PT-symmetric lattices with a local degree of freedom

Harsha Vemuri and Yogesh N. Joglekar

PT symmetric lattices with a local degree of freedom

Harsha Vemuri and Yogesh N. Joglekar

Department of Physics, Indiana University Purdue University Indianapolis (IUPUI), Indianapolis, Indiana 46202, USA

Recently, open systems with balanced, spatially separated loss and gain have been realized and studied using non-Hermitian Hamiltonians that are invariant under the combined parity and time-reversal ($\mathcal{PT}$) operations. Here, we model and investigate the effects of a local, two-state, quantum degree of freedom, called a pseudospin, on a one-dimensional tight-binding lattice with position-dependent tunneling amplitudes and a single pair of non-Hermitian, $\mathcal{PT}$-symmetric impurities. We show that if the resulting Hamiltonian is invariant under exchange of two pseudospin labels, the system can be decomposed into two uncoupled systems with tunable threshold for $\mathcal{PT}$ symmetry breaking. We discuss implications of our results to systems with specific tunneling profiles, and open or periodic boundary conditions.

Introduction: A non-Hermitian Hamiltonian $H \neq H^\dagger$ that is invariant under the combined parity and time-reversal ($\mathcal{PT}$) operations is called $\mathcal{PT}$ symmetric. Since the groundbreaking discovery of such Hamiltonians in continuum models fifteen years ago [1], significant research has been carried out to identify and characterize the properties of $\mathcal{PT}$-symmetric Hamiltonians that, typically, are decomposed into a Hermitian kinetic term and a non-Hermitian, $\mathcal{PT}$-symmetric potential term, $V(x) = V^*(-x) \neq V^\dagger(x)$ [2, 3]. The spectrum ϵ_λ of such a non-Hermitian Hamiltonian is purely real over a region of the parameter space, called the $\mathcal{PT}$-symmetric phase; in this region, its (non-orthonormal) eigenvectors $|\phi_\lambda\rangle$ are simultaneous eigenfunctions of the $\mathcal{PT}$ operation. For parameters outside the $\mathcal{PT}$-symmetric phase, the eigenvalues of the Hamiltonian occur in complex conjugate pairs, and due to the anti-linear nature of the time-reversal operator $\mathcal{T}$, the corresponding eigenfunctions are not simultaneous eigenfunctions of the $\mathcal{PT}$ operation. This emergence of complex eigenvalues is called $\mathcal{PT}$-symmetry breaking. $\mathcal{PT}$-symmetric Hamiltonians are ideally suited to model non-equilibrium phenomena that transition from a quasi steady-state behavior ($\mathcal{PT}$-symmetric phase) to loss of reciprocity (broken $\mathcal{PT}$-symmetry) [4, 5].

In the past three years, experiments on coupled optical waveguides [6–9], coupled electrical circuits [10], and coupled pendulums [11] have shown that instead of being a mathematical curiosity, $\mathcal{PT}$-symmetric Hamiltonians represent open (quantum) systems with spatially separated, balanced, loss and gain. The discrete nature of these systems has also sparked new interest in the properties of $\mathcal{PT}$-symmetric tight-binding lattice models with different topologies [12, 13]; such lattice models are most readily realized in evanescently coupled optical waveguides [14, 15]. Recent theoretical work has led to the identification of robust and fragile $\mathcal{PT}$-symmetric phases in a lattice with open boundary conditions [16–19], tunable $\mathcal{PT}$-symmetric threshold in a lattice with periodic boundary conditions [20], and substantially strengthened $\mathcal{PT}$-symmetric phase in finite lattices with position-dependent tunneling profile [21, 22]. All of this work is, however, restricted to systems in one spatial dimension where the parity operation is defined as $\mathcal{P} : x \rightarrow -x$ in the continuum case (with a suitably defined origin) and $\mathcal{P} : k \rightarrow \bar{k} = N+1-k$ in a lattice with N sites. In particular, the properties of $\mathcal{PT}$-symmetric Hamiltonians in two (or higher) dimensions have been barely explored [23].

In this paper, we investigate $\mathcal{PT}$-symmetric lattices with a local, two-state, quantum degree of freedom labeled by a pseudospin $\sigma = \pm 1$. We present a class of models that can be mapped onto one-dimensional lattice models that have been investigated in the past, and thus are solvable in a straightforward manner. Such degree of freedom can represent, for example, two orthogonal polarizations of a mode in a single elliptical waveguide [24] in an array of coupled elliptical waveguides. Thus, although we use the term "pseudospin" to denote this degree of freedom, we emphasize that its time-reversal properties are unspecified. By using physically motivated $\mathcal{PT}$-symmetric potentials (at only two sites) and tunneling amplitude profiles, we show that the local degree of freedom leads to a robust, tunable $\mathcal{PT}$-symmetric phase. Although our results are applicable to general $\mathcal{PT}$-symmetric systems, in the following, we use a language that is applicable to coupled optical waveguides [15].

Tight-binding Model: We consider a lattice of N coupled waveguides with open boundary conditions, described by the following tight-binding Hamiltonian,

$$H_0 = - \sum_{i=1}^{N-1} \left[a_i^\dagger T(i) a_{i+1} + a_{i+1}^\dagger T^\dagger(i) a_i \right]. \quad (1)$$

Here $a_k^\dagger = (a_{k,+}^\dagger, a_{k,-}^\dagger)$ are the creation operators for the two modes $|k, +\rangle$ and $|k, -\rangle$ at site k respectively. $[T(k)] = t_s(k)\mathbb{1} + t_d(k)\tau_x$ is the 2×2 tunneling matrix that couples the two modes at site k to the two modes at site $k+1$, τ_x is the x-Pauli matrix, and $t_s \geq 0$ ($t_d \geq 0$) denote the tunneling amplitude for processes that preserve (flip) the local degree of freedom (Fig. 1). We choose a parity-symmetric, real tunneling function $t(k) = t(N-k)$ to ensure that H_0 commutes with the combined $\mathcal{PT}$ operator [21]. Note that we have chosen tunneling matrix so that Eq.(1) is invariant under the exchange of pseudospin labels $\sigma \leftrightarrow -\sigma$. For a pair of balanced loss or gain impurities at mirror-symmetric locations, the potential is
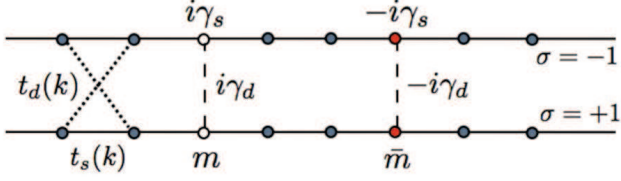
FIG. 1. (color online) Tight binding lattice with a local degree of freedom represented by pseudospin $\sigma = \pm 1$. The tunneling amplitudes $t_s(k)$ and $t_d(k)$ denote processes that preserve or change the pseudospin respectively, while the tunneling from site k to $k + 1$. Similarly, $\pm i\gamma_s$ and $\pm i\gamma_d$ are on-site, non-Hermitian potentials that represent gain (white site) or loss (red site). This system is $\mathcal{PT}$-symmetric irrespective of the time-reversal properties of the pseudospin.

given by

$$V = a_m^\dagger i\Gamma a_m - a_{\bar{m}}^\dagger i\Gamma a_{\bar{m}} \qquad (2)$$

where $\bar{m} = N + 1 - m$, $i[\Gamma] = i\gamma_s\mathbb{1} + i\gamma_d\tau_x$ denotes the non-Hermitian gain matrix at site m, and $0 \leq \gamma_d \leq \gamma_s$ denote the gain amplitudes for mode preserving and mode exchanging processes. The potential V is also invariant under the exchange of pseudospin labels, and is $\mathcal{PT}$ symmetric irrespective of the time-reversal properties of the pseudospin.

The eigenvalue difference equation obeyed by a two-component eigenfunction $\Psi(k) = (f_k, g_k)^T$ with energy ϵ is given by

$$-T(k-1)\Psi(k-1) - T(k)\Psi(k+1) + (\delta_{k,m} - \delta_{k,\bar{m}})i\Gamma\Psi(k) = \epsilon\Psi(k). \quad (3)$$

where $k = 1, \cdots, N$. We note that open boundary conditions are implemented by assigning $T(0) = 0 = T(N)$ whereas periodic boundary conditions imply $T(0) = T(N) \neq 0$. Using the symmetric and antisymmetric basis that *diagonalizes the tunneling matrix $T(k)$ at every site*, it is straightforward to obtain the following decoupled equations,

$$-(t_k^S f_{k+1}^S + t_{k-1}^S f_{k-1}^S) + i\gamma^S f_k^S(\delta_{k,m} - \delta_{k,\bar{m}}) = \epsilon f_k^S, (4)$$

$$-(t_k^A f_{k+1}^A + t_{k-1}^A f_{k-1}^A) + i\gamma^A f_k^A(\delta_{k,m} - \delta_{k,\bar{m}}) = \epsilon f_k^A.(5)$$

Here $t_k^{S(A)} = [t_s(k) \pm t_d(k)]$ are the symmetric and antisymmetric combinations of the tunneling rates, $\gamma^{S(A)} = (\gamma_s \pm \gamma_d)$, and $f_k^{S(A)} = (f_k \pm g_k)$ are the eigenfunction components in the symmetric-antisymmetric basis.

Eqs.(4)-(5) show that the $\mathcal{PT}$-symmetric Hamiltonian is *a direct sum of Hamiltonians for two lattices with no internal structure*: $H = H_0 + V = H_S \oplus H_A$ where H_S is the $\mathcal{PT}$-symmetric Hamiltonian with tunneling profile t_k^S and a pair of non-Hermitian impurities at mirror-symmetric locations $(m, \bar{m})$ with strength γ^S, and H_A is obtained correspondingly. We emphasize that this decomposition into uncoupled problems is valid for arbitrary, position-dependent tunneling profiles $t_s(k)$, mode-
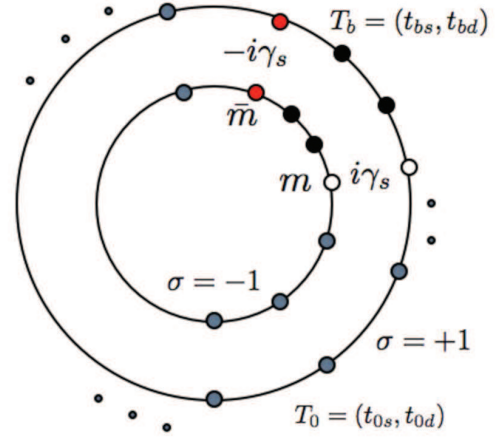

FIG. 2. (color online) Tight binding ring with a local degree of freedom represented by pseudospin $\sigma = \pm 1$ with a pair of $\mathcal{PT}$-symmetric impurities $\pm i\gamma$ located at arbitrary sites $(m, \bar{m})$. The tunneling matrix T is constant, but different, along the two paths that go from the gain site m (white site) to the loss site $\bar{m}$ (red site). This system will map onto two, uncoupled rings with constant tunnelings $t_0^{S(A)} = (t_{0s} \pm t_{0d})$ and $t_b^{S(A)} = (t_{bs} \pm t_{bd})$ along the two, outer and in-between, paths from site m to site $\bar{m}$.

mixing amplitudes $t_d(k)$, open or periodic boundary conditions, and arbitrary loss or gain strengths, as long as the underlying Hamiltonian is invariant under the exchange pseudospin indices.

Specific Cases and Numerical Results: When there is no mixing between the two pseudospin states, $t_d = 0 = \gamma_d$, the problem is trivial. In general, the $\mathcal{PT}$-symmetric threshold for H_0 is equal to the smaller of the corresponding thresholds for H_S and H_A.

When $\gamma_d = 0$, the loss (or gain) potential couples maximally to the pseudospin eigenmodes $\sigma = \pm 1$, and *not to a linear combination of them*. In this case, if the tunneling is constant, the $\mathcal{PT}$-symmetric phase diagram $\gamma_{PT}(m)$ is given by a U-shaped curve, obtained in Ref. [18], with the maximum value $\gamma_{PT} = (t_s - t_d)$. For parity-symmetric, non-constant tunneling profiles, the appreciably strong $\mathcal{PT}$- symmetric threshold, obtained in Ref. [22], is now selectively suppressed by increasing the mode-mixing tunneling amplitude $t_d(k)$. For a lattice with periodic boundary conditions, we consider the model with tunneling matrices $T_0 = t_{0s}\mathbb{1} + t_{0d}\tau_x$ and $T_b = t_{bs}\mathbb{1} + t_{bd}\tau_x$ that are constant along each of the two paths that connect the gain site to the loss site (Fig. 2). It then follows that the $\mathcal{PT}$-symmetric threshold is independent of the distance between the loss and gain sites, as discussed in Ref. [20], and is given by the smaller of the two combinations, $(t_0^S - t_b^S)$ and $(t_0^A - t_b^A)$. Thus, a $\mathcal{PT}$-symmetric ring with a local degree of freedom offers significant threshold tunability independent of the distance between the loss and gain impurities.

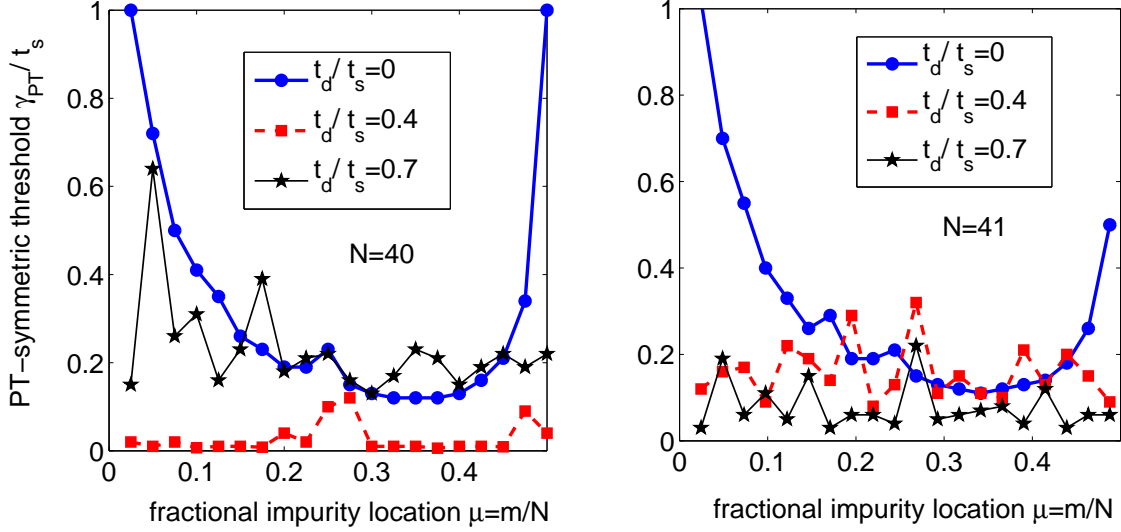When $\gamma_d \neq 0$, the analysis carried out here predicts

FIG. 3. (color online) Typical $\mathcal{PT}$-symmetric phase diagram for even ($N = 40$, left panel) and odd ($N = 41$ right panel) lattices with a local degree of freedom σ. The vertical axis shows the $\mathcal{PT}$-threshold $\gamma_{PT}(\mu)$, measured in units of t_s and the horizontal axis shows the fractional position $\mu = m/N$ of the impurity. The tunneling is given by $t_s \geq t_d \geq 0$, and impurity at site m acts as a gain for $\sigma = +1$ and a loss for $\sigma = -1$. When the mode-mixing $t_d = 0$ (blue circles), decoupled pseudospins lead to earlier results [18]. As t_d increases (red squares and black stars), the $\mathcal{PT}$-symmetric threshold $\gamma_{PT}(\mu)$ generally decreases.

bounds on the gain matrix, given by $(\gamma_s + \gamma_d) \leq (t_s + t_d)$ and $(\gamma_s - \gamma_d) \leq (t_s - t_d)$; however, these bounds do not determine the individual thresholds for γ_s and γ_d. In the extreme case of $\gamma_s = \gamma_d$ (meaning the gain potential only couples to the symmetric combination), we find that $\gamma^A = 0$, H_A is a purely *Hermitian* Hamiltonian and therefore, the $\mathcal{PT}$-symmetric threshold is solely determined by the Hamiltonian H_S. Note that, in general, a direct-sum decomposition of the Hamiltonian H is possible *if and only if* the tunneling matrix $T(k)$ *at every site k* and the non-Hermitian potential matrix $i\Gamma$ can be simultaneously diagonalized.

Finally we consider the case where the full Hamiltonian H cannot be decomposed into two non-interacting pieces. Generically, for an open lattice or a ring with constant tunneling matrix T and a single pair of gain/loss matrix $i\Gamma$, wave function matching approach [17, 18] leads to a characteristic equation for eigenvalues of H that results from the determinant of a 6×6 matrix. It is, thus, of little analytical value to calculate the $\mathcal{PT}$-symmetric threshold $\gamma_{PT}(m)$ and instead, we obtain the $\mathcal{PT}$-symmetric phase diagram numerically. We restrict ourselves to the simplest case of a constant-tunneling Hamiltonian H_0 and an impurity potential matrix $i\Gamma = i\gamma_s \tau_z$ where τ_z is the z-Pauli matrix. In contrast to the previous cases, where the losses or gains for both modes occurred in the same waveguide, this non-Hermitian potential represents gain for one mode, $\sigma = +1$, and loss for the other, $\sigma = -1$, at site m.

Figure 3 shows the numerically obtained results for the threshold $\gamma_{PT}(\mu)/t_s$ as a function of the fractional

location $\mu = m/N$ of the first impurity for three different values of mode-mixing tunneling $t_d/t_s = \{0, 0.4, 0.7\}$. The left-hand panel shows the results for an even lattice with $N = 40$. When $t_d = 0$ (solid blue circles), the two degrees of freedom are uncoupled and $\mathcal{PT}$-symmetric phase diagram is identical to that for an open lattice with no internal degree of freedom [18]. As t_d/t_s increases (solid red squares and black stars), generically, we find that the critical $\gamma_{PT}(\mu)$ is non-monotonically suppressed for different values of impurity locations μ. The right-hand panel shows corresponding results for an odd lattice with $N = 41$. When $t_d = 0$ (solid blue circles), the threshold impurity strength is given by $\gamma_{PT}/t_s = \sqrt{1 + 1/N} \approx 1.012$ when $m = 1$ [17], and therefore, does not appear in the figure. Once again, when t_d increases, the $\mathcal{PT}$-symmetric phase is (mostly) suppressed in a non-monotonic way. These results suggest that $\mathcal{PT}$-symmetry breaking in such systems shows a rich behavior that cannot be described with a simple analytical model.

Discussion: In this paper, we have introduced $\mathcal{PT}$-symmetric lattices with a local, two-state, quantum degree of freedom. By imposing invariance requirements on the Hermitian tunneling term, and $\mathcal{PT}$-symmetric potential term that represents spatially separated gain and loss impurities, we have shown that a broad class of such lattice systems can be expressed as the direct sum of two, uncoupled, $\mathcal{PT}$-symmetric systems. In such cases, we have predicted that $\mathcal{PT}$-symmetric threshold can be tuned by mode-mixing tunneling amplitude. Since the mapping is exact, all signatures of $\mathcal{PT}$-symmetry

breaking, such as the ubiquitous, maximal chirality at $\mathcal{PT}$-symmetry breaking threshold [20], the even-odd effect [22], tunable amplification [25], etc. will be applicable in these cases as well.

Since we have used the mode polarization as an example of the local degree of freedom, a microscopic calculation of the mode structure and the overlap between modes in adjacent waveguides is necessary to obtain typical tunneling matrix elements. Similarly a detailed study of the selection rules for different polarizations will be necessary to characterize the relative strengths of elements of the gain matrix γ_s and γ_d.

[1] C.M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[2] C.M. Bender, D.C. Brody, and H.F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).
[3] For a review, see C.M. Bender, Rep. Prog. Phys. **70**, 947 (2007) and references therein.
[4] T. Kottos, Nat. Phys. **6**, 192 (2010).
[5] M.C. Zheng, D.N. Christodoulides, R. Fleischmann, and T. Kottos, Phys. Rev. A **82**, 010103(R) (2010).
[6] A. Guo, G.J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G.A. Siviloglou, and D.N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).
[7] C.E. Rüter, K.G. Makris, R. El-Ganainy, D.N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. **6**, 192 (2010).
[8] L. Feng, M. Ayache, J. Huang, Y.-L. Xu, M.-H. Lu, Y.-F. Chen, Y. Fainman, and A. Scherer, Science **333**, 729 (2011).
[9] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D.N. Christodoulides, and U. Peschel, Nature **488**, 167 (2012).
[10] J. Schindler, A. Li, M.C. Zheng, F.M. Ellis, and T. Kottos, Phys. Rev. A **84**, 040101(R) (2011).
[11] C.M. Bender, B.K. Berntson, D. Parker, and E. Samuel, arXiv:1206.4972.
[12] M. Znojil, Phys. Rev. A **82**, 052113 (2010).
[13] M. Znojil, Phys. Lett. A **375**, 3435 (2011).
[14] A. Yariv, IEEE J. Quantum Electron. **9**, 919 (1973).
[15] D.N. Christodoulides, F. Lederer, and Y. Silberberg, Nature (London) **424**, 817 (2003).
[16] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. **103**, 030402 (2009).
[17] L. Jin and Z. Song, Phys. Rev. A **80**, 052107 (2009).
[18] Y.N. Joglekar, D. Scott, M. Babbey, and A. Saxena, Phys. Rev. A **82**, 030103(R) (2010).
[19] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, J. Phys. A **43**, 265305 (2010).
[20] D.D. Scott and Y.N. Joglekar, Phys. Rev. A **85**, 062105 (2012).
[21] Y.N. Joglekar and A. Saxena, Phys. Rev. A **83**, 050101(R) (2011).
[22] D.D. Scott and Y.N. Joglekar, Phys. Rev. A **83**, 050102(R) (2011).
[23] C.M. Bender and D.J. Weir, J. Phys. A: Math Theor. **45**, 425303 (2012); B.P. Mandal, B.K. Mourya, and R.K. Yadav, arXiv:1301.2387.
[24] See, for example, C. Yeh, J. App. Phys. **33**, 3235 (1962).
[25] H. Vemuri, V. Vavilala, T. Bhamidipati, and Y.N. Joglekar, Phys. Rev. A **84**, 043826 (2011).