ (blue solid line) in comparison to the distribution $D_{A|BC} - D_{A|B} - D_{A|C}$ (red dash-dotted line) for the two quantum states, where although the QD is not monogamous as pointed out in Refs. [25, 26], we can see that the SQD is monogamous.

For the further verification on the theorem, we analyze the standard form of three-qubit pure states [34]

$$\begin{aligned} |\Psi\rangle_{ABC} \ = \ & \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \\ & + \lambda_4|111\rangle, \end{aligned} \tag{24}$$

where the real number λ_i ranges in $[0, 1]$ with the condition $\sum \lambda_i^2 = 1$, and the relative phase ϕ changes in $[0, \pi]$. Without loss of generality, we set $\lambda_0 = \cos\theta_0$, $\lambda_1 = \sin\theta_0\cos\theta_1$, $\lambda_2 = \sin\theta_0\sin\theta_1\cos\theta_2$, $\lambda_3 = \sin\theta_0\sin\theta_1\sin\theta_2\cos\theta_3$, and $\lambda_4 = \sin\theta_0\sin\theta_1\sin\theta_2\sin\theta_3$, respectively. In Fig.2, the quantum correlation distribution of SQD is plotted as a function of parameters $\theta_0, \theta_1, \theta_2$, and θ_3 (the relative phase is set to $\phi = 0$), where θ_i ranges in $[0, \pi/2]$ with equal interval being $\pi/40$. Again, we can see that the SQD is monogamous.

C. A genuine three-qubit quantum correlation measure with the hierarchy structure

A quantum correlation measure should satisfy the following necessary criteria: (i) it should be a non-negative real number; (ii) it is invariant under local unitary operations [30, 32]; and (iii) it is zero in an n-partite quantum state if and only if the state is a product state in any bipartite cut [35].

Based on our previous analysis on the quantum correlation distribution of SQD, we define a tripartite quantum correlation measure as

$$Q_3(A|BC) = D^2_{A|BC} - D^2_{A|B} - D^2_{A|C}, \tag{25}$$

which characterizes the genuine three-qubit quantum correlation in a pure state $|\psi_{ABC}\rangle$. The nonnegative property of Q_3 is satisfied due to the SQD being monogamous. The tripartite correlation Q_3 is invariant under local unitary operations because the SQDs are unchanged under the transformation.

For the third requirement, we first prove that the measure $Q_3(A|BC)$ is zero if a three-qubit state is a product state in any bipartite cut. When the quantum state has the form $|\psi_{ABC}\rangle = |\varphi_A\rangle \otimes |\varphi_{BC}\rangle$, the SQD $D^2_{A|BC} = S^2(A) = 0$ due to the product property under this partition. The SQD $D^2_{A|B} = 0$ because we have $\sum(I_A \otimes E^B_j)\rho_{AB}(I_A \otimes E^{B\dagger}_j) = \rho_{AB}$ with E^B_j being the projector composed of the eigenvector of ρ_B. The case for $D^2_{A|C} = 0$ is similar. So, the genuine tripartite quantum correlation $Q_3(A|BC) = 0$. For the product state $|\psi'_{ABC}\rangle = |\varphi_{AB}\rangle \otimes |\varphi_C\rangle$, we also have $Q_3(A|BC) = 0$, since $D^2_{A|BC} = D^2_{A|B} = S^2(A)$ and $D^2_{A|C} = 0$. Similarly, we can derive $Q_3(A|BC) = 0$ for $|\psi''_{ABC}\rangle = |\varphi_{AC}\rangle \otimes |\varphi_B\rangle$. Therefore, $Q_3(A|BC)$ is zero when the three-qubit pure state is a product state in any bipartite cut.

Next, we prove that when the three-qubit pure state is not bipartite product under any partition, the measure Q_3 is always nonzero. Based on the correlation distribution in Eq. (8), it is sufficient to prove the term $T_1 = E^2_f(C^2_{A|BC}) - E^2_f(C^2_{AB}) - E^2_f(C^2_{AC}) > 0$ since the second term is nonnegative. For a non-product state $|\omega_{ABC}\rangle$, its bipartite concurrence $C_{A|BC}$ is a positive value and we have the CKW relation $C^2_{A|BC} \geq C^2_{A|B} + C^2_{A|C}$. When $C^2_{A|B} \neq 0$ and $C^2_{A|C} \neq 0$, we can obtain that $T_1(E^2_f) > 0$ because the entanglement $E^2_f(C^2)$ is a monotonically increasing and convex function of the concurrence C^2. When one of the two-qubit concurrence is zero, for example $C^2_{AC} = 0$, the CKW relation is $C^2_{A|BC} > C^2_{A|B}$. According to the monotonic property, we have $T_1(E^2_f) > 0$. It should be noted that $C^2_{A|BC} = C^2_{A|B}$ should be removed simply because it corresponds to the case that the three-qubit pure state is a product one under the partition $AB|C$. Therefore, $T_1(E^2_f) > 0$ if ever the three-qubit state is of non-product, implying that the measure $Q_3(A|BC)$ is positive.

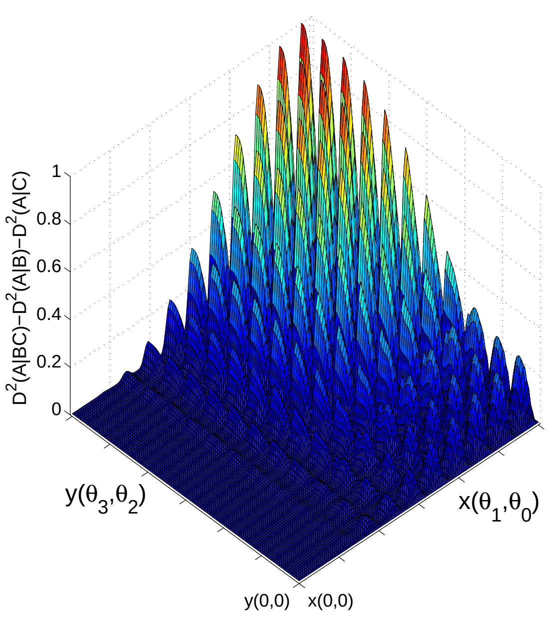

FIG. 2: (Color online) The monogamy property of SQD for the standard form of three-qubit pure states in Eq. (24). The distribution of SQD is plotted as a function of $x(\theta_1, \theta_0)$ and $y(\theta_3, \theta_2)$ where θ_i ranges in $[0, \pi/2]$ with equal interval being $\pi/40$ and the relative phase is set to $\phi = 0$.

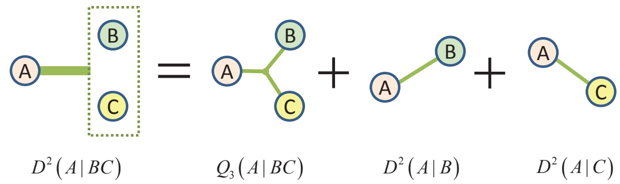

FIG. 3: (Color online) The hierarchy structure of quantum correlations in a three-qubit pure state.

So far, we have shown that the introduced tripartite quantum correlation measure $Q_3(A|BC)$ satisfies all the three necessary criteria. Furthermore, the measure may be understood as the monogamy score difference of SQD between the given state and a bipartite product state, *i.e.*,

$$\begin{aligned} Q_3(A|BC) &= ||\psi_{ABC} - \varphi_A \otimes \varphi_{BC}||_{MD2} \\ &= M_{D2}(\psi_{ABC}) - M_{D2}(\varphi_A \otimes \varphi_{BC}), \end{aligned} \tag{26}$$

where monogamy score is $M_{D2}(ABC) = D^2_{A|BC} - D^2_{A|B} - D^2_{A|C}$. When $Q_3(A|BC)$ is nonzero, the quantum state is not product state and its monogamy score is larger than that of any bipartite product state. The score difference is just the residual SQD. The larger the value of $Q_3(A|BC)$ is, the farther the monogamy distance between the give state and the bipartite product state is. Therefore the measure $Q_3(A|BC)$ can characterize the genuine three-qubit quantum correlation and has a physical explanation in terms of the monogamy score difference.

In addition, for a three-qubit pure state $|\psi_{ABC}\rangle$, we can obtain a hierarchy structure of quantum correlations. As depicted schematically in Fig.3, Eq. (25) can be rewritten as

$$D^2_{A|BC} = D^2_{A|B} + D^2_{A|C} + Q_3(A|BC), \tag{27}$$

where $D^2_{A|BC}$ quantifies the total quantum correlation in the partition $A|BC$, $D^2_{A|B}$ and $D^2_{A|C}$ quantify two-qubit quantum correlations, and $Q_3(A|BC)$ characterizes the genuine three-qubit quantum correlation under the partition $A|BC$.

As an application, we consider generalized GHZ and W states, which are two inequivalent classes under stochastic local operations and classical communication [36]. The generalized GHZ state has the form $|G_3\rangle = \alpha|000\rangle + \beta|111\rangle$. Its two-qubit quantum correlations are zero because the reduced density matrices ρ_{ij} are classical states. Therefore, there is only the genuine three-qubit quantum correlation $Q_3(A|BC) = S^2(A)$ in the generalized GHZ state. For the generalized W state $|W_3\rangle = a|001\rangle + b|010\rangle + c|100\rangle$, both two-qubit and three-qubit quantum correlations are nonzero when parameters a, b, and c are nonzero. When $a = b = 1/2$ and $c = \sqrt{2}/2$, the tripartite quantum correlation has the maximal value $Q_3(A|BC) \simeq 0.2779$.

Also noting that the QD is asymmetric for different measurement parties, the tripartite quantum correlation under qubit permutation is not equivalent to each other: $Q_3(A|BC) \neq Q_3(B|AC) \neq Q_3(C|AB)$ for a generic quantum state. From this consideration, we may define a new tripartite quantum correlation measure

$$Q_3(|\psi_{ABC}\rangle) = \frac{1}{3}\sum_{i,j,k} Q_3(i|jk), \tag{28}$$

where $i \neq j \neq k \in \{A, B, C\}$, and the measure may be referred to as the three-qubit mean SQD. This mean SQD not only satisfies all three conditions for a multipartite correlation measure, but also is independent of bipartite partitions, reflecting really the global tripartite quantum correlation in a three-qubit pure state $|\psi_{ABC}\rangle$.

D. Tripartite correlation indicator in mixed states

In three-qubit mixed states, the quantum correlation distribution of SQD is not always monogamous. As an example, we analyze the quantum state

$$\rho_{ABC}(W) = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \tag{29}$$

where the non-normalized pure state components are $|\psi_1\rangle = a|100\rangle + b|010\rangle + c|001\rangle$ and $|\psi_2\rangle = d|000\rangle$, respectively. Using the Koashi-Winter formula, we have the discord

$$D_{A|BC} = E_f(AE) - S(A|BC) \tag{30}$$

where subsystem BC is equivalent to a logic qubit and the subsystem E is the environment degree of freedom purifying the mixed state. Due to $\rho_{ABC}(W)$ is a rank-2 quantum state, the environment subsystem is equivalent to a logic qubit. In Eq. (29), we set the parameters $a = \cos\theta_1$, $b = \sin\theta_1\sin\theta_2\cos\theta_3$, $c = \sin\theta_1\sin\theta_2\sin\theta_3$, and $d = \sin\theta_1\cos\theta_2$. When the parameters $\theta_1 = \theta_2 = \theta_3 = 0.4\pi$, we can get $E_f(AE) = 0.06942$ by using the Wootters formula [23], which results in $D^2_{A|BC} = 0.10845$. Similarly, we have $D^2_{A|B} = 0.02368$ and $D^2_{A|C} = 0.08994$. Substituting these SQDs into the correlation distribution $D^2_{A|BC} - D^2_{A|B} - D^2_{A|C}$, we can evaluate the value of the distribution is -0.00517.

Although the quantum correlation distribution can be negative, we can still introduce a tripartite quantum correlation indicator whenever the distribution in a mixed state ρ_{ABC} is always monogamous (an example of these case will be presented in the next section). In this case, we may define the indicator as

$$\mathcal{Q}_3(\rho_{i|jk}) = D^2_{i|jk} - D^2_{i|j} - D^2_{i|k}, \tag{31}$$

where $i \neq j \neq k \in \{A, B, C\}$. Furthermore, we can introduce a symmetric tripartite correlation indicator

$$\mathcal{Q}_3(\rho_{ABC}) = \frac{1}{3}\sum_{i\neq j\neq k} \mathcal{Q}_3(i|jk), \tag{32}$$

which indicates the global tripartite quantum correlation in a three-qubit mixed state.

III. MULTIPARTITE QUANTUM CORRELATION INDICATORS IN FOUR-QUBIT SYSTEMS

In four-qubit pure states, the structure of quantum correlation distributions is more complicated than that in three-qubit states. In general, these distributions are not monogamous. However, if the distributions of SQD are monogamous in a given four-qubit system, we can also construct an indicator of the four-body correlation with the components

$$\begin{aligned}\mathcal{Q}_4^{(1*3)} &= D^2_{A|BCD} - D^2_{A|B} - D^2_{A|C} - D^2_{A|D} \\ \mathcal{Q}_4^{(2*2)} &= D^2_{AB|CD} - D^2_{A|C} - D^2_{A|D} - D^2_{B|C} - D^2_{B|D}\end{aligned} \tag{33}$$

where the superscript $(1*3)$ means that the correlation distribution lies in the partition between one qubit and the other three qubits and the case for $(2*2)$ is the distribution between two two-qubit subsystems. Under qubit permutations, $\mathcal{Q}_4^{(1*3)}$ and $\mathcal{Q}_4^{(2*2)}$ have four and six inequivalent components, respectively. The non-zero component indicates the genuine multipartite quantum correlation in the designated partition of a given state. For example, in the generalized four-qubit GHZ state $|G_4\rangle = \alpha|0000\rangle + \beta|1111\rangle$, the correlation distribution is always nonnegative, and we have $\mathcal{Q}_4^{(1*3)} = \mathcal{Q}_4^{(2*2)} = S^2(A)$. Another example is the cluster state $|C_4\rangle = (|0000\rangle - |0111\rangle - |1010\rangle + |1101\rangle)/2$ [37], in which we have $\mathcal{Q}_4^{(1*3)} = 1$ and $\mathcal{Q}_4^{(2*2)} = 2$.

At this stage, as an interesting example, we consider the dynamical property of quantum correlations in a real quantum system. As is known, the dynamical property of two-qubit quantum correlation has been widely investigated both theoretically and experimentally (see, for example, Refs. [38–44] and references therein). However, the dynamical property of multipartite quantum correlations is still very challenging. We now use the multipartite correlation indicators to analyze the dynamical evolution in four-partite cavity-reservoir systems. The system is composed of two entangled cavity photons being affected by the dissipation of two individual N-mode reservoirs, where the interaction of a single cavity-reservoir system is described by Hamiltonian [45]

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\sum_{k=1}^{N}\omega_k\hat{b}_k^\dagger\hat{b}_k + \hbar\sum_{k=1}^{N} g_k(\hat{a}\hat{b}_k^\dagger + \hat{b}_k\hat{a}^\dagger). \tag{34}$$

The initial state is $|\Phi_0\rangle = (\alpha|00\rangle + \beta|11\rangle)_{c_1c_2}|00\rangle_{r_1r_2}$, where the dissipative reservoirs are in the vacuum state. In the limit of $N \to \infty$ for a reservoir with a flat spectrum, the output state of the cavity-reservoir system has the form [45]

$$|\Phi_t\rangle = \alpha|0000\rangle_{c_1r_1c_2r_2} + \beta|\phi_t\rangle_{c_1r_1}|\phi_t\rangle_{c_2r_2}, \tag{35}$$

where $|\phi_t\rangle = \xi(t)|10\rangle + \chi(t)|01\rangle$ with the amplitudes being $\xi(t) = \exp(-\kappa t/2)$ and $\chi(t) = [1 - \exp(-\kappa t)]^{1/2}$. For the output state, we analyze its relevant components of the three- and four-partite quantum correlation indicators $\mathcal{Q}_3$ and $\mathcal{Q}_4$ given in Eqs. (31) and (33). Here, we use the method introduced by Chen *et al* for calculating the quantum discord of two-qubit X states (see the calculation in Appendix) [46].

In Fig.4, we plot different components of multipartite quantum correlation indicators as a function of the time evolution parameter κt and the initial state amplitude α. It is noted that all the correlation distributions are non-negative and we have

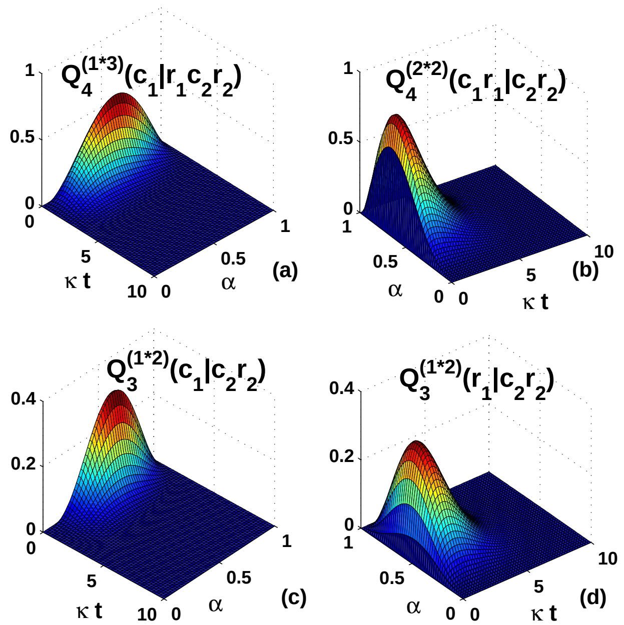

FIG. 4: (Color online) Different components of multipartite quantum correlation indicators in cavity-reservoir systems as a function of the time evolution κt and the initial state amplitude α, where all the correlation distributions are non-negative and detect the genuine multipartite quantum correlations.

$\mathcal{Q}_4 \geq 0$ and $\mathcal{Q}_3 \geq 0$ for these components. When the time $\kappa t = 0$, the quantum state is a product state and these indicators are zero. Along with the time evolution, they first increase to their maxima, and then decay asymptotically. When the parameter $\kappa t \to \infty$, the output state evolves to a product state again and all the multipartite quantum correlations disappear.

In the cavity-reservoir system, its multipartite entanglement evolution was investigated in Refs. [45, 47, 48]. The genuine multipartite entanglement can be characterized by a series of entanglement indicators. Here, in our analysis, we consider the

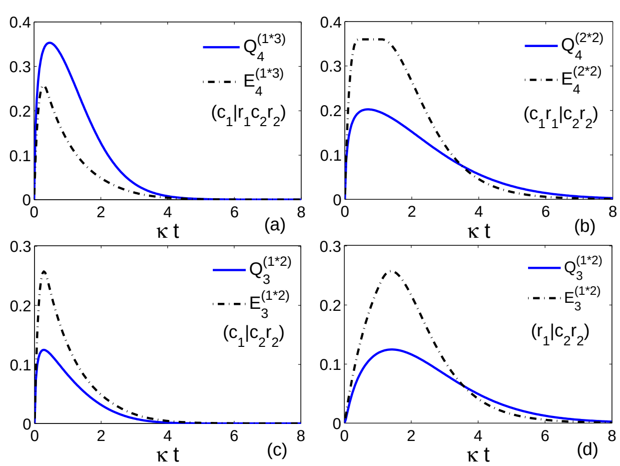

FIG. 5: (Color online) The multipartite quantum correlation indicators (blue solid lines) as a function of the time evolution parameter κt in comparison to the multipartite entanglement indicators (black dash-dotted lines) in the output state $|\Phi_t\rangle$ with the initial state parameter $\alpha = 1/\sqrt{10}$.

following components

$$
\begin{aligned}
E_4^{(1*3)}(|\Phi_t\rangle) &= C^2_{c_1|r_1c_2r_2} - C^2_{c_1r_1} - C^2_{c_1c_2} - C^2_{c_1r_2} \\
E_4^{(2*2)}(|\Phi_t\rangle) &= C^2_{c_1r_1|c_2r_2} - C^2_{c_1c_2} - C^2_{r_1r_2} - \sum C^2_{c_ir_j} \\
E_3^{(1*2)}(\rho_{c_1c_2r_2}) &= C^2_{c_1|c_2r_2} - C^2_{c_1c_2} - C^2_{c_1r_2} \\
E_3^{(1*2)}(\rho_{r_1c_2r_2}) &= C^2_{r_1|c_2r_2} - C^2_{r_1c_2} - C^2_{r_1r_2},
\end{aligned} \tag{36}
$$

where C^2 is the square of concurrence and the subscripts $i \neq j$ in the second equation. The component $E_4^{(1,3)}$ can be used to characterize the genuine multipartite entanglement in the partition $c_1|r_1c_2r_2$, and $E_4^{(2,2)}$ can indicate the genuine block-block entanglement in the partition $c_1r_1|c_2r_2$ [47]. Moreover, the component $E_3^{(1,2)}$ is used to quantify the qubit-block entanglement in three-qubit mixed states [48–50].

In Fig.5, we plot the relevant components of multipartite quantum correlation indicators $\mathcal{Q}_4$ and $\mathcal{Q}_3$ in comparison to these multipartite entanglement indicators E_4 and E_3 for the output state $|\Phi_t\rangle$. As seen from the figure, the multipartite quantum correlation is correlated with the multipartite entanglement in every partition structure. However, the peaks of correlation and entanglement do not coincide completely. The reason is that quantum correlation and quantum entanglement are not equivalent in general. Particularly, in the dynamical procedure, the evolution of two-qubit entanglement can exhibit the phenomenon of entanglement sudden death [51–53], but the corresponding evolution of quantum correlation is always asymptotic. In addition, the peak values of quantum correlation indicators can be greater (Fig. 5a) or less (Fig. 5b-d) than those of quantum entanglement indicators. This is due to that different measures of quantum states are lack of the same ordering [54–56]. Although the quantum correlation can be greater than entanglement in separable states, the ordering may change in a generic quantum state. For example, quantum discord is not always greater than the entanglement of formation even in two-qubit quantum states [57].

IV. DISCUSSION AND CONCLUSION

The QD is very difficult to compute because of the minimization over all positive operator-valued measures. Till now, the analytical result of QD is still an open problem except for some specific classes of quantum states [46, 57–63]. However, in three-qubit pure states, we can calculate two-qubit QD via the Wootters formula [23] and Koashi-Winter relation [33]. In this case, the analytical formula of genuine tripartite quantum correlation is available and can be rewritten as

$$
\begin{aligned}
Q_3(A|BC) \;=\; & S(A)^2 - [E_f(AC) - S(A|B)]^2 \\
& -[E_f(AB) - S(A|C)]^2.
\end{aligned} \tag{37}
$$

Therefore, in three-qubit pure states, not only the hierarchy structure of quantum correlation holds but also all the quantum correlations can be calculated analytically.

In conclusion, we have explored multipartite quantum correlations with the monogamy of SQD and answered the two important questions. We have proven that the SQD is monogamous in three-qubit pure states and the residual correlation is a reasonable measure for genuine three-qubit quantum correlation, which gives a clear hierarchy structure for quantum correlations. For three-qubit mixed states, although the distribution of SQD is not always monogamous, we have constructed an effective indicator which can detect the genuine tripartite quantum correlation in a specific class of states. For four-qubit pure states, the monogamy property of SQD may still be used to construct effective indicators for measuring genuine multipartite quantum correlations. As an interesting example, we have addressed the evolution of multipartite cavity-reservoir systems. The present work may shed a light on understanding of quantum correlations in multipartite systems.

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Appendix: calculation of the discord in cavity-reservoir systems

The density matrix of two-qubit X state can be written in

$$\rho_X^{AB} = \begin{pmatrix} a_{00} & 0 & 0 & a_{03} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{12}^* & a_{22} & 0 \\ a_{03}^* & 0 & 0 & a_{33} \end{pmatrix}. \tag{38}$$

When the elements satisfy the following relations [46]:

$$\begin{aligned} &|a_{12} + a_{03}| \geq |a_{12} - a_{03}|, \\ &|\sqrt{a_{00}a_{33}} - \sqrt{a_{11}a_{22}}| \leq |a_{12}| + |a_{03}|, \end{aligned} \tag{39}$$

Chen *et al* proved that the optimal measurement for the quantum discord is σ_x. In the output state $|\Phi_t\rangle$, we find the optimal measurement is σ_x for state $\rho_{c_1c_2}$. Then, according to the definition of the quantum discord in Eq. (4), we can get the value of $D^2_{c_1|c_2}$. For other two-qubit quantum discords in the correlation distributions, we can obtain that the optimal measurement is also σ_x, where we use the property that subsystem c_ir_i ($i = 1, 2$) is equivalent to a logic qubit. In a similar way, we can calculate these SQDs.

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