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Enhancement of parameter estimation precision in noise systems by dynamical decoupling pulses

Qing-Shou Tan,[1] Yixiao Huang,[1] Xiaolei Yin,[1] Le-Man Kuang,[2] and Xiaoguang Wang[1]

[1]*Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China*
[2]*Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, and Department of Physics, Hunan Normal University, Changsha 410081, China*

We present a scheme to enhance precision of parameter estimation (PPE) in noise systems via employing dynamical decoupling pulses. The exact analytical expression for the estimation precision of unknown parameter is obtained by using the transfer matrix and time-dependent Kraus operators. We show that PPE in noise systems can be preserved in the Heisenberg limit by controlling of dynamical decoupling pulses. It is found that larger number of pulses and longer reservoir correlation time can remarkably protect the PPE.

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I. INTRODUCTION

The ultra-precise estimation of parameters plays an important role in quantum metrology such as quantum frequency standards, measurement of gravity accelerations, clock synchronization [1–5], and so on. In the quantum metrology field, quantum Fisher information (QFI) [6–13] is a key concept, which gives a theoretical-achievable limit on the precision when estimating an unknown parameter φ. According to the quantum Cramér-Rao theorem [9, 14], the mean square fluctuation of φ becomes

$$\Delta\varphi \geq \Delta\varphi_{QCB} \equiv \frac{1}{\sqrt{\nu F(\varphi)}}, \tag{1}$$

where ν represents the number of the independent measurements and $F(\varphi)$ is QFI with respect to the unknown parameter φ

$$F[\rho(\varphi)] = \mathrm{Tr}[\rho(\varphi)L_\varphi^2], \tag{2}$$

where L_φ is the so-called symmetric logarithmic derivative determined by equation $\frac{\partial\rho(\varphi)}{\partial\varphi} = \frac{1}{2}[\rho(\varphi)L_\varphi + L_\varphi\rho(\varphi)]$. Equation (1) implies that large Fisher information means a high precision of estimation. Thus increasing the Fisher information has important theoretical and practical value to enhance the precision of parameter estimation (PPE)[15].

Recently, many works have demonstrated that entangled states can improve the precision of parameter estimation [15–26]. When using a maximally entangled state the precision can be improved to the Heisenberg limit (HL) (proportional to $1/N$), where N is the particle number. Thus limit is better resolution than the best estimation limit with separable states, called the standard quantum limit (SQL) (proportional to $1/\sqrt{N}$)[22, 23, 28].

Until recently, most of the work in quantum metrology involved isolated systems undergoing unitary evolution [1]. However, under realistic physical conditions, the unavoidable interaction with the environment leads to decoherence. In a recent paper [1], Escher and coworkers proposed a general framework for quantum metrology of noisy systems, and obtained useful analytic bounds for optical interferometry and atomic spectroscopy. It is found that in the presence of decoherence, even with the use of entanglement the Heisenberg level estimation cannot be achieved, since the entangled states are very sensitive to the action of environment [25, 27, 29, 30]. In Ref [4] the authors have studied the QFI of Greenberger-Horne-Zeilinger (GHZ) state under three typical types noise sources [i.e., Amplitude-damping channel (ADC), Phase-damping channel and Depolarizing channel] by means of Kraus operators, respectively. They found when the decoherence strength is sufficiently large, in all these channels precision higher than SQL level cannot be achieved. Therefore, it is important to find strategies to suppress decoherence in order to obtain the ultra-precise measurement.

In quantum information theory, there are two main classes of ideas to overcome decoherence: passive techniques, in which quantum information is encoded within decoherence free subspace [31, 32] which does not decoherent because of reasons of symmetry; and active approaches, such as quantum error correction [33] and dynamical decoupling (DD) techniques [34–45]. In Ref [3], the author used the decoherence free subspace to suppress the collective dephasing of N ions which are stored in a linear Paul trap and proved that quantum enhancement can readily be achieved in the presence of noise. DD strategies as another protocol to protect quantum information, aim at averaging the unwanted interaction with the environment to zero by means of dynamical control field. DD methods have been studied in connection to a wide range of applications and become particularly popular in area of quantum information. In Ref [10], the authors investigated how to extract maximum information from the noise quantum system, in their work the DD scheme was considered for recovering the lost information of a single qubit in a heat bath of bosons.

In this paper we propose a scheme to enhance PPE in N-qubit noise systems employing DD pulses. By the use of the transfer matrix and exact time-dependent Kraus operators, we attain the analytic expressions of the QFI and error precision in the presence of DD pulses. We show that the use of the DD pulse sequences is very effective to protect PPE and it is found that the HL can be achieved in the presence of noise as long as the number of pulses is large enough.

The paper is organized as follow. In Sec. II, we derive time-dependent Kraus operators of ADC case in the presence of DD pulses, and present exact solution of N-qubit reduced density matrix. In Sec. III, we study the effects of the DD pulses on protecting PPE, and indicate that the precision on HL can be achieved even in noise systems. Finally, a summary is provided in the last section.

II. DYNAMICS OF N-QUBIT IN NOISE SYSTEM UNDER DD PULSE SEQUENCES

In this section, we investigate the dynamics of N-qubit in noise system under DD-pulse sequences. We consider the independent reservoir case in which each qubit interacts with a reservoir. Suppose there is no interaction between the N pairs of 'qubit+reservoir' system, the dynamics of the whole system can be obtained simply from the evolution of the individual pairs.

A. Controlled Hamiltonian

Hamiltonian $H(t)$ of one single qubit interacting with its own reservoir with controlled pulses is given by

$$H = H_S(t) + H_B + H_I, \tag{3}$$

with

$$H_B = \sum_j \omega_j a_j^\dagger a_j, \qquad H_I = \sum_j g_j(\sigma_- a_j^\dagger + \sigma_+ a_j). \tag{4}$$

the Hamiltonian of reservoir and qubit-reservoir interaction. And the Hamiltonian of the qubit is

$$\begin{aligned} H_S(t) &= H_S + H_c(t) \\ &= \frac{\omega_0}{2}\sigma_z + \frac{\pi}{2}\sum_{n=1}^{\infty}\delta(t-nT)\sigma_z, \end{aligned} \tag{5}$$

which consists of two parts. The first term is the free Hamiltonian, the second term is the control part, which comprises a train of instantaneous π pulses (the width of each pulse is sufficiently short), where T is the time interval between two consecutive pulses. The effect of each pulse on the qubit is simply a rotation around the z-axis with π, which is described by the operator $U_c = -i\sigma_z$.

Choosing $U(t) = \hat{\mathrm{T}}\exp\left[-i\int_0^t dt' H_c(t')\right]$, then the effective Hamiltonian $H_{\rm eff}$ of the total system in the present of control pulses is given by

$$\begin{aligned} H_{\rm eff} &= U^\dagger(H_B + H_S + H_I)U(t) \\ &= \omega_0|e\rangle\langle e| + \sum_j \omega_j a_j^\dagger a_j + \sum_j g_j(-1)^n(\sigma_- a_j^\dagger + \sigma_+ a_j), \end{aligned} \tag{6}$$

where $n = [t/T]$ is the number of pulses denoted by the integer part of t/T. To get the above equation, we have used the relation $\sigma_z\sigma_\pm\sigma_z = -\sigma_\pm$ and omitted a constant for convenience. From the above equation we can see clearly that the control pulses only change the sign of g_j periodically, leading to $\langle H_I\rangle = 0$.

B. Model Solution

At zero-temperature, Hamiltonian (6) can be exactly solved under the single excitation approximation of the environment. Here, we first assume the initial state of the system plus environment is of the form

$$|\Psi(0)\rangle = \left[C_e(0)|e\rangle + C_g(0)|g\rangle\right]|0\rangle_E, \tag{7}$$

which evolves after time t into the state

$$|\Psi(t)\rangle = \left[C_e(t)|e\rangle + C_g(t)|g\rangle\right]|0\rangle_E + \sum_j C_j(t)|g\rangle|1_j\rangle_E, \tag{8}$$

where $|1_j\rangle$ denotes that only the jth mode of the bath is excited. Note that the basis $|g\rangle|0\rangle_E$ does not evolve under the rotating-wave approximation.

Substituting Eqs. (6) and (8) into Schrödinger equation, we can obtain the following coupled equations

$$\begin{aligned} \dot{C}_e(t) &= -i\omega_0 C_e(t) - i\sum_j g_j(-1)^{[\frac{t}{T}]}C_j(t), \\ \dot{C}_j(t) &= -i\omega_j C_j(t) - ig_j(-1)^{[\frac{t}{T}]}C_e(t). \end{aligned} \tag{9}$$

To obtain $\dot{C}_e(t)$ and $\dot{C}_j(t)$, we can go to the rotating frame, define $c_e(t) = C_e(t)e^{i\omega_0 t}$, $c_j(t) = C_j(t)e^{i\omega_j t}$ [44], and get

$$\begin{aligned} \dot{c}_e(t) &= -i\sum_j g_j(-1)^{[\frac{t}{T}]}e^{i(\omega_0-\omega_j)t}c_j(t), \\ \dot{c}_j(t) &= -ig_j(-1)^{[\frac{t}{T}]}e^{-i(\omega_0-\omega_j)t}c_e(t). \end{aligned} \tag{10}$$

Assuming that $c_j(0) = C_j(0) = 0$, we can get a closed equation for $c_e(t)$, namely

$$\dot{c}_e(t) = -\int_0^t dt_1 f(t-t_1)c_e(t_1) \tag{11}$$

and the correlation function $f(t-t_1)$ is related to the spectral density $J(\omega)$ of the reservoir. Assuming that the qubit is in resonance with the cavity mode, the spectral density is the Lorentzizan spectral distribution [46]

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0\lambda^2}{(\omega_0-\omega)^2+\lambda^2}, \tag{12}$$

where λ reflects the spectral width of the coupling, which is connected to the reservoir correlation time τ_B by $\tau_B = \lambda^{-1}$ and γ_0 is related to the decay of the excited state of the atom in the Markovian limit connected to the relaxation time $\tau_R = \gamma_0^{-1}$. Then, we have

$$f(t-t_1) = \frac{1}{2}(-1)^{\left[\frac{t}{T}\right]+\left[\frac{t_1}{T}\right]}\gamma_0\lambda e^{-\lambda(t-t_1)}. \tag{13}$$

Here, the factor $(-1)^{[\frac{t}{T}]+[\frac{t_1}{T}]}$ is induced by the sequence of π pulses. If $n = 0$ ($T\to\infty$), we have $\lim_{T\to\infty}(-1)^{[\frac{t}{T}]+[\frac{t_1}{T}]} = 1$.

When $t \in [nT, (n+1)T)$, the general solution of Eq. (11) can be derived as (see Appendix A for details)

$$c_e(t) = \begin{cases} e^{-\lambda t/2}[2\Delta_n F_1(n) + (1+\lambda\Delta_n)F_2(n)]c_e(0), & \lambda = 2\gamma_0, \\ e^{-\lambda t/2}[A_n\cosh(\Delta_n d) + B_n\sinh(\Delta_n d)]c_e(0), & \lambda \neq 2\gamma_0, \end{cases} \tag{14}$$

where $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$ and $\Delta_n = (t - nT)/2$. The coefficients F_1 and F_2 are given as

$$F_1 = \frac{\lambda^2 T(p_+^n - p_-^n)}{4\sqrt{(\lambda T)^2 + 4}}, \qquad F_2 = \frac{p_+^n + p_-^n}{2} + \frac{\lambda^2 T}{4} F_1 \tag{15}$$

with $p_\pm = \frac{1}{2}[1 \pm \sqrt{(\lambda T)^2 + 4}]$. Later, we main study the case of $\lambda \neq 2\gamma_0$. In this case the constant coefficients A_n and B_n ($n \geq 1$) in the presence of control pulses can be attained as

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \mathcal{M}^n \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}, \tag{16}$$

with the initial values $A_0 = 1$ and $B_0 = \lambda/d$ [46], corresponding to the case in absence of pulses. And the transfer matrix in the presence of control pulses is given by

$$\mathcal{M} = \begin{pmatrix} \cosh(\tau) & \sinh(\tau) \\ \frac{2\lambda}{d}\cosh(\tau) - \sinh(\tau) & \frac{2\lambda}{d}\sinh(\tau) - \cosh(\tau) \end{pmatrix}, \tag{17}$$

where we have introduced $\tau = Td/2$. Through diagonalizing the transfer matrix, we can obtain

$$A_n = \alpha_+ m_+^n + \alpha_- m_-^n, \qquad B_n = \beta_+ m_+^n + \beta_- m_-^n, \tag{18}$$

where

$$\begin{aligned} \alpha_\pm &= \frac{1}{2}\left[1 \pm \cosh(\tau)/\Theta\right], \qquad m_\pm = \frac{\lambda}{d}\sinh(\tau) \pm \Theta, \\ \beta_\pm &= \alpha_\pm [m_\pm - \cosh(\tau)]/\sinh(\tau), \end{aligned} \tag{19}$$

with $\Theta = \sqrt{1 + \left[\frac{\lambda}{d}\sinh(\tau)\right]^2}$. Furthermore, for finite time t and λ in the limit $n \to \infty$ ($T \to 0$), we have $\cosh(\tau) \simeq 1$ and $\sinh(\tau) \simeq \tau$, then we arrive at $A_n \approx \left(1 + \frac{\lambda T}{2}\right)^n$ and $B_n \approx \frac{\lambda}{d}\left[A_n - \frac{\lambda T}{2}\left(\frac{\lambda T}{2} - 1\right)^n\right]$. Therefore, we can obtain $c_e(t) \approx c_e(0)$, which means the decoherence effect can be nearly completely suppressed in this case.

By defining decay rate $\kappa(t) \equiv \frac{c_e(t)}{c_e(0)} \in [0, 1]$, we can now express the reduced density matrix $\rho_S(t)$ of the qubit system in the form of Kraus operators [26](see Appendix B)

$$\rho_S(\varphi, t) = \sum_i K_i(\varphi, t)\rho_S(0)K_i^\dagger(\varphi, t) \equiv \mathcal{E}_\varphi(t)[\rho_S(0)], \tag{20}$$

with $\varphi = \omega_0 t$. The time-dependent Kraus operators can be expressed as

$$\begin{aligned} K_1(\varphi, t) &= e^{i\varphi/2}\kappa(t)|e\rangle\langle e| + e^{-i\varphi/2}|g\rangle\langle g|, \\ K_2(\varphi, t) &= \sqrt{1 - \kappa(t)^2}e^{-i\varphi/2}|g\rangle\langle e|. \end{aligned} \tag{21}$$

which corresponds to the ADC model. When $\kappa(t) \to 1$, we have $K_1(\varphi, t) \to e^{-i\varphi\sigma_z/2}$ and $K_2(\varphi, t) \to 0$.

With the help of these Kraus operators, the time evolution of N-qubit reduced density operator can be given by

$$\rho(t) = \sum_{\mu_1, \cdots, \mu_N} \left[\otimes_{i=1}^N K_{\mu_i}(t)\right]\rho(0)\left[\otimes_{i=1}^N K_{\mu_i}^\dagger(t)\right], \tag{22}$$

where $K_{\mu_i}(t)$ denotes the Kraus operators for the ith qubit.

III. PPE ENHANCEMENT BY π–PULSE SEQUENCES

Now we study how to protect the QFI and improve the estimation precision of unknown parameter φ, induced by the noise channel $\mathcal{E}_\varphi$. The schematic we propose is shown in Fig. 1, which consists of N qubits in independent reservoirs. Each qubit interacts with a reservoir which leads decoherence precess. In order to suppress decoherence and enhance the precision of estimation, a sequence of π pulses are applied to each qubit simultaneously. The Hamiltonian of each qubit-reservoir system has given in Eq. (3).

To obtain the maximal QFI, the input state is initially prepared in the GHZ state

$$|\psi_{\rm in}(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right), \tag{23}$$

where $\sigma_z|0\rangle = |0\rangle, \sigma_z|1\rangle = -|1\rangle$. Such a state is a maximally

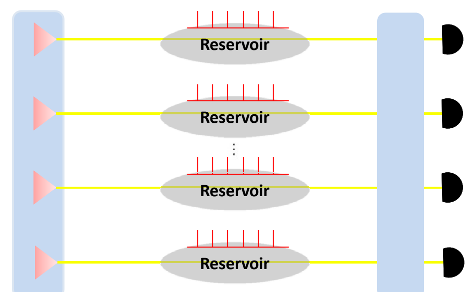

FIG. 1: (Color online) The Schematic representation of parameter estimation for a N-qubit noise system in the presence of dynamical decoupling. The total evolution procedure can be described by the tensor product $\mathcal{E}_\varphi^{\otimes N}$.

entangled state, which can improve the precision to $1/N$, i.e. the HL.

According to Eq. (22), the reduced density matrix of the total system at time t reads [4]

$$\begin{aligned} \rho_S(t) &= \frac{1}{2}\Big[\mathcal{E}_\varphi(t)(|0\rangle\langle 0|)^{\otimes N} + \mathcal{E}_\varphi(t)(|0\rangle\langle 1|)^{\otimes N} \\ &\quad + \mathcal{E}_\varphi(t)(|1\rangle\langle 0|)^{\otimes N} + \mathcal{E}_\varphi(t)(|1\rangle\langle 1|)^{\otimes N}\Big] \\ &= \rho_1 \oplus \rho_2, \end{aligned} \tag{24}$$

where $\mathcal{E}_\varphi(t)$ represents the noise channel with DD pulses for a single qubit. And

$$\begin{aligned} \rho_1 &= \frac{1}{2}\sum_{m=1}^{N-1}\kappa(t)^{2(N-m)}\left[1 - \kappa(t)^2\right]^m |0\rangle\langle 0|^{\otimes(N-m)}|1\rangle\langle 1|^{\otimes m}, \\ \rho_2 &= \frac{1}{2}\Big[\kappa(t)^{2N}|0\rangle\langle 0|^{\otimes N} + [1 + (1 - \kappa(t)^2)^N]|1\rangle\langle 1|^{\otimes N} \\ &\quad + \kappa(t)^N\left(e^{-iN\varphi}|0\rangle\langle 1|^{\otimes N} + e^{iN\varphi}|1\rangle\langle 0|^{\otimes N}\right)\Big]. \end{aligned} \tag{25}$$

The diagonal matrix ρ_1 is independent of parameter φ, thus we only need to consider ρ_2 when estimating the value of parameter φ. In order to estimate φ, we first calculate the QFI.

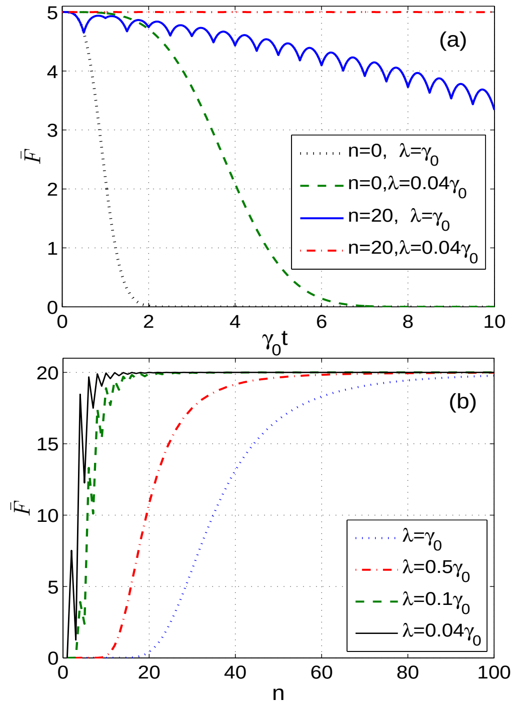

FIG. 2: (Color online) Mean QFI $\bar{F}$ (a) as a function of time $\gamma_0 t$ with $N = 5$, (b) with respect to pulse number n at a given time $\gamma_0 t = 10$ and $N = 20$.

To calculate the QFI, we first diagonalize the φ dependent density matrix ρ_2 as $\rho_2 = \sum_i p_i(t) |\psi_i\rangle\langle\psi_i|$, where $\{|\psi_i\rangle\}$ are the eigenstates of ρ_2 with eigenvalues $\{p_i\}$. In this diagonal representation, the explicit expression of the QFI is given by (as shown in Appendix C)

$$F[\rho(\varphi, t)] = \frac{4N^2\kappa(t)^{2N}}{[1 + (1 - \kappa(t)^2)^N + \kappa(t)^{2N}]^2}. \quad (26)$$

According to the quantum Cramér-Rao bound, minimal variance of the estimation of the parameter φ can be obtained as

$$\Delta\varphi_{\min}(t) \equiv \frac{1}{\sqrt{F[\rho(\varphi, t)]}} = \frac{1 + [1 - \kappa(t)^2]^N + \kappa(t)^{2N}}{2N\kappa(t)^N}. \quad (27)$$

Here, we have set the measurement times $\nu = 1$. Equations (26) and (27) are functions of decay rate $\kappa(t)$ and number of qubits N. From the above equations, we can find in the initial time $\kappa(0) = 1$, presented $F[\rho(\varphi, 0)] = N^2$ and $\Delta\varphi_{\min}(t) = 1/N$, which is the HL on precision of estimation. In this case, the large number of particles N, helps improve the precision of estimation. However, if $\kappa(t) \to 0$ (i.e., without DD pulses and $t \to \infty$), $F[\rho(\varphi, t)] \to 0$, the QFI-based parameter φ lost completely, and the parameter φ can not be estimated in this case, i.e. $\Delta\varphi_{\min}(t) \to \infty$. Worse of all, with the increase of N, the values of $[\kappa(t)]^N$ decay rapidly. Thus, there exist competitive relation between $[\kappa(t)]^N$ and N. To take the advantage of large number of particles N, we need to suppress the decay of $\kappa(t)$. Luckily, the nearly unit value of $\kappa(t)$ can be obtained in the present of DD pulses when $n \to \infty$ ($T \to 0$), as analyzed in the previous section. And the precision on HL can be achieved in this limit. In order to observe

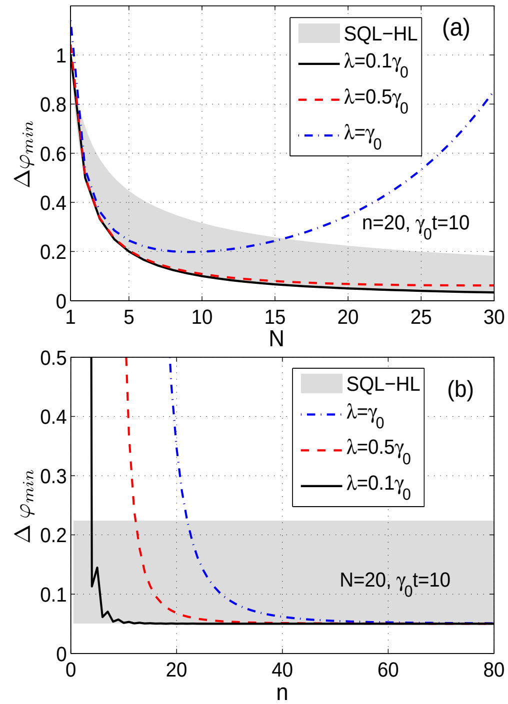

FIG. 3: (Color online) Comparing the values of $\Delta\varphi_{\min}$ at fixed time $\gamma_0 t = 10$ with different λ (a) with respect to the number of qubits N and (b) control pulses n. Shaded area: Region between HL and SQL.

the effect of DD pulses on the estimation precision clearly, in Fig. 2 and Fig. 3 we have plotted mean QFI $\bar{F} = F/N$ and estimation precision $\Delta\varphi_{\min}$ with respect to the evolution time t, the number of pulses n and qubits N, with different values of λ corresponding to the correlation time ($\tau_B = \lambda^{-1}$) of the reservoir.

Figure 2(a) shows the time evolution of mean QFI $\bar{F}$, which is a sufficient condition for entanglement when $\bar{F} > 1$ [4, 24]. The different curves are obtained for different values of λ and number of pulses n, when fixing $N = 5$. We notice a very different behavior with ($n = 20$) and without ($n = 0$) pulses, for different values of λ. In the absence of control pulses $\bar{F}$ reach to zero quickly for $\lambda = \gamma_0$. And control pulses do improve the situation, but does not recover the lost information completely. In contrast, the situation is more better for $\lambda = 0.04\gamma_0$, since exist a longer correlation time τ_B of the reservoir. In this case most of the lost information is recovered by applying the sequence of π pulses.

To investigate the effect of the reservoir correlation time τ_B on the efficiency of control pulses. In Fig. 2(b) we compare the values of $\bar{F}$ for different values of λ at a fixed time $\gamma_0 t = 10$ as a function of the number of pulses n when $N = 20$. We can find that the smaller λ, the less number of control pulses is needed to recover the lost information. For instance, for $\lambda = 0.04\gamma_0$ about 20 pulses, the QFI can recover to its initial value $\bar{F} = N = 20$, while for case of $\lambda = \gamma_0$ more than 100 pulses are needed.

A comparison of the values of $\Delta\varphi_{\min}$ at fixed time $\gamma_0 t = 10$ with respect to the number of qubits N and control pulses n for different values of λ has been shown in Fig. 3. The lower and upper boundaries of the shade regions are the HL and the SQL, respectively. With the increase of the number of qubits N , the theoretical error limit values of estimation (both SQL and HL) decrease monotonically.

Figure 3(a) indicates that $\Delta\varphi_{\min}$ reflects different dependence on the number of qubits N for different values of λ. The precision of HL level can be achieved with increase of N for $\lambda = 0.1\gamma_0$ at fixed time $\gamma_0 t = 10$ and $n = 20$. However, the behavior is completely different for $\lambda = \gamma_0$. In the case of $\lambda = \gamma_0$, the precision higher than SQL can be reached only for the case of $N < 16$, and the error value increase rapidly when $N > 16$. This means that large N may increase the error value of estimation in this case at fixed n.

The values of $\Delta\varphi_{\min}$ at fixed time $\gamma_0 t = 10$ as a function of pulse number n are also given in Fig. 3(b). We can see as long as the number of control pulses is large enough the precision can be improved to the nearby HL for difference λ. However, the larger λ the more pulses are needed to reach the same precision of estimation. In a word, the DD scheme is fully effective as long as $T \ll \tau_B (= \lambda^{-1})$.

IV. CONCLUSION

In conclusion, we have proposed a scheme to enhance PPE in noise systems by employing dynamical decoupling pulses. In our scheme N qubits are embedded into independent reservoirs. The unknown parameter φ to be estimated is induced by the channel $\mathcal{E}_\varphi(t)$. Resorting to the transfer matrix method and time-dependent Kraus operators, the exact analytical expression for the estimation precision of φ has been derived. Using of this expression, we have demonstrated that PPE in N qubits noise systems can be preserved in the HL by controlling of dynamical decoupling pulses. It has been found that larger number of pulses and longer reservoir correlation time can protect PPE more effectively.

Finally, it should be pointed out that these results we have obtained in this paper are based on ideal π pulses, which can be treated as δ functions. This means that the effects of the duration time and errors of the pulses are neglected. However, experimentally, this idealized situation maybe not realistic. The real pulses are always of finite amplitude and of finite length [41, 42]. These imperfect pulses will accumulate an extra phase, which increases with the number of pulses and affects the PPE. To reduce the error as much as possible, we can apply the optimized π pulse sequences[40]. Although it is more sophisticated in form, the same PPE can be attained with less number of pulses. Therefore, less amount of phase errors is accumulated. Detailed consideration of these impacts will be interesting. We hope that the scheme proposed in present paper might have promising applications in quantum information processing and quantum metrology.

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Appendix A: Derivation of Eq. (14)

In this Appendix, we present details of the derivation of Eq. (14).

When $t \in [nT, (n+1)T)$, Eq. (11) can be rewritten as

$$\begin{aligned}\dot{c}_e(t) &= -\frac{\gamma_0\lambda}{2}\int_0^t (-1)^{[\frac{t}{T}]+[\frac{t'}{T}]} e^{-\lambda(t-t')} c_e(t')dt' \\ &= -\frac{\gamma_0\lambda}{2}(-1)^n \left\{\sum_{k=1}^{n}(-1)^{k-1}\int_{(k-1)T}^{kT} e^{-\lambda(t-t')}c_e(t')dt' \right. \\ &\quad \left. +(-1)^n \int_{nT}^{t} e^{-\lambda(t-t')}c_e(t')dt'\right\},\end{aligned} \tag{A1}$$

where $n = [\frac{t}{T}]$, $k = [\frac{t'}{T}]$, and the symbol $[x]$ represents the largest integer not greater than x. We differentiate with respect to t and obtain

$$\begin{aligned}\ddot{c}_e(t) &= -\frac{\gamma_0\lambda}{2}(-1)^n \left\{-\lambda\left[\sum_{k=1}^{n}(-1)^{k-1}\int_{(k-1)T}^{kT} e^{-\gamma(t-t')}c_e(t')dt' \right.\right. \\ &\quad \left.\left. +(-1)^n\int_{nT}^{t} e^{-\gamma(t-t')}c_e(t')dt'\right] + (-1)^n c_e(t)\right\} \\ &= -\lambda\dot{c}_e(t) - \frac{\gamma_0\lambda}{2}c_e(t).\end{aligned} \tag{A2}$$

This equation is an ordinary differential equation which is local in time, and contains only $\ddot{c}_e(t), \dot{c}_e(t)$ and $c_e(t)$.

In the following, we solve the solution of $c_e(t)$ in two different cases: $\lambda = 2\gamma_0$ and $\lambda \neq 2\gamma_0$.

(i) The Case of $\lambda = 2\gamma_0$. In this case, we have $d = \sqrt{\lambda^2 - 2\gamma_0\lambda} = 0$, when $t \in [nT, (n+1)T)$ the general solution of $c_e(t)$ can be derived as

$$c_e(t) = (C_1 t + C_2)e^{-\frac{\lambda t}{2}}, \tag{A3}$$

and C_1 and C_2 are given as

$$\begin{aligned}C_1 &= e^{\frac{\lambda nT}{2}}\left[\dot{c}_e(nT) + \frac{\lambda}{2}c_e(nT)\right], \\ C_2 &= e^{\frac{\lambda nT}{2}}\left[-nT\dot{c}_e(nT) + \left(1 - \frac{\lambda nT}{2}\right)c_e(nT)\right].\end{aligned} \tag{A4}$$

Then we have

$$\begin{pmatrix} c_e(t) \\ \dot{c}_e(t) \end{pmatrix} = e^{-\frac{\lambda(t-nT)}{2}} \begin{pmatrix} 1+\frac{\lambda(t-nT)}{2} & t-nT \\ -\frac{\lambda^2(t-nT)}{4} & 1-\frac{\lambda(t-nT)}{2} \end{pmatrix} \begin{pmatrix} c_e(nT_+) \\ \dot{c}_e(nT_+) \end{pmatrix}$$
$$= e^{-\frac{\lambda(t-nT)}{2}} \begin{pmatrix} 1+\frac{\lambda(t-nT)}{2} & t-nT \\ -\frac{\lambda^2(t-nT)}{4} & 1-\frac{\lambda(t-nT)}{2} \end{pmatrix} \sigma_z \begin{pmatrix} c_e(nT_-) \\ \dot{c}_e(nT_-) \end{pmatrix}, \tag{A5}$$

where

$$\begin{pmatrix} c_e(nT_-) \\ \dot{c}_e(nT_-) \end{pmatrix} = e^{-\lambda T/2} \begin{pmatrix} 1+\frac{\lambda}{2}T & T \\ -\frac{\lambda^2 T}{4} & 1-\frac{\lambda T}{2} \end{pmatrix} \sigma_z \begin{pmatrix} c_e[(n-1)T_-] \\ \dot{c}_e[(n-1)T_-] \end{pmatrix}. \tag{A6}$$

Here we have used the boundary conditions $c_e(nT_-) = c_e(nT_+)$, $\dot{c}_e(nT_-) = -\dot{c}_e(nT_+)$ and σ_z is the pauli matrix. Using the recurrence relation, we can easily obtain the following expression after an n-pulse sequence:

$$\sigma_z \begin{pmatrix} c_e[(n-1)T_-] \\ \dot{c}_e[(n-1)T_-] \end{pmatrix} = e^{-\frac{\lambda nT}{2}} (\sigma_z)^n \begin{pmatrix} 1+\frac{\lambda}{2}T & T \\ -\frac{\lambda^2 T}{4} & 1-\frac{\lambda T}{2} \end{pmatrix}^n \begin{pmatrix} c_e(0) \\ 0 \end{pmatrix}, \tag{A7}$$

here we have used the initial condition $\dot{c}_e(0) = 0$. Thus, we have

$$\begin{pmatrix} c_e(t) \\ \dot{c}_e(t) \end{pmatrix} = e^{-\lambda t/2} \begin{pmatrix} 1+\frac{\lambda(t-nT)}{2} & t-nT \\ -\frac{\lambda^2(t-nT)}{4} & 1-\frac{\lambda(t-nT)}{2} \end{pmatrix} \times \mathrm{M}^n \begin{pmatrix} c_e(0) \\ 0 \end{pmatrix}. \tag{A8}$$

The transfer matrix

$$\mathrm{M} = \begin{pmatrix} 1+\frac{\lambda}{2}T & T \\ \frac{\lambda^2 T}{4} & \frac{\lambda T}{2}-1 \end{pmatrix} \tag{A9}$$

can be diagonalized as $P^{-1}\mathrm{M}P = \mathrm{Diag}[p_+, p_-]$ with $p_\pm = \frac{1}{2}[\lambda T \pm \sqrt{(\lambda T)^2+4}]$. Where matrixes P and P^{-1} are given as

$$P = \begin{pmatrix} T & -T \\ \frac{1}{2}\sqrt{(\lambda T)^2+4}-1 & \frac{1}{2}\sqrt{(\lambda T)^2+4}+1 \end{pmatrix},$$
$$P^{-1} = \frac{1}{T\sqrt{(\lambda T)^2+4}} \begin{pmatrix} \frac{1}{2}\sqrt{(\lambda T)^2+4}+1 & T \\ -\frac{1}{2}\sqrt{(\lambda T)^2+4}-1 & T \end{pmatrix}. \tag{A10}$$

Then, we have

$$\mathrm{M}^n = P \begin{pmatrix} p_+^n & 0 \\ 0 & p_-^n \end{pmatrix} P^{-1} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{A11}$$

where

$$m_{11} = \frac{p_+^n + p_-^n}{2} + \frac{p_+^n - p_-^n}{\sqrt{(\lambda T)^2+4}}, m_{12} = \frac{T(p_+^n - p_-^n)}{\sqrt{(\lambda T)^2+4}},$$
$$m_{22} = \frac{p_+^n + p_-^n}{2} - \frac{p_+^n - p_-^n}{\sqrt{(\lambda T)^2+4}}, m_{21} = \frac{\lambda^2 T(p_+^n - p_-^n)}{4\sqrt{(\lambda T)^2+4}}. \tag{A12}$$

Hence, Eq. (A. 8) can be rewritten as

$$\begin{pmatrix} c_e(t) \\ \dot{c}_e(t) \end{pmatrix} = e^{-\lambda t/2} \begin{pmatrix} 1+\frac{\lambda(t-nT)}{2} & t-nT \\ -\frac{\lambda^2(t-nT)}{4} & 1-\frac{\lambda(t-nT)}{2} \end{pmatrix} \times \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} c_e(0) \\ 0 \end{pmatrix}. \tag{A13}$$

Therefore, the population of excite state in the present of decoupling pulses can be obtained as

$$c_e(t) = e^{-\lambda t/2} \left\{ (t-nT)m_{21} + \left[1 + \frac{\lambda(t-nT)}{2}\right] m_{11} \right\} c_e(0). \tag{A14}$$

Where m_{21} and m_{11} are replaced by $F_1(n)$ and $F_2(n)$ in Eq. (14), respectively.

(ii) The Case of $\lambda \neq 2\gamma_0$. In this case, we have $d = \sqrt{\lambda^2 - 2\gamma_0\lambda} \neq 0$, and the general solution of $c_e(t)$ can be derived as

$$c_e(t) = e^{-\lambda t/2} \left[A_n \cosh\left(\frac{(t-nT)d}{2}\right) + B_n \sinh\left(\frac{(t-nT)d}{2}\right) \right] c_e(0), \tag{A15}$$

with

$$A_n \equiv e^{\frac{\lambda nT}{2}} c_e(nT_+),$$
$$B_n \equiv e^{\frac{\lambda nT}{2}} \left[\frac{\lambda c_e(nT_+)}{d} + \frac{2\dot{c}_e(nT_+)}{d} \right]. \tag{A16}$$

When $t \in [(n-1)T, nT)$, we also have

$$c_e(t) = e^{-\lambda t/2} \left\{ A_{n-1} \cosh\left(\frac{[t-(n-1)T]d}{2}\right) + B_{n-1} \sinh\left(\frac{[t-(n-1)T]d}{2}\right) \right\}. \tag{A17}$$

Using the boundary condition $c_e(nT_-) = c_e(nT_+)$ and $\dot{c}_e(nT_-) = -\dot{c}_e(nT_+)$, we have

$$A_n = A_{n-1}\cosh(\tau) + B_{n-1}\sinh(\tau),$$
$$B_n = \frac{2\lambda}{d}\left[A_{n-1}\cosh(\tau) + B_{n-1}\sinh(\tau)\right] - \left[A_{n-1}\sinh(\tau) + B_{n-1}\cosh(\tau)\right]. \tag{A18}$$

Thanks to the recurrence relations of constant coefficients A_n and B_n, we can obtain Eq. (16). The transfer matrix $\mathcal{M}$ can be diagonalized as $\tilde{P}^{-1}\mathcal{M}\tilde{P} = \mathrm{Diag}[m_+, m_-]$. The matrix $\tilde{P}$ and $\tilde{P}^{-1}$ are given as

$$\tilde{P} = \begin{pmatrix} \sinh(\tau) & \sinh(\tau) \\ m_+ - \cosh(\tau) & m_- - \cosh(\tau) \end{pmatrix},$$
$$\tilde{P}^{-1} = \frac{1}{|\tilde{P}|_{\mathrm{det}}} \begin{pmatrix} m_- - \cosh(\tau) & -\sinh(\tau) \\ \cosh(\tau) - m_+ & \sinh(\tau) \end{pmatrix}. \tag{A19}$$

Then, we have

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \tilde{P} \begin{pmatrix} m_+^n & 0 \\ 0 & m_-^n \end{pmatrix} \tilde{P}^{-1} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}, \tag{A20}$$

hence Eq. (18) is attained.

Appendix B: Noise channel $\mathcal{E}_\varphi(t)$

Here, we will give a derivation of Eq. (20). Corresponding to Eq. (8), we can rewritten the finial state of the system plus environment as

$$\begin{aligned}|\Psi(t)\rangle &= \left[e^{-i\omega_0 t}c_e(t)|e\rangle + C_g(0)|g\rangle\right]|0\rangle_E \\ &\quad + \sum_j e^{-i\omega_j t}c_j(t)|g\rangle|1_j\rangle_E.\end{aligned} \tag{B1}$$

Then the reduced density matrix of the qubit system can be read as

$$\begin{aligned}\rho_S(t) &= \mathrm{Tr}_E(|\Psi(t)\rangle\langle\Psi(t)|) \\ &= \begin{pmatrix} C_e^2(0)\kappa(t)^2 & e^{-i\omega_0 t}C_e(0)C_g(0)\kappa(t) \\ e^{i\omega_0 t}C_e(0)C_g(0)\kappa(t) & 1-C_e^2(0)\kappa(t)^2\end{pmatrix} \\ &= e^{-i\varphi\sigma_z/2}\begin{pmatrix}\rho_{ee}(0)\kappa(t)^2 & \rho_{eg}(0)\kappa(t) \\ \rho_{ge}(0)\kappa(t) & 1-\rho_{ee}(0)\kappa(t)^2\end{pmatrix}e^{i\varphi\sigma_z/2} \\ &= e^{-i\varphi\sigma_z/2}\sum_i E_i\rho(0)E_i^\dagger e^{i\varphi\sigma_z/2} \\ &= \sum_i K_i(\varphi,t)\rho(0)K_i^\dagger(\varphi,t) \equiv \mathcal{E}_\varphi(t)\rho(0),\end{aligned} \tag{B2}$$

where

$$E_1(t) = \kappa(t)|0\rangle\langle 0| + |1\rangle\langle 1|, \qquad E_2(t) = \sqrt{1-\kappa(t)^2}|1\rangle\langle 0|, \tag{B3}$$

and $K_i(\varphi,t)$ have given in Eq. (21).

Appendix C: Quantum Fisher information

In this Appendix, we will calculate the QFI, which have given in Eq. (26). In the basis of $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$, $\rho_2(t)$ can be written as

$$\rho_2(t) = \frac{1}{2}\begin{pmatrix}\kappa^{2N} & e^{-iN\varphi}\kappa^N \\ e^{iN\varphi}\kappa^N & 1+(1-\kappa^2)^N\end{pmatrix}. \tag{C1}$$

In order to calculate the QFI, we first diagonalize $\rho_2(t)$ as

$$\rho_2(t) = \sum_i p_i(t)\left|\psi_i(t)\right\rangle\left\langle\psi_i(t)\right|. \tag{C2}$$

The corresponding eigenvalues and eigenvectors are given by

$$\begin{aligned}p_{1,2}(t) &= \frac{1}{4}\Big[1+\kappa^{2N}+(1-\kappa^2)^N \\ &\quad \pm\sqrt{(1+\kappa^{2N}+(1-\kappa^2)^N)^2-4\kappa^{2N}(1-\kappa^2)^N}\Big],\end{aligned} \tag{C3}$$

and

$$\begin{aligned}\left|\psi_1(t)\right\rangle &= \sin\alpha(t)\,|1\rangle^{\otimes N} + e^{-iN\varphi}\cos\alpha(t)\,|0\rangle^{\otimes N}, \\ \left|\psi_2(t)\right\rangle &= \cos\alpha(t)\,|1\rangle^{\otimes N} - e^{-iN\varphi}\sin\alpha(t)\,|0\rangle^{\otimes N},\end{aligned} \tag{C4}$$

where

$$\alpha(t) = \arctan\frac{2\kappa^N}{\kappa^{2N}-1-(1-\kappa^2)^N+\Xi} \tag{C5}$$

with $\Xi = \sqrt{(1+\kappa^{2N}+(1-\kappa^2)^N)^2-4\kappa^{2N}(1-\kappa^2)^N}$.

In this diagonal representation, the matrix elements of the symmetric logarithmic derivative (SLD) is

$$L_{ij} = \frac{2\langle\psi_i|\partial_\varphi\rho_2|\psi_j\rangle}{p_i+p_j}. \tag{C6}$$

Then $L(t)$ is obtained explicitly as

$$L(t) = \frac{2iN\kappa^N}{1+\kappa^{2N}+(1-\kappa^2)^N}\left[|\psi_1\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_1|\right]. \tag{C7}$$

Thus the QFI can be calculated as

$$\begin{aligned}F &= \frac{1}{2}\mathrm{Tr}\left[\rho_2 L^2 + L^2\rho_2\right] \\ &= \frac{4N^2\kappa(t)^{2N}}{[1+(1-\kappa(t)^2)^N+\kappa(t)^{2N}]^2}.\end{aligned} \tag{C8}$$

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