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Serdar Elhatisari, Sebastian König, Dean Lee, and H.-W. Hammer

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Causality, universality, and effective field theory for van der Waals interactions

Serdar Elhatisari,[1,*] Sebastian König,[2,†] Dean Lee,[1,‡] and H.-W. Hammer[2,§]

[1]*Department of Physics, North Carolina State University, Raleigh, NC 27695, USA*

[2]*Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, 53115 Bonn, Germany*

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We analyze low-energy scattering for arbitrary short-range interactions plus an attractive $1/r^6$ tail. We derive the constraints of causality and unitarity and find that the van der Waals length scale dominates over parameters characterizing the short-distance physics of the interaction. This separation of scales suggests a separate universality class for physics characterizing interactions with an attractive $1/r^6$ tail. We argue that a similar universality class exists for any attractive potential $1/r^\alpha$ for $\alpha \geq 2$. We also discuss the extension to multi-channel systems near a magnetic Feshbach resonance. We discuss the implications for effective field theory with attractive singular power law tails.

I. INTRODUCTION

Low-energy universality appears when there is a large separation between the short-distance scale of the interaction and the physically relevant long-distance scales. Some well-known examples include the unitarity limit of two-component fermions [1–5] and the Efimov effect in three-body and four-body systems [6–15]. See Ref. [16, 17] for reviews of the subject and literature. There have been many theoretical studies of low-energy phenomena and universality for interactions with finite range. These studies have direct applications to nuclear physics systems such as cold dilute neutron matter or light nuclei such as the triton and alpha particle. To a good approximation, the van der Waals interactions between alkali atoms can also be treated as a finite-range interaction.

However, there are some differences. For potentials with an attractive $1/r^\alpha$ tail and $\alpha > 2$, the s-wave scattering phase shift near threshold has been formulated in Ref. [18]. For $\alpha > 3$, the modified scattering parameters for an s-wave Feshbach resonance were derived in Ref. [19] using coupled-channel calculations. Analytical expressions for the s-wave scattering length and effective range for two neutral atoms and $\alpha = 6$ have been derived in Ref. [20]. However, the applicability of the effective range theory is limited for interactions with attractive tails. In order to define the scattering length for angular momentum $L \geq 2$ and the effective range for $L \geq 1$ a modified version of effective range theory known as quantum defect theory is needed [21, 22]. Furthermore, scattering parameters of magnetically tunable multi-channel

*Electronic address: selhati@ncsu.edu

†Electronic address: koenig@hiskp.uni-bonn.de

‡Electronic address: dean_lee@ncsu.edu

§Electronic address: hammer@hiskp.uni-bonn.de

systems have been studied in the context of the multichannel quantum-defect theory [23, 24]. See Ref. [25] for a very recent development of the multichannel quantum-defect theory for higher partial waves. There is also growing empirical evidence that there exists a new type of low-energy universality that ties together all interactions with an attractive $1/r^6$ tail. This might seem surprising since there is no such analogous behavior for interactions with a Coulomb tail. In this paper we derive the theoretical foundations for this van der Waals universality at low energies by studying the near-threshold behavior and the constraints of causality. We also show that this universality extends to any power law interaction $1/r^\alpha$ with $\alpha \geq 2$ in any number of dimensions. Our analysis applies to energy-independent interactions. We first consider a single scattering channel but then also consider multi-channel systems near a magnetic Feshbach resonance. The full analysis for the multi-channel problem off resonance will be discussed in future publications.

In our analysis we assume that the two-body potential has a long-distance attractive tail of the form $-C_6/r^6$. We define the van der Waals length scale, β_6, as

$$\beta_6 = (2\mu C_6)^{\frac{1}{4}} \,, \tag{1.1}$$

where μ is the reduced mass of the scattering particles. For simplicity we use atomic units [a.u.] throughout our discussion. So, in particular, we set $\hbar = 1$. In Ref. [26, 27] it was noticed that an approximate universal relationship exists between the effective range and inverse scattering length for S-wave scattering in many different pairs of scattering alkali atoms. If we write A_0 as the scattering length and R_0 as the effective range, the relation is

$$R_0 \approx \frac{\beta_6 \Gamma(1/4)^2}{3\pi} - \frac{4\beta_6^2}{3A_0} + \frac{8\pi\beta_6^3}{3\Gamma(1/4)^2 A_0^2} \,. \tag{1.2}$$

This approximate relation becomes exact for a pure $-C_6/r^6$ potential. What is surprising about Eq. (1.2) is that the van der Waals length β_6 dominates over other length scales which characterize the short-distance repulsive force between alkali atoms. This approximate universality suggests there is some separation of scales between the van der Waals length β_6 and the length scales of the short-range forces. This separation of scales will become more transparent later in our analysis when we determine the coefficients of the short-range K-matrix. It would be useful to exploit the separation of scales as an effective field theory with an explicit van der Waals tail plus contact interactions. In this paper, we discuss the constraints on such a van der Waals effective field theory.

We note that a similar dominance of the van der Waals length β_6 has been discovered for the three-body parameter in the Efimov effect [28–30]. In this paper we focus only on two-body systems. However, our analysis should be useful in developing the foundations for van der Waals effective field theory. This in turn could be used to investigate the Efimov effect and other low-energy phenomena in a model-independent way. An extension of our analysis may be useful to understand the recently observed universality of the three-body parameter for narrow Feshbach resonances [31].

The organization of our paper is as follows. We first discuss the connection between causality bounds and effective field theory. Next we consider asymptotic solutions of the Schrödinger equation. After that we derive causality bounds for the short-range K-matrix and consider the impact of these results on van der Waals effective field theory. Then we discuss quantum defect theory and calculate causal ranges for several examples of single-channel S-wave scattering in alkali atoms. We also consider the constraints of causality near a magnetic Feshbach resonances. We then conclude with a summary and discussion.

II. CAUSALITY BOUNDS AND EFFECTIVE FIELD THEORY

For an effective field theory with local contact interactions, the range of the interactions is controlled by the momentum cutoff scale. Problems with convergence can occur if the cutoff scale is set higher than the scale of the new physics not described by the effective theory. It is useful to have a quantitative measure of when problems may or may not appear, and this is where the causality bound provides a useful diagnostic tool. For each scattering channel we use the physical scattering parameters to compute a quantity called the causal range, R^b. R^b is the minimum range for the interactions consistent with the requirements of causality and unitarity. For any fixed cutoff scale, the causality bound marks a branch cut of the effective theory when viewed as a function of physical scattering parameters [32, 33]. The coupling constants of the effective theory become complex for scattering parameters violating the causality bound. These branch cuts do not appear in perturbation theory, but they can spoil the convergence pattern of the perturbative expansion.

Wigner was the first to recognize the constraints of causality and unitarity for two-body scattering with finite-range interactions [34]. The time delay of a scattered wave packet is given the energy derivative of the phase shift,

$$\Delta t = 2\frac{d\delta}{dE}\,. \tag{2.1}$$

It is clear that the incoming wave packet must first reach the interacting region before the outgoing wave packet can leave. So the causality bound can be viewed as a lower bound on the time delay, Δt. When applied to wave packets near threshold, the causality bound becomes an upper bound on the effective range parameter. Phillips and Cohen derived this bound for S-wave scattering with finite-range interactions [35]. Some constraints on nucleon–nucleon scattering and the chiral two-pion exchange potential were considered in Ref. [36], and relations between the scattering length and effective range have been explored for one-boson exchange potentials [37]. As mentioned above, the same authors studied the relationship between the scattering length and effective range for the van der Waals interaction [26, 27]

In Refs. [38, 39] the causality and unitarity bounds for finite-range interactions were extended to an arbitrary number of spacetime dimensions or value of angular momentum. A complementary discussion based upon conformal symmetry and scaling dimensions can be found in Ref. [40]. Coupled channel systems with partial wave mixing were first studied in Ref. [32], and the interactions with attractive and repulsive Coulomb tails were first considered in Ref. [33].

III. ASYMPTOTIC SOLUTIONS OF THE SCHRÖDINGER EQUATION

We consider a system of two spinless particles interacting via a spherically symmetric potential in the center-of-mass frame. As noted in the introduction, we use atomic units where $\hbar = 1$. The total wavefunction in the relative coordinate $\vec{r}$ can be separated into the radial and the angular parts as

$$\Psi^{(k)}_{L,M}(\vec{r}) = R^{(k)}_L(r)\, Y^M_L(\hat{r})\,, \tag{3.1}$$

where k is the magnitude of the spatial momentum and $Y_L^M(\hat{r})$ are the spherical harmonics. We define the rescaled radial wavefunction $U_L^{(k)}(r)$ as

$$U_L^{(k)}(r) = r\, R_L^{(k)}(r)\,. \tag{3.2}$$

We first review the case where the only interaction between the two particles has a finite range, R. This means that the interactions are exactly zero when the two particles exceed a distance R. Let μ denote the reduced mass of the two-body scattering system. The radial Schrödinger equation for a scattering state with energy $E = k^2/(2\mu)$ is then

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + k^2\right] U_L^{(k)}(r) = 2\mu \int_0^R dr'\, W(r,r') U_L^{(k)}(r')\,. \tag{3.3}$$

We have written the interaction as a rotationally-invariant operator with real kernel $W(r,r')$ to avoid any assumption regarding the locality or non-locality of the potential. In our analysis we consider only interactions which are energy independent. The finite-range condition implies that $W(r,r') = 0$ if $r > R$ or $r' > R$. We assume that the interaction is sufficiently well-behaved at the origin to admit a regular solution. This assumption imposes the restriction that at short distances the potential is not too singular such that the radial wavefunction satisfies the regularity condition,

$$\lim_{r\to 0} U_L^{(k)}(r) \frac{d}{dr} U_L^{(k)}(r) = 0\,. \tag{3.4}$$

In Ref. [41] it is proven that this condition is fulfilled by a class of potentials $V(r)$ provided that

$$\int_0^R r'\,|V(r')|\,dr' < \infty\,. \tag{3.5}$$

We choose a normalization such that for $r > R$, the radial wavefunction has the form

$$U_L^{(k)}(r) = k^{L+\frac{1}{2}} \sqrt{\frac{\pi r}{2}} \left[\cot\delta_L(k) J_{L+\frac{1}{2}}(kr) - N_{L+\frac{1}{2}}(kr)\right]\,, \tag{3.6}$$

where $J_{L+\frac{1}{2}}(kr)$ and $N_{L+\frac{1}{2}}(kr)$ are the Bessel functions of the first and second kind, and $\delta_L(k)$ is the scattering phase shift.

We now discuss the main problem of interest where the interactions have a long-range van der Waals tail. In addition to the non-singular finite-range interaction parameterized by $W(r,r')$, we assume that there is a long-range local potential $-C_6/r^6$ for $r > R$. The van der Waals length scale β_6 was defined in Eq. (1.1). The radial Schrödinger equation is

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + \frac{\beta_6^4}{r^6}\theta(r-R) + k^2\right] U_L^{(k)}(r) = 2\mu \int_0^R dr'\, W(r,r') U_L^{(k)}(r')\,. \tag{3.7}$$

The step function $\theta(r-R)$ cuts off the long-range potential at distances less than R. This ensures that we satisfy the regularity condition in Eq. (3.4) and avoids mathematical problems associated with unregulated singular potentials [42]. The general form of the solutions for Eq. (3.7) has been discussed by Gao in Ref. [43].

In order to simplify some of the more lengthy expressions to follow, we introduce dimensionless rescaled variables $r_s = r/\beta_6$, $k_s = \beta_6 k$, and $\rho_s = 1/(2r_s^2)$. In the outer region, $r > R$, the Schrödinger equation reduces to

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + \frac{\beta_6^4}{r^6} + k^2\right] U_L^{(k)}(r) = 0\,, \tag{3.8}$$

or

$$\left[\frac{d^2}{dr_s^2} - \frac{L(L+1)}{r_s^2} + \frac{1}{r_s^6} + k_s^2\right] U_L^{(k)}(r) = 0\,. \tag{3.9}$$

The exact solutions for Eq. (3.9) have been studied in detail in Ref. [44] using the formalism of quantum defect theory [45–47].

The van der Waals wavefunctions F_L and G_L are linearly independent solutions of Eq. (3.9). In order to write these out we first need several functions defined in Appendix A. The van der Waals wavefunctions F_L and G_L can be written as summations of Bessel functions,

$$F_L(k,r) = \frac{r_s^{1/2}}{X_L^2(k_s) + Y_L^2(k_s)}\Bigg[X_L(k_s)\sum_{m=-\infty}^{\infty} b_m(k_s)\, J_{\nu+m}(\rho_s) - Y_L(k_s)\sum_{m=-\infty}^{\infty} b_m(k_s)\, N_{\nu+m}(\rho_s)\Bigg]\,, \tag{3.10}$$

$$G_L(k,r) = \frac{r_s^{1/2}}{X_L^2(k_s) + Y_L^2(k_s)}\Bigg[X_L(k_s)\sum_{m=-\infty}^{\infty} b_m(k_s)\, N_{\nu+m}(\rho_s) + Y_L(k)\sum_{m=-\infty}^{\infty} b_m(k_s)\, J_{\nu+m}(\rho_s)\Bigg]\,. \tag{3.11}$$

The function X_L is defined in Eq. (A11), and Y_L is defined in Eq. (A12). For $m \geq 0$ the function b_m is given in Eq. (A6) while b_{-m} is given in Eq. (A7a). The offset ν appearing in the order of the Bessel functions is given by the solution of Eq. (A10) in Appendix A. For notational convenience, however, we omit writing the explicit k_s dependence of ν. Let us define $\delta_L^{(\mathrm{short})}(k)$ to be the phase shift of the van der Waals wavefunctions due to the scattering from the short-range interaction. The normalization of $U_L^{(k)}(r)$ is chosen so that for $r > R$,

$$U_L^{(k)}(r) = F_L(k,r) - \tan\delta_L^{(\mathrm{short})}(k)\, G_L(k,r)\,. \tag{3.12}$$

Our van der Waals wavefunctions are related to the functions f_L^{c0} and g_L^{c0} defined of Ref. [43] by the normalization factors $F_L = f_L^{c0}/\sqrt{2}$ and $G_L = -g_L^{c0}/\sqrt{2}$. Henceforth we write all expressions in terms of the short-range reaction matrix

$$\hat{K}_L = \tan\delta_L^{(\mathrm{short})}(k)\,, \tag{3.13}$$

which is related to the short-range scattering matrix via

$$\hat{S}_L = e^{2i\delta_L^{(\mathrm{short})}} = \frac{i - \hat{K}_L}{i + \hat{K}_L}\,. \tag{3.14}$$

For any finite-range interaction $\hat{K}_L$ is analytic in k^2 and can be calculated by matching solutions for $r \leq R$ and $r > R$ at the boundary. It can be written in compact form as

$$\hat{K}_L = \left. \frac{W(U_L^{(k)}, F_L^{(k)})}{W(U_L^{(k)}, G_L^{(k)})} \right|_{r=R} , \tag{3.15}$$

where $U_L^{(k)}$ is the solution of Eq. (3.7) that is regular at the origin, and W denotes the Wronskian of two functions,

$$W(f, g) = fg' - f'g \, .$$

IV. CAUSALITY BOUNDS FOR SHORT-RANGE K-MATRIX $\hat{K}_L$

In this section we derive causality bounds for the short-range K-matrix $\hat{K}_L$. For this we need to expand the wavefunction $U_L^{(k)}(r)$ in powers of k^2. The steps we follow are analogous to those used in Refs. [32, 33, 38, 39]. We first expand $\hat{K}_L$,

$$\hat{K}_L = \tan \delta_L^{(\text{short})}(k) = \sum_{n=0}^{\infty} K_{L,2n} \, k^{2n} \, . \tag{4.1}$$

The first two terms $K_{L,0}$ and $K_{L,2}$ are analogous to the inverse scattering length and effective range parameters in the usual effective range expansion. The higher-order terms can be regarded as analogs of the shape parameters. Next we expand the van der Waals wavefunctions in powers of k^2,

$$F_L(k, r) = f_{L,0}(r) + f_{L,2}(r) \, k^2 + \mathcal{O}(k^4) \, , \tag{4.2}$$

$$G_L(k, r) = g_{L,0}(r) + g_{L,2}(r) \, k^2 + \mathcal{O}(k^4) \, . \tag{4.3}$$

In the following we define

$$\nu_0 = \frac{1}{4}(2L + 1),$$

which corresponds to the value of ν at threshold. Using the low-energy expansions in Appendix B, we find that the coefficients in Eq. (4.2) are

$$f_{L,0}(r) = r_s^{1/2} J_{\nu_0}(\rho_s) \, , \tag{4.4}$$

and

$$\begin{aligned} f_{L,2}(r) = \frac{\Gamma(\nu_0)\Gamma(2\nu_0 - 1)}{\Gamma(\nu_0 + 1)\Gamma(2\nu_0)} \frac{\beta_6^2}{16} r_s^{1/2} \big[J_{\nu_0 - 1}(\rho_s) + N_{\nu_0}(\rho_s) \big] \\ - \frac{\Gamma(\nu_0)\Gamma(2\nu_0 + 1)}{\Gamma(\nu_0 + 1)\Gamma(2\nu_0 + 2)} \frac{\beta_6^2}{16} r_s^{1/2} \big[J_{\nu_0 + 1}(\rho_s) - N_{\nu_0}(\rho_s) \big] \, . \end{aligned} \tag{4.5}$$

Similarly the coefficients in Eq. (4.3) are

$$g_{L,0}(r) = r_s^{1/2} N_{\nu_0}(\rho_s) \, , \tag{4.6}$$

and

$$g_{L,2}(r) = \frac{\Gamma(\nu_0)\Gamma(2\nu_0-1)}{\Gamma(\nu_0+1)\Gamma(2\nu_0)}\frac{\beta_6^2}{16}r_s^{1/2}\big[N_{\nu_0-1}(\rho_s) - J_{\nu_0}(\rho_s)\big] - \frac{\Gamma(\nu_0)\Gamma(2\nu_0+1)}{\Gamma(\nu_0+1)\Gamma(2\nu_0+2)}\frac{\beta_6^2}{16}r_s^{1/2}\big[N_{\nu_0+1}(\rho_s) + J_{\nu_0}(\rho_s)\big]\,. \quad (4.7)$$

Using Eq. (3.12), we can now express $U_L^{(k)}(r)$ as an expansion in powers of k^2. For $r > R$, we have

$$U_L^{(k)}(r) = f_{L,0}(r) - K_{L,0}\, g_{L,0}(r) + k^2\big[f_{L,2}(r) - K_{L,0}\, g_{L,2}(r) - K_{L,2}\, g_{L,0}(r)\big] + \mathcal{O}(k^4)\,. \quad (4.8)$$

We now consider two solutions of the Schrödinger equation $U_L^{(k_a)}(r)$ and $U_L^{(k_b)}(r)$ with momenta k_a and k_b, respectively. We have

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + \frac{\beta_6^4}{r^6}\theta(r-R) + k_a^2\right] U_L^{(k_a)}(r) = 2\mu\int_0^R dr'\, W(r,r')U_L^{(k_a)}(r')\,, \quad (4.9)$$

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + \frac{\beta_6^4}{r^6}\theta(r-R) + k_b^2\right] U_L^{(k_b)}(r) = 2\mu\int_0^R dr'\, W(r,r')U_L^{(k_b)}(r')\,. \quad (4.10)$$

Following the same steps as in Eq. (31)-(36) in Ref. [39], we obtain the Wronskian integral formula

$$\frac{W[U_L^{(k_b)}, U_L^{(k_a)}](r)}{k_b^2 - k_a^2} = \int_0^r U_L^{(k_a)}(r')U_L^{(k_b)}(r')\, dr'\,, \quad (4.11)$$

for any $r > R$. Using Eq. (4.8) for momenta k_a and k_b we find

$$\begin{aligned}\frac{W[U_L^{(b)}, U_L^{(a)}](r)}{k_b^2 - k_a^2} = {}& W[f_{L,2}, f_{L,0}](r) - K_{L,0}\big\{W[g_{L,2}, f_{L,0}](r) + W[f_{L,2}, g_{L,0}](r)\big\} \\ & + K_{L,0}^2 W[g_{L,2}, g_{L,0}](r) - K_{L,2}W[g_{L,0}, f_{L,0}](r) + \mathcal{O}(k_a^2, k_b^2)\end{aligned} \quad (4.12)$$

In the Wronskian integral formula (4.11) we set $k_a = 0$ and take the limit $k_b \to 0$. With the wavefunction at zero energy written as $U_L^{(0)}$, the result is

$$K_{L,2} = b_L(r) - \frac{\pi}{4}\int_0^r \left[U_L^{(0)}(r')\right]^2 dr'\,, \quad (4.13)$$

where

$$\begin{aligned} b_L(r) = {}& \frac{\pi}{4}W[f_{L,2}, f_{L,0}](r) + \frac{\pi}{4}K_{L,0}^2 W[g_{L,2}, g_{L,0}](r) \\ & - \frac{\pi}{4}K_{L,0}\big\{W[g_{L,2}, f_{L,0}](r) + W[f_{L,2}, g_{L,0}](r)\big\}\,. \end{aligned} \quad (4.14)$$

The Wronskians appearing in Eq. (4.14) can be written out explicitly as

$$\begin{aligned} W[f_{L,2}, f_{L,0}](r) = {}& \frac{\beta_6\rho_s}{16\nu_0(2\nu_0-1)}\left[J_{\nu_0-2}(\rho_s)\; J_{\nu_0}(\rho_s) - J_{\nu_0-1}^2(\rho_s)\right] \\ & + \frac{\beta_6\rho_s}{16\nu_0(2\nu_0+1)}\left[J_{\nu_0+2}(\rho_s)\; J_{\nu_0}(\rho_s) - J_{\nu_0+1}^2(\rho_s)\right] \\ & + \frac{\beta_6\rho_s}{4(2\nu_0-1)(2\nu_0+1)}\left[J_{\nu_0+1}(\rho_s)\; J_{\nu_0-1}(\rho_s) - J_{\nu_0}^2(\rho_s) + \frac{4}{\pi\rho_s}\right]\,, \end{aligned} \quad (4.15)$$

$$W[g_{L,2}, g_{L,0}](r) = \frac{\beta_6 \rho_s}{16\nu_0(2\nu_0-1)}\left[N_{\nu_0-2}(\rho_s)\ N_{\nu_0}(\rho_s) - N_{\nu_0-1}^2(\rho_s)\right] + \frac{\beta_6 \rho_s}{16\nu_0(2\nu_0+1)}\left[N_{\nu_0+2}(\rho_s)\ N_{\nu_0}(\rho_s) - N_{\nu_0+1}^2(\rho_s)\right] + \frac{\beta_6 \rho_s}{4(2\nu_0-1)(2\nu_0+1)}\left[N_{\nu_0+1}(\rho_s)\ N_{\nu_0-1}(\rho_s) - N_{\nu_0}^2(\rho_s) + \frac{4}{\pi\rho_s}\right], \quad (4.16)$$

and

$$W[g_{L,2}, f_{L,0}](r) = W[f_{L,2}, g_{L,0}](r) = \frac{\beta_6 \rho_s}{16\nu_0(2\nu_0-1)}\left\{J_{\nu_0-1}(\rho_s)\left[N_{\nu_0+1}(\rho_s) - N_{\nu_0-1}(\rho_s)\right] - N_{\nu_0}(\rho_s)\left[J_{\nu_0}(\rho_s) - J_{\nu_0-2}(\rho_s)\right]\right\} - \frac{\beta_6 \rho_s}{16\nu_0(2\nu_0+1)}\left\{J_{\nu_0+1}(\rho_s)\left[N_{\nu_0+1}(\rho_s) - N_{\nu_0-1}(\rho_s)\right] - N_{\nu_0}(\rho_s)\left[J_{\nu_0+2}(\rho_s) - J_{\nu_0}(\rho_s)\right]\right\}. \quad (4.17)$$

The fact that the integral on the right-hand side of Eq. (4.13) is positive semi-definite sets an upper bound on the short-range parameter $K_{L,2}$. We find that

$$K_{L,2} \leq b_L(r) \quad (4.18)$$

for any $r > R$.

V. IMPACT ON EFFECTIVE FIELD THEORY

In this section we discuss the impact of our causality bounds for an effective field theory with short-range interactions and an attractive $1/r^6$ tail. In Fig. 1 we plot the $L = 0$ Wronskians $W[f_{0,2}, f_{0,0}]$, $W[g_{0,2}, g_{0,0}]$, and $W[g_{0,2}, f_{0,0}]$ for $\beta_6 = 50$ [a.u.]. Figures 2 and 3 show the analogous plots for $L = 1$ and $L = 2$, respectively.

We note that all of the Wronskian functions in Fig. 1, 2, 3 vanish in the limit $r \to 0$. This stands in clear contrast to what one finds for purely finite-range interactions [38, 39]. In that case the effective range parameter, r_L, satisfies the upper bound

$$r_L \leq b_L^{\text{free}}(r)\,, \quad (5.1)$$

where the function $b_L^{\text{free}}(r)$ is (*cf.* Eq. (60) in Ref. [39])

$$b_L^{\text{free}}(r) = -\frac{2\Gamma(L-\frac{1}{2})\Gamma(L+\frac{1}{2})}{\pi}\left(\frac{r}{2}\right)^{-2L+1} - \frac{4}{L+\frac{1}{2}}\frac{1}{a_L}\left(\frac{r}{2}\right)^2 + \frac{2\pi}{\Gamma(L+\frac{3}{2})\Gamma(L+\frac{5}{2})}\frac{1}{a_L^2}\left(\frac{r}{2}\right)^{2L+3}, \quad (5.2)$$

and a_L is the scattering length. Near $r = 0$ the behavior of $b_L^{\text{free}}(r)$ is

$$b_L^{\text{free}}(r) = -\frac{2\Gamma(L-\frac{1}{2})\Gamma(L+\frac{1}{2})}{\pi}\left(\frac{r}{2}\right)^{-2L+1} + \mathcal{O}(r^2)\,. \quad (5.3)$$

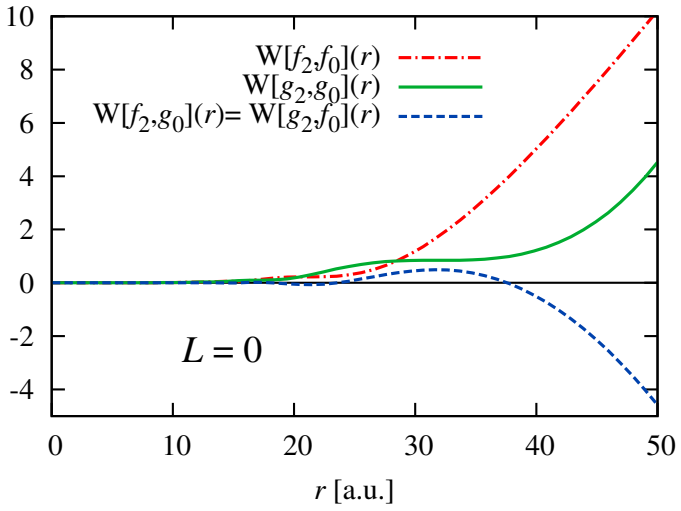

FIG. 1: (Color online)Plot of $W[f_{0,2}, f_{0,0}](r)$, $W[g_{0,2}, g_{0,0}](r)$, and $W[g_{0,2}, f_{0,0}](r)$ as a function of r for $L = 0$ and $\beta_6 = 50$ [a.u.].

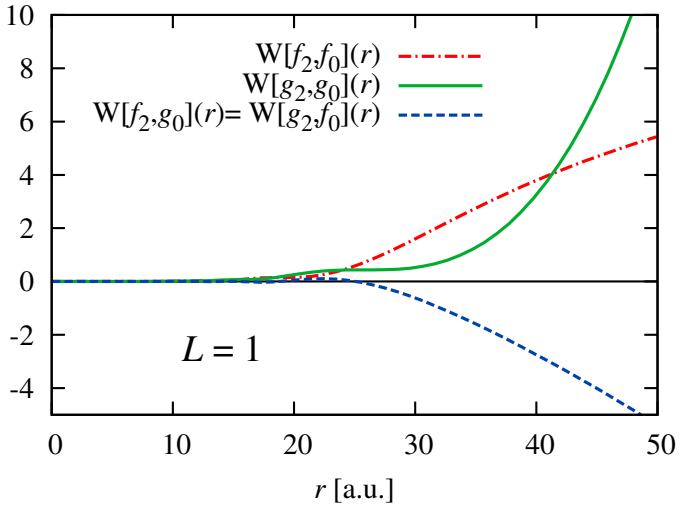

FIG. 2: (Color online)Plot of $W[f_{1,2}, f_{1,0}](r)$, $W[g_{1,2}, g_{1,0}](r)$, and $W[g_{1,2}, f_{1,0}](r)$ as a function of r for $L = 1$ and $\beta_6 = 50$ [a.u.].

We see that $b_L^{\rm free}(r)$ diverges to negative infinity as $r \to 0$ for $L \geq 1$. The causality bound on r_L also drives r_L to negative infinity for $L \geq 1$,

$$r_L \leq -\frac{2\Gamma(L-\frac{1}{2})\Gamma(L+\frac{1}{2})}{\pi}\left(\frac{r}{2}\right)^{-2L+1} + \mathcal{O}(r^2)\,. \tag{5.4}$$

For an effective field theory with local contact interactions, the range of the interactions are controlled by the momentum cutoff scale. No matter the values for a_L and r_L, it is not possible to take the momentum cutoff scale arbitrarily high without violating the causality bound for channels with angular momentum $L \geq 1$. For finite-range interactions with an

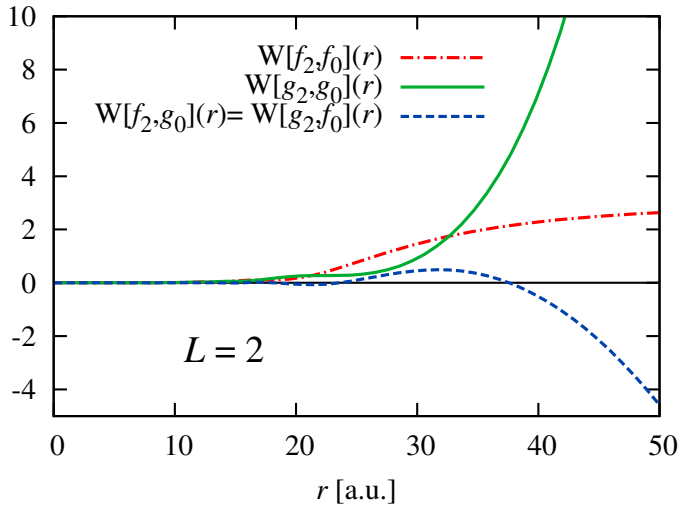

FIG. 3: (Color online)Plot of $W[f_{2,2}, f_{2,0}](r)$, $W[g_{2,2}, g_{2,0}](r)$, and $W[g_{2,2}, f_{2,0}](r)$ as a function of r for $L = 2$ and $\beta_6 = 50$ [a.u.].

additional attractive or repulsive Coulomb tail, one finds the same leading behavior [33][1]

$$b_L^{\text{Coulomb}}(r) = -\frac{2\Gamma(L-\frac{1}{2})\Gamma(L+\frac{1}{2})}{\pi}\left(\frac{r}{2}\right)^{-2L+1} + \mathcal{O}(r^{-2L})\,, \tag{5.5}$$

i.e., the only difference in the causality bound relation for the Coulomb-modified effective range is the subleading $\mathcal{O}(r^{-2L})$ pole term which is absent in the purely finite-range case. Hence, also for an effective field theory with contact interactions and long-range Coulomb tail, it is not possible to take the momentum cutoff scale arbitrarily high for $L \geq 1$ without violating the causality bound.

There is no such divergence in $b_L(r)$ at $r = 0$ for the attractive $1/r^6$ interaction. For an effective field theory with contact interactions and van der Waals tail, the causality bound does not impose convergence problems as long as $K_{L,2}$ is less than or equal to zero. This holds true for any L. There is no constraint from causality and unitarity preventing one from taking the cutoff momentum to be arbitrarily large. The key difference between the van der Waals interaction and the Coulomb interaction is that, when extended all the way to the origin, the attractive $1/r^6$ interaction is singular and the spectrum is unbounded below. An essential singularity appears at $r = 0$, and both van der Waals wavefunctions F_L and G_L vanish at the origin.

These exact same features appear in any attractive $1/r^\alpha$ interaction for $\alpha > 2$ in any number of spatial dimensions. The same can be said about an attractive $1/r^2$ interaction when the coupling constant is strong enough to form bound states. The key point is that in the zero-range limit of these attractive singular potentials, the spectrum of bound states

[1] To get this analogy, we are using here the normalization of the Coulomb-modified effective range expansion found in Eq. (28) of Ref. [33] and insert it in Eqs. (64), (A.6), and (A.7) of the same paper, which give the explicit expressions for the Coulomb-modified causality bound functions for $L = 0, 1, 2$. The statement for general L then follows by generalization.

extends to arbitrarily large negative energies. As a consequence, the scattering wavefunctions above threshold must vanish at the origin in order to satisfy orthogonality with respect to all such bound-state wavefunctions localized near the origin. In all of these cases the function $b_L(r)$ remains finite as $r \to 0$ for any L. We conclude that for an effective field theory with contact interactions and attractive singular power law interactions, we can take the cutoff momentum arbitrarily large for any L without producing a divergence in the coefficient $K_{L,2}$ of the short-range K-matrix.

VI. QUANTUM DEFECT THEORY AND THE MODIFIED EFFECTIVE RANGE EXPANSION

Up to now we have been discussing the short-range phase shift of K-matrix for scattering relative to the van der Waals wavefunctions F_L and G_L. For power law interactions $1/r^\alpha$ with $\alpha > 2$, we also have the option to define phase shifts relative to the Bessel functions of the free wave equation. The problem though is that the usual effective range expansion,

$$k^{2L+1} \cot \delta_{L,d}(p) = -\frac{1}{a_L} + \frac{1}{2} r_L k^2 + \sum_{n=0}^{\infty} (-1)^{n+1} \mathcal{P}_L^{(n)} k^{2n+4} \,, \tag{6.1}$$

is spoiled by non-analytic terms as a function of k^2. For the van der Waals interactions in the $L = 0$ channel, the leading non-analytic term is proportional to k^3, and so the scattering parameters a_0 and r_0 are well-defined, but the shape parameters $\mathcal{P}_L^{(n)}$ are not. For $L = 1$ the leading non-analytic term is proportional to k and so only the scattering length a_1 is well-defined. For $L \geq 2$ none of the low-energy scattering parameters are well defined. To resolve these problems, a modified form of the effective range expansion is used which is known as quantum defect theory [45–47].

In quantum defect theory for attractive $1/r^6$ potentials, one defines an offset for the phase shift [22],

$$\eta_L = \frac{\pi}{2}(\nu - \nu_0) \,. \tag{6.2}$$

The modified effective range expansion is then

$$k^{2L+1} \cot\left(\delta_L + 2\eta_L\right) = -\frac{1}{A_L} + \frac{1}{2} R_L \, k^2 + \mathcal{O}\left(k^4 \ln k\right) \,, \tag{6.3}$$

where A_L and R_L are the generalized scattering length and effective range parameters. These definitions coincide with the usual scattering length a_L for $L = 0, 1$ and the usual effective range r_L for $L = 0$. The generalized scattering length and effective range can be written in terms of the short range K-matrix parameters as

$$A_L = \frac{\pi^2 \beta_6^{2L+1}}{2^{4L+1}[\Gamma(\frac{L}{2} + \frac{1}{4})\Gamma(L + \frac{3}{2})]^2} \left[(-1)^L - \frac{1}{K_{L,0}}\right] \tag{6.4}$$

and

$$R_L = -\frac{2^{4L+2}\Gamma\left(\frac{L}{2} + \frac{1}{4}\right)^2 \Gamma\left(L + \frac{3}{2}\right)^2 \beta_6^{-2L-1}}{\pi^2 \left(K_{L,0}(-1)^L - 1\right)^2} \left[\frac{\beta_6^2 \left(K_{L,0}^2 + 1\right)}{4L^2 + 4L - 3} - K_{L,2}\right] \,. \tag{6.5}$$

From these results we see that the short-range parameter $K_{L,2}$ appears in combination with β_6^2. But in nearly all single-channel scatterings between pairs of alkali atoms, from the following equation,

$$K_{L,2} = \frac{\beta_6^2}{4L^2+4L-3}\left\{1+\left[(-1)^L - A_L\frac{2^{4L+1}\Gamma\left(\frac{L}{2}+\frac{1}{4}\right)^2\Gamma\left(L+\frac{3}{2}\right)^2}{\pi^2\beta_6^{2L+1}}\right]^{-2}\right\}$$
$$+R_L A_L^2\frac{2^{4L}\Gamma\left(\frac{L}{2}+\frac{1}{4}\right)^2\Gamma\left(L+\frac{3}{2}\right)^2}{\pi^2\beta_6^{2L+1}}\left[(-1)^L - A_L\frac{2^{4L+1}\Gamma\left(\frac{L}{2}+\frac{1}{4}\right)^2\Gamma\left(L+\frac{3}{2}\right)^2}{\pi^2\beta_6^{2L+1}}\right]^{-2}, \quad (6.6)$$

one quantitatively finds that $K_{L,2}$ is at least one order of magnitude smaller than β_6^2. This separation of scales is the reason for the approximate universality found in Ref. [26, 27].

The dominance of β_6^{2n} over the subleading coefficients $K_{L,2n}$ in Eq. (4.1) for $n \geq 1$ holds for nearly all cases of single-channel scattering between alkali atoms [21, 48]. This phenomenological fact explains the absence of short-distance length scales in the universality relation in Eq. (1.2). Furthermore, Gao has shown that when short-range interactions arise from a repulsive central potential, the fact that the K-matrix is nearly independent of energy means that the K-matrix is also nearly independent of angular momentum L [48]. This produces a surprisingly rich class of universal physics for single-channel van der Waals interactions where $K_{L,2n}$ is negligible compared to β_6^{2n} for all L, and $K_{L,0}$ is approximately the same for all L. Therefore β_6 and the S-wave scattering length will determine, to a good approximation, the threshold scattering behavior for all values of L.

VII. CAUSAL RANGE FOR SINGLE-CHANNEL SCATTERING

We have shown that for negative $K_{L,2} \leq 0$, the range R of the short-range interaction can be taken all the way down to zero. But when $K_{L,2}$ is positive, there is a constraint on R and we use Eq. (4.18) to determine a minimum value for R. We call this minimum range the causal range R^b, and we determine R^b as the solution to the equation

$$K_{L,2} = b_L(R^b)\,. \quad (7.1)$$

As pointed out in Ref. [33], one can show *a priori* that $b_L(r)$ is a monotonically increasing function of r. Therefore, if a real solution to Eq. (7.1) exists, then it is unique. If however there is no real solution, then there is no constraint on the interaction range and we define R^b to be zero. For an effective field theory with contact interactions and van der Waals tail, the cutoff momentum can be made as large as $\sim 1/R^b$ before the causality bound is violated.

In the following analysis we extract the single-channel S-wave effective range parameters a_0 and r_0 for several different pairs of alkali atoms ^{7}Li, ^{23}Na, and ^{133}Cs in singlet and triplet channels. The data is taken from Refs. [20, 21, 49–51]. The reduced masses for $^7Li_2$, $^{23}Na_2$ and $^{133}Cs_2$ are $\mu = 6394.7, 20954, 121100$ [a.u.], respectively. The van der Waals coupling constants for $^7Li_2$, $^{23}Na_2$ and $^{133}Cs_2$ are $C_6 = 1388, 1472, 7020$ [a.u.]. We calculate the corresponding K-matrix parameters using Eq. (6.4) and Eq. (6.6) and then compute the resulting causal ranges. We recall that for $L = 0$ we simply have $A_0 = a_0$ and $R_0 = r_0$. The results for the scattering parameters and causal ranges are given in columns II, V and VI of Table I. The discrepancies in R_0 are due to the fact that in the analytic studies in

TABLE I: Scattering parameters and causal ranges for S-wave scattering of ^{7}Li,^{23}Na, and ^{133}Cs pairs. The scattering data collection is taken from Ref. [20]. In columns I and IV the scattering data for ^{7}Li are from Ref. [50], the scattering data for ^{23}Na are from Ref. [49, 50], and data for ^{133}Cs are from Ref. [51]. In column III the effective range parameters, R_0, are calculated analytically in Refs. [20, 21]. In column IV, the R_0 are obtained from numerical calculations. The scattering parameters in columns II and V are calculated using Eqs. (6.4) and (6.6), and the causal ranges in column VI are obtained from Eq. (7.1).

			I	II	III	IV	V	VI
Atoms	State	β_6	A_0	$K_{0,0}$	R_0	R_0	$K_{0,2}$	R^b
^{7}Li–^{7}Li	$^1\Sigma_g$	64.9097	36.9	-5.282	66.3	66.5	$2 \sim 124$	$7 \sim 19$
^{7}Li–^{7}Li	$^3\Sigma_u$	64.9097	-17.2	0.643	1006.3	1014.8	$0 \sim 17$	$3 \sim 25$
^{23}Na–^{23}Na	$^1\Sigma_g$	88.624	34.936	5.705	187.317	187.5	$0 \sim 86$	$4 \sim 20$
^{23}Na–^{23}Na	$^3\Sigma_u$	88.624	77.286	-1.213	62.3756	62.5	$2 \sim 13$	$16 \sim 24$
^{133}Cs–^{133}Cs	$^1\Sigma_g$	203.62	68.216	3.365	624.013	624.55	$0 \sim 146$	$7 \sim 45$

Ref. [20, 21] $K_{0,2}$ is neglected, while the numerical calculations of Ref. [49–51] include the short-range contribution from $K_{0,2}$.

In column V of Table I, we present an approximate range for $K_{0,2}$ for each atomic pair using the values for R_0 in columns III and IV. Since $K_{0,2}$ is positive, we cannot go all the way to the zero range limit. However, in each case $K_{0,2}$ is at least one order of magnitude smaller than β_6^2.[2] Although we cannot take the zero range limit, the causal ranges are small in comparison to β_6. In each case R^b is a less than one third of the size of β_6. Hence one can probe these interactions in a van der Waals effective field theory with cutoff momentum up to roughly three times $1/\beta_6$ without violating the causality bound.

VIII. CAUSAL RANGE NEAR A MAGNETIC FESHBACH RESONANCE

In Ref. [52] the multi-channel problem of scattering around a magnetic Feshbach resonance is reduced to a description by an effective single-channel K-matrix that depends on the applied magnetic field B. The behavior around the resonance is described by several parameters. $B_{0,L}$ is the position of the resonance, while $g_{\rm res}$ parameterizes the width of the Feshbach resonance. $K_L^{\rm bg}$ is a background value for the K-matrix, and the scale $d_{B,L}$ is introduced to define a dimensionless magnetic field. We write the effective single-channel K-matrix as

$$\hat{K}_L^{\rm eff}(k,B) = -K_L^{\rm bg}\left[1 + \frac{g_{\rm res}}{k^2\beta_6^2 - g_{\rm res}\,(B_s+1)}\right], \tag{8.1}$$

with

$$B_s = \frac{(B - B_{0,L})}{d_{B,L}}\,. \tag{8.2}$$

The parametrization given above corresponds to Eq. (18) in Ref. [52]. Note that we have changed the notation slightly and are using a different sign convention.

[2] Note that $K_{L,2}$ has the dimension of an area (in the appropriate atomic units).

By expanding the right-hand side of Eq. (8.1) in k^2, it is straightforward to determine the K-matrix expansion parameters $K_{L,0}$ and $K_{L,2}$. A short calculation yields that

$$K_{L,0}^{\rm eff} = -K_L^{\rm bg}\left(1+\frac{1}{B_s+1}\right)\,, \tag{8.3}$$

$$K_{L,2}^{\rm eff} = \frac{\beta_6^2 K_L^{\rm bg}}{g_{\rm res}(B_s+1)^2}\,. \tag{8.4}$$

As noted in Ref. [52], the parameters $K_L^{\rm bg}$ and $g_{\rm res}$ are constrained by the condition

$$K_L^{\rm bg} g_{\rm res} < 0\,. \tag{8.5}$$

From this we directly see that $K_{L,2}^{\rm eff}$ given by Eq. (8.4) is always negative. From the causality bound in Eq. (4.18), it follows that where this effective single-channel description is applicable and correctly captures the entire energy-dependence of the short-range K-matrix, the causal range will be zero when the interaction is tuned close to a Feshbach resonance.

IX. SUMMARY AND DISCUSSION

In this paper we have analyzed two-body scattering with arbitrary short-range interactions plus an attractive $1/r^6$ tail. We derived the constraints of causality and unitarity for the short-range K-matrix,

$$\hat{K}_L = \tan\delta_L^{\rm (short)}(k) = \sum_{n=0}^{\infty} K_{L,2n}\, k^{2n}\,. \tag{9.1}$$

For any r larger than the range of the short-range interactions, R, we find that $K_{L,2}$ satisfies the upper bound

$$K_{L,2} \le b_L(r)\,, \tag{9.2}$$

where $b_L(r)$ is

$$\begin{aligned} b_L(r) = {}& \frac{\pi}{4} W[f_{L,2}, f_{L,0}](r) + \frac{\pi}{4} K_{L,0}^2 W[g_{L,2}, g_{L,0}](r) \\ & - \frac{\pi}{4} K_{L,0}\left\{ W[g_{L,2}, f_{L,0}](r) + W[f_{L,2}, g_{L,0}](r) \right\}\,, \end{aligned} \tag{9.3}$$

and the Wronksians are given in Eq. (4.15), (4.16), (4.17).

In clear contrast with the case for finite-range interactions only [38, 39] or with Coulomb tails [33], the function $b_L(r)$ does not diverge but rather vanishes as $r \to 0$ for all L. When $K_{L,2} \le 0$, there is no constraint derived from causality and unitarity that prevents the use of an effective field theory with zero range contact interactions plus an attractive $1/r^6$ tail. This holds true for any angular momentum value L. For the phenomenologically important case of a multi-channel system near a magnetic Feshbach resonance, the effective value for $K_{L,2}$ is negative and so the short-range interaction can be taken to have zero range.

The van der Waals interaction is qualitatively different from the Coulomb interaction where $b_L^{\rm Coulomb}(r)$ diverges for $L \ge 1$. The key difference is that both van der Waals wavefunctions F_L and G_L vanish at the origin. This phenomenon also occurs for an attractive

$1/r^\alpha$ interaction for $\alpha > 2$ in any number of spatial dimensions. It is also valid for an attractive $1/r^2$ interaction when the coupling constant is strong enough to form bound states. For an effective field theory with contact interactions and attractive singular power law tail, the cutoff momentum can be made arbitrarily large for any L without producing a divergence in the coefficient $K_{L,2}$ of the short-range K-matrix.

When $K_{L,2}$ is positive there is a lower bound on the range of the short-range interactions. We define the causal range R^b as this minimum value for the range, given by the condition

$$K_{L,2} = b_L(R^b)\,. \tag{9.4}$$

We have analyzed several examples of S-wave scattering in alkali atoms in Table I. We find that the $K_{L,2}$ is at least one order of magnitude smaller than β_6^2. As a result we find that the causal ranges are small in comparison with β_6.

In summary, we find that β_6 dominates over distance scales parameterizing the short-range interactions. The origin of this van der Waals universality can be explained by two facts. The first fact is the phenomenological observation that, in single-channel scattering between alkali atoms, there is significant separation between the typical length scales of the short-distance physics and β_6. This can be seen by the small size of the short-range parameter $K_{L,2}$ compared with β_6^2. As Gao has shown, this also leads to the approximate universal relation that $K_{L,0}$ is the same for all L [48]. Therefore, to a good approximation, β_6 and the S-wave scattering length will determine the threshold scattering behavior for all values of L. For the multi-channel case near a magnetic Feshbach resonance, we find that the effective $K_{L,2}^{\rm eff}$ is no longer negligible. However $K_{L,2}^{\rm eff}$ is negative, and this means that there is no constraint from causality preventing the zero-range limit for the short-distance interactions.

The second fact underlying the van der Waals universality is that the zero-range limit of short-distance interactions is well-behaved with regard to scattering near threshold. We note, however, that there is still no scale-invariant limit for $L \geq 1$ since the effective range parameter will diverge to negative infinity as β_6 goes to zero. This can be seen from the β_6^{-2L+1} behavior with negative coefficient for $L \geq 1$ in Eq. (6.5).

The analysis in this paper should be useful in developing an effective field theory with an attractive $1/r^6$ tail and contact interactions. Similarly one can also construct effective field theories for other attractive singular potentials $1/r^\alpha$ for $\alpha \geq 2$. These effective field theories could be used to investigate the Efimov effect and other low-energy phenomena in a model-independent way.

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Appendix A: Van der Waals wavefunctions

In this section we derive the van der Waals wavefunctions F_L and G_L, following the steps in Ref. [44]. We first redefine the radial function as $U_L(r) = \sqrt{r_s}\, Z(\rho_s)$. This rearrangement puts Eq. (3.9) into the form of an inhomogeneous Bessel equation,

$$\mathcal{L}_{\nu_0} Z(\rho_s) = \left[\rho_s^2 \frac{d^2}{d\rho_s^2} + \rho_s \frac{d}{d\rho_s} - \nu_0^2 + \rho_s^2\right] Z(\rho_s) = -\frac{k_s^2}{8} \frac{Z(\rho_s)}{\rho_s}\,, \tag{A1}$$

with

$$\nu_0 = \frac{1}{4}(2L+1).$$

The idea, introduced in Ref. [53], is now to consider $Z_\nu(\rho_s)$ as a series expansion of solutions,

$$Z(\rho_s) = \sum_{n=0}^{\infty} k_s^{2n}\, \varphi^{(n)}(\rho_s)\,, \tag{A2}$$

and to use perturbation theory to obtain a solution for $Z_\nu(\rho_s)$. Substituting Eq. (A2) into Eq. (A1) leads to an infinite number of differential equations,

$$\begin{aligned}\mathcal{L}_{\nu_0}\, \varphi^{(0)}(\rho_s) \; + k_s^2 \left[\mathcal{L}_{\nu_0}\, \varphi^{(1)}(\rho_s) \; + \frac{1}{8\rho_s}\, \varphi^{(0)}(\rho_s)\right] \\ + k_s^4 \left[\mathcal{L}_{\nu_0}\, \varphi^{(2)}(\rho_s) \; + \frac{1}{8\rho_s}\, \varphi^{(1)}(\rho_s)\right] + \cdots = 0\,.\end{aligned} \tag{A3}$$

The zeroth-order differential equation is homogenous while all other orders are inhomogeneous. This procedure generates a secular perturbation in all inhomogeneous differential equations as well as driving terms. The secular terms here refer to the solutions of the zeroth-order differential equation, which are Bessel functions.

Following Ref. [44], we introduce a function $Z_\nu(\rho_s)$ which has an expansion in terms of Bessel functions with momentum-dependent coefficients,

$$Z_\nu(\rho_s) = \sum_{m=-\infty}^{\infty} b_m(k_s)\, \mathcal{J}_{\nu+m}(\rho_s)\,. \tag{A4}$$

We insert this as an ansatz into Eq. (A1) with ν yet to be determined. Here $\mathcal{J}_n$ denotes collectively the Bessel functions of first and second kind, J_n and N_n. Substitution of Eq. (A4) into Eq. (A1) yields a three-term recurrence relation for the b_m functions with $-\infty < m < \infty$,

$$[(\nu+m)^2 - \nu_0^2] b_m(k_s) + \frac{k_s^2}{16(m+\nu-1)} b_{m-1}(k_s) + \frac{k_s^2}{16(m+\nu+1)} b_{m+1}(k_s) = 0\,. \tag{A5}$$

Solving these equations for $b_m(k_s)$ yields

$$b_m(k_s) = (-1)^m \left(\frac{k_s}{4}\right)^{2m} \frac{\Gamma(\nu)\Gamma(\nu-\nu_0+1)\Gamma(\nu+\nu_0+1)}{\Gamma(\nu+m)\Gamma(\nu-\nu_0+m+1)\Gamma(\nu+\nu_0+m+1)}\, c_m(\nu) \tag{A6}$$

and

$$b_{-m}(k_s) = (-1)^m \left(\frac{k_s}{4}\right)^{2m} \frac{\Gamma(\nu-m+1)\Gamma(\nu-\nu_0-m)\Gamma(\nu+\nu_0-m)}{\Gamma(\nu+1)\Gamma(\nu-\nu_0)\Gamma(\nu+\nu_0)}\, c_m(-\nu) \tag{A7a}$$

for $m \geq 0$. The functions $c_m(\pm\nu)$ are defined as

$$c_m(\pm\nu) = \prod_{s=0}^{m-1} Q(\pm\nu + s)\, b_0(k_s)\,, \tag{A8}$$

where $Q(\nu)$ is given by

$$Q(\nu) = \frac{1}{1 - \dfrac{k_s^2}{16(\nu+1)[(\nu+1)^2 - \nu_o^2](\nu+2)[(\nu+2)^2 - \nu_o^2]} Q(\nu+1)}\,. \tag{A9}$$

The coefficient $b_0(k_s)$ only determines the overall normalization and is simply set to one in the following. Eq. (A5) for $m = 0$ determines the shift ν in the order of the Bessel functions. We determine ν using the constraint

$$(\nu^2 - \nu_0^2) - \frac{Q(-\nu)}{16^2\nu(\nu-1)[(\nu-1)^2 - \nu_0^2]}\, k_s^4 - \frac{Q(\nu)}{16^2\nu(\nu+1)[(\nu+1)^2 - \nu_0^2]}\, k_s^4 = 0\,. \tag{A10}$$

In general there are several roots which become complex beyond a critical scaled momentum k_s, and one must be careful to choose the physical solution. For a detailed discussion of this point, see Refs. [22, 44].

Choosing either $\mathcal{J}_n = J_n$ or $\mathcal{J}_n = N_n$ already yields a pair of linearly independent solutions. However, in order to get a pair with energy-independent normalization as $r_s \to 0$ (which ensures analyticity in the energy), we furthermore define

$$X_L(k_s) = \cos\eta_L \sum_{m=-\infty}^{\infty} (-1)^m b_{2m}(k_s) - \sin\eta_L \sum_{m=-\infty}^{\infty} (-1)^m b_{2m+1}(k_s) \tag{A11}$$

and

$$Y_L(k_s) = \sin\eta_L \sum_{m=-\infty}^{\infty} (-1)^m b_{2m}(k_s) + \cos\eta_L \sum_{m=-\infty}^{\infty} (-1)^m b_{2m+1}(k_s)\,, \tag{A12}$$

with

$$\eta_L = \frac{\pi}{2}(\nu - \nu_0)\,.$$

Combining everything, we arrive at the van der Waals wavefunctions,

$$F_L(k,r) = \frac{r_s^{1/2}}{X_L^2(k_s) + Y_L^2(k_s)} \left[X_L(k_s) \sum_{m=-\infty}^{\infty} b_m(k_s)\, J_{\nu+m}(\rho_s) - Y_L(k_s) \sum_{m=-\infty}^{\infty} b_m(k_s)\, N_{\nu+m}(\rho_s) \right], \tag{A13}$$

$$G_L(k,r) = \frac{r_s^{1/2}}{X_L^2(k_s) + Y_L^2(k_s)} \left[X_L(k_s) \sum_{m=-\infty}^{\infty} b_m(k_s)\, N_{\nu+m}(\rho_s) + Y_L(k) \sum_{m=-\infty}^{\infty} b_m(k_s)\, J_{\nu+m}(\rho_s) \right]. \tag{A14}$$

Appendix B: Low-energy expansions

In this Appendix we expand all functions relating to the van der Waals wavefunctions in powers of momentum. We first consider ν, the shift in the order of the Bessel functions in Eq. (3.10) and Eq. (3.11). Using Eq. ($A10$) in Appendix A, we find

$$\nu = \nu_0 - \frac{3}{2^8 \nu_0 (4\nu_0^2 - 1)(\nu_0^2 - 1)} k_s^4 + \mathcal{O}(k_s^8)\,, \tag{B1}$$

where $\nu_0 = (2L+1)/4$. Using the expansion in Eqs. (A6), (A7a), and (A9), we get

$$b_m(k_s) = (-1)^m \frac{\Gamma(\nu_0)\Gamma(2\nu_0+1)}{m!\,\Gamma(\nu_0+m)\Gamma(2\nu_0+m+1)} \left(\frac{k_s}{4}\right)^{2m} + \mathcal{O}(k_s^{2m+2}) \tag{B2}$$

and

$$b_{-m}(k_s) = \frac{\Gamma(\nu_0-m+1)\Gamma(2\nu_0-m)}{m!\,\Gamma(\nu_0+1)\Gamma(2\nu_0)} \left(\frac{k_s}{4}\right)^{2m} + \mathcal{O}(k_s^{2m+2}) \tag{B3}$$

for $m \geq 0$. Substituting these expressions into Eqs. (A11) and (A12) we obtain

$$X_L(k_s) = 1 + \mathcal{O}(k_s^4)\,, \tag{B4}$$

$$Y_L(k_s) = -\left[\frac{\Gamma(\nu_0)\Gamma(2\nu_0-1)}{\Gamma(\nu_0+1)\Gamma(2\nu_0)} + \frac{\Gamma(\nu_0)\Gamma(2\nu_0+1)}{\Gamma(\nu_0+1)\Gamma(2\nu_0+2)}\right] \left(\frac{k_s}{4}\right)^2 + \mathcal{O}(k_s^4)\,. \tag{B5}$$

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