Transparent control of an exactly solvable two-level system via combined modulations

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Transparent control of an exactly solvable two-level system via combined modulations

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Analytically exact solutions of a driven two-level system are extremely important to many areas of physics (E. Barnes and S. Das Sarma, Phys. Rev. Lett. 109, 060401, 2012), we here present a simple method of synchronous and asynchronous combined-modulations to generate a lot of new exact solutions, and apply the simplest several solutions to perform the coherent control of quantum states. The physical interesting phenomena such as the arbitrary population transfers between the different and/or asymptotic stationary states and the controlled coherent population oscillation are achieved analytically and numerically. Simplicities of the exact solutions and the modulated fields render the control strategies more transparent and the qubit manipulation more convenient in experiments.

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The analytically exact and perturbed solutions of the two-level problem have played a central role in studying a number of important physical phenomena, such as quantum computing [1], qubit control [2–4] and coherent manipulations of various quantum states [5–7]. Research attempting to control quantum phenomena has been underway for a long time [8–12]. In practice one may achieve the control goal by introducing external fields, e.g. laser light to transfer an initial state to a desired final state. Such laser field can be designed with help of the quantum optimal control theory [11, 13]. It is well-known that exact analytical solutions can provide deeper understanding of the underlying physics than straight numerical calculations [14]. Therefore we are interested in the situations where the exact analytical solutions of simple forms exist, rendering the control strategies more transparent [13].

In an analytically solvable driven two-level system, the forms of driving fields contain an infinite variety. The most frequently used drives are of sinusoidal [15] and hyperbolic forms [16], which adjust the corresponding amplitude- and frequency-modulated functions. The single [17–20] and/or combined [14, 21–25] modulations have been applied for producing an unlimited number of exact solutions in terms of complicated functions which include the Gauss hypergeometric function [21–23], Weber function [18, 19] and some more specific functions [17, 20, 25]. It is worth noting that the physical interesting phenomena are very finite in a two-level system, which may be described by a few simple solutions associated with simple driving forms.

In this paper, we present a simple method of synchronous and asynchronous combined-modulations to construct a lot of new exact solutions of the Schrödinger equation describing a driven two-level (or double-well) system. It is shown that most of the interesting physical phenomena could be conveniently described only by several simplest solutions that creates the possibility of the transparent controls. In our transparent controls, the simple combined modulations lead to the controlled coherent population oscillation (CCPO) and the arbitrary coherent population transfer (ACPT) between the different and/or asymptotic stationary states, including the incoherent destruction of tunneling (IDT) of the Floquet states which enable us to control the system to an arbitrary stationary state, the coherent destruction of tunneling (CDT) between the superposition states, and the controlled coherent population inversion (CPI). The analytical results are confirmed numerically and good agreements are displayed. The results can be applied to many branches of contemporary physics and could be observed particularly as physical realizations of qubits and quantum logic gates.

Two-level model with combined modulations. We consider the bosonic representation [26] of the two-level (or double-well) Hamiltonian in the general form with combined modulations [22]

$$H(t) = G(t)(a_1^\dagger a_1 - a_2^\dagger a_2) + J(t)(a_1^\dagger a_2 + a_2^\dagger a_1), \quad (1)$$

where $a_j(a_j^\dagger)$ are annihilation (creation) operators for the atom in j-th state or well with $j = 1, 2$, the real functions $J(t)$ and $G(t)$ correspond to the amplitude- and frequency-modulated functions respectively. The bosonic representation (1) is equivalent to the two-state representation [26], and also can be expressed in terms of the angular momentum operators [7, 17] through the formulas [27] $\sigma_x = a_1^\dagger a_2 + a_2^\dagger a_1$, $\sigma_z = a_1^\dagger a_1 - a_2^\dagger a_2$. The system (1) is dimensionless by adopting the Planck constant $\hbar = 1$ and using the reference frequency $\omega_0 = 100$Hz to normalize the energy and the field parameters in $J(t)$, $G(t)$, and using ω_0^{-1} to normalize time t [28].

Taking the localized states $|1\rangle$ and $|2\rangle$ as the bases, we expand the quantum state $|\psi\rangle$ of system (1) as

$$|\psi\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle, \quad (2)$$

where C_j for $j = 1, 2$ denote the time-dependent probability amplitudes in the states (wells) 1 and 2. Inserting Eqs. (1) and (2) into the Schrödinger equation

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$i\frac{\partial|\psi\rangle}{\partial t}=H|\psi\rangle$ yields the coupled equation

$$\begin{aligned} i\dot{C}_1(t) &= G(t)C_1(t)+J(t)C_2(t), \\ i\dot{C}_2(t) &= -G(t)C_2(t)+J(t)C_1(t), \end{aligned} \tag{3}$$

which governs time evolution of the probability amplitudes. For different forms of the amplitude- and frequency-modulated functions, some complicated exact solutions have been constructed [14, 21–25]. We here focus on simple exact solutions and their rich physics, which are associated with some special driving forms and parameters.

Analytically exact solution for the synchronous modulations. By the synchronous modulations we mean that the amplitude and frequency modulated functions are in the same form, namely the former is proportional to the latter, $J(t)=J_0G(t)$ with proportionality constant J_0 and any function $G(t)$. In such a case, we can adopt the new "time variable"

$$\tau=\int G(t)dt \tag{4}$$

to make Eq. (3) the linear differential equation of τ with constant coefficients. Its simple exact solution of form $C_j(t)=C_j[\tau(t)]$ reads

$$\begin{aligned} C_1(t) &= D_1e^{-i\sqrt{0.25+J_0^2}\ \tau}+D_2e^{i\sqrt{0.25+J_0^2}\ \tau}, \\ C_2(t) &= \frac{1}{J_0}[D_1(\sqrt{0.25+J_0^2}-0.5)e^{-i\sqrt{0.25+J_0^2}\ \tau} \\ &+ D_2(\sqrt{0.25+J_0^2}+0.5)e^{i\sqrt{0.25+J_0^2}\ \tau}]. \end{aligned} \tag{5}$$

The general solution is a coherent superposition of the two special solutions $e^{\pm i\sqrt{0.25+J_0^2}\ \tau}$ with the complex superposition constants D_j for $j=1,2$ being adjusted by the initial conditions and normalization. Generally, Eq. (5) leads Eq. (2) to be the coherent state $|\psi\rangle=|\psi_1\rangle+|\psi_2\rangle$ with $|\psi_1\rangle=|\psi\rangle_{D_2=0}$, $|\psi_2\rangle=|\psi\rangle_{D_1=0}$, which describes controlled coherent population oscillation (CCPO) of the particle. Once the initial setup is set to give $D_j=0$ for $j=1$ or 2, Eq. (2) becomes the incoherent stationary state $|\psi_2\rangle$ or $|\psi_1\rangle$ with a constant norm, which displays the incoherent destruction of tunneling (IDT) for a suitable constant J_0. Analytic solutions for some pulses and oscillatory fields include an infinite variety of amplitude and frequency modulations [22]. Here by selecting some simple modulated functions $G(t)$ and combining Eqs. (4) with (5), we perform the interesting transparent control of the quantum tunnel (or transition).

Controlled coherent population oscillation and incoherent destruction of tunneling. At first, we take the usual periodic modulation as an example to show CCPO of the non-Floquet states and IDT of the Floquet states. Let the function $G(t)$ be in the form $G_0+G_1\cos\omega t$ such that Eq. (4) gives $\tau=G_0t+\frac{G_1}{\omega}\sin\omega t$ and Eq. (5) implies that $|\psi_1\rangle$ and $|\psi_2\rangle$ are two Floquet states with quasi-energies $E_\pm=\pm\sqrt{0.25+J_0^2}\ G_0$, and their linear superposition $|\psi\rangle$ is a non-Floquet state [28]. Setting the initial probability amplitudes as the real constants $C_1(0)=\sqrt{0.99}$ and $C_2(0)=\sqrt{0.01}$, and the proportionality constant as $J_0=1$, from Eq. (5) we obtain the constants $D_1=0.7647, D_2=0.2303$. Then we take $\omega=1.2$ and two sets of different parameters G_0 and G_1 to plot time evolutions of the population imbalance $z(t)=P_1(t)-P_2(t)$ with $P_1(t)=|C_1(t)|^2$ and $P_2(t)=|C_2(t)|^2$ as in Fig. 1. For the zero constant-modulation, $G_0=0$, Fig. 1(a) exhibits perfect agreement between the numerical [curves from Eq. (3)] and analytical [circular points from Eq. (5)] results. For the nonzero constant-modulation, $G_0=0.5$, Fig. 1(b) indicates small deviation between the numerical and analytical results [29]. Both the figures show that the amplitude and frequency of the population oscillations can be controlled by adjusting the driving parameters.

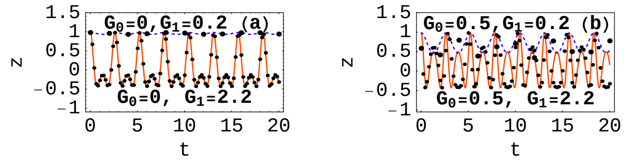

FIG. 1: (Color online) Time evolutions of the population imbalance showing the CCPO for the parameters $J_0=1$, $\omega=1.2$ and (a) $G_0=0$; (b) $G_0=0.5$ with the solid curves corresponding to $G_1=0.2$ and the dashed curves to $G_1=2.2$. Hereafter all the quantities plotted in the figures are dimensionless, and all the circular points indicate the analytical solutions and the curves represent the numerical results.

When one of D_1 and D_2 is equal to zero, Eq. (5) gives the Floquet solutions $C_1(t)$ and $C_2(t)$ with constant probabilities $|C_1|^2=P_1$, $|C_2|^2=P_2=1-P_1$. This means the decoherence between Floquet states $|\psi_1\rangle$ and $|\psi_2\rangle$ and results in the exact IDT. As an instance, we take the real initial values $C_1(0)=D_1=\sqrt{P_1}$ and $C_2(0)=\frac{\sqrt{P_1}}{2J_0}(2\sqrt{0.25+J_0^2}-1)=\pm\sqrt{1-P_1}$. Combining these with Eq. (5) leads to $D_2=0$, $|\psi\rangle=|\psi_1\rangle$ and the restriction relation between J_0 and P_1,

$$P_{1\pm}=0.5(1\pm1/\sqrt{1+4J_0^2}). \tag{6}$$

As the functions of J_0, $P_{1\pm}$ is plotted in Fig. 2(a). Given Eq. (6), for a fixed J_0 value we have two allowable values of the probability in state (well) one, P_{1+} and P_{1-}. If initial value is one of the both, the exact IDT infers that the initial population is kept and no quantum tunnel occurs such that the system is in a kind of new stationary states. Taking the parameters $G_0=0.5$, $G_1=2.2$, $\omega=1.2$ yields $\tau=0.5t+\frac{2.2}{1.2}\sin(1.2\ t)$. Then setting $D_2=0$ and three different groups of $(J_0,\ D_1)$ values, from Eq. (5) we plot time evolutions of the population imbalance $z(t)$, as the circular points in Fig. 2(b). The corresponding numerical results are shown by the curves of Fig. 2(b), which are in good agreement with the analytical ones.

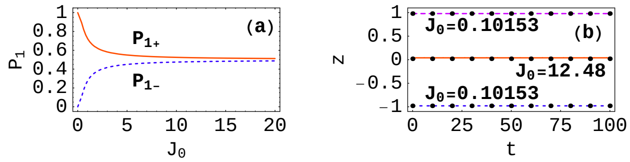

FIG. 2: (Color online) (a) In the Floquet state $|\psi\rangle = |\psi_1\rangle$, the probability P_1 versus the field parameter J_0. (b) Time evolutions of the population imbalance showing the IDT for the parameters $G_0 = 0.5$, $G_1 = 2.2$, $\omega = 1.2$ and $(J_0,\ D_1) = (12.48,\ \sqrt{0.52})$ (solid line), $(0.10153,\ \sqrt{0.99})$ (dashed line), $(0.10153,\ \sqrt{0.01})$ (dotted line).

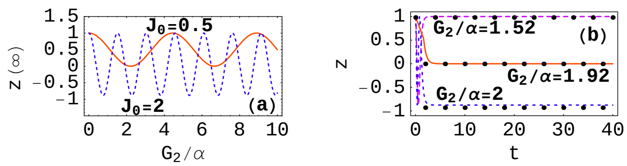

FIG. 3: (Color online) (a) Plots of the asymptotic population imbalance versus $\frac{G_2}{\alpha}$ for the initial conditions $C_1(0) = 1,\ C_2(0) = 0$ and the parameters $J_0 = 0.5,\ D_1 = 0.85355,\ D_2 = 0.14645$ (solid curve), and $J_0 = 2,\ D_1 = 0.6212678,\ D_2 = 0.37873$ (dashed curve). (b) Time evolution of the population imbalance $z(t)$ for three values $\frac{G_2}{\alpha} = 1.52,\ 1.92,\ 2$ on the dashed curve of Fig. 3(a).

Therefore, we can control the system to arbitrary stationary states, through the IDT.

Arbitrary coherent population transfers and exact coherent destruction of tunneling between the asymptotic stationary states. In the case of the synchronous modulations, by using the square sech-shaped $G(t) = G_2\mathrm{sech}^2(\alpha t)$ instead of the periodic $G(t)$, Eq. (4) gives the "time variable" $\tau = \frac{G_2}{\alpha}\tanh(\alpha t)$, where G_2 and α are the driving strength and inverse pulse width. Applying such a τ to Eqs. (5) and (2), we obtain the exact asymptotic stationary state in which the atomic population imbalance $z(\infty)$ is a constant at time $t \to \infty$ and $\tau \to \frac{G_2}{\alpha}$. Assuming the initial state to be $|\psi(0)\rangle = |1\rangle$ with conditions $C_1(0) = 1$ and $C_2(0) = 0$, for the parameter $J_0 = 0.5$ (or 2) from Eq. (5) we get the constants $D_1 = 0.85355$ (or 0.6212678) and $D_2 = 0.14645$ (or 0.37873). Employing such parameters and Eq. (5) results in the asymptotic population imbalance as the functions of $\frac{G_2}{\alpha}$, which are plotted in Fig. 3(a). Then we take three values $\frac{G_2}{\alpha} = 1.52,\ 1.92$ and 2 on the dashed curve of Fig. 3(a) to plot time evolutions of the population imbalance $z(t)$, as the circular points in Fig. 3(b). The corresponding numerical results are given as the solid, dashed and dotted lines for the three different $\frac{G_2}{\alpha}$ values, respectively. The both results agreeably show that one can adjust the external field parameters J_0 and $\frac{G_2}{\alpha}$ to make the ACPT from a fixed initial state to any desired stationary state, including the exact CDT for $\frac{G_2}{\alpha} = 1.52$ case of Fig. 3(b).

Controlled coherent population inversion for the asynchronous modulations. In the case of asynchronous modulation, we take the sech-shaped amplitude modulation $G(t) = G_3\tanh(\alpha t)$ and tanh-shaped frequency modulation $J(t) = J_0\mathrm{sech}(\alpha t)$ as an example to exhibit the transparent control. The complicated solutions in terms of the Gauss hypergeometric function are well-known for the similar system [14, 22, 25]. We here demonstrate that for a set of suitable field parameters $G_0,\ J_0$ and α a simple solution exists, which exactly describes the controlled coherent population inversion (CPI). To do this, we first make a function transformation $C_1(t) = \sqrt{\mathrm{sech}}\ C(t)$ and from Eq. (3) establish the second-order equation

$$
\begin{aligned}
i\ddot{C}(t) &= [U_0 + U_1(t)]C(t), \quad U_0 = 0.5(\alpha^2 - 2J_0^2 - i\alpha G_3),\\
U_1(t) &= \frac{1}{4}(4J_0^2 - \alpha^2 - G_3^2)\tanh^2(\alpha t). \qquad (7)
\end{aligned}
$$

This equation is mathematically equivalent to the Schrödinger equation of square tanh-shaped potential, and its n exact solutions in terms of hypergeometric functions have been found [30]. It is important to note that by selecting the field parameters

$$
4J_0^2 - \alpha^2 - G_3^2 = 0, \quad U_0 = [0.5(\alpha - iG_3)]^2, \qquad (8)
$$

one can produce the simple solutions of Eq. (7) as $C_\pm = D_\pm e^{\pm 0.5(\alpha - iG_3)t}$ with $D_\pm$ being constants. Given Eq. (8), the general solution of Eq. (7) is the coherent superposition $C(t) = C_+ + C_-$. Applying $C_1(t) = \sqrt{\mathrm{sech}(\alpha t)}\ C(t)$ and Eqs. (3) and (8), we arrive at the simple general solutions

$$
\begin{aligned}
C_1(t) &= \sqrt{\mathrm{sech}(\alpha t)}(D_- e^{-0.5(\alpha - iG_3)t} + D_+ e^{0.5(\alpha - iG_3)t}),\\
C_2(t) &= (i\dot{C}_1 - 0.5GC_1)/J = \frac{1}{2J_0}(i\alpha + G_3)\sqrt{\mathrm{sech}(\alpha t)}\\
&\quad \times (D_- e^{0.5(\alpha + iG_3)t} - D_+ e^{-0.5(\alpha + iG_3)t}). \qquad (9)
\end{aligned}
$$

From Eq. (8) the relation among the parameters G_0 and J_0 is displayed in Fig. 4(a) for three different values of α. It is quite interesting to note that Eqs. (9) and (8) imply $P_1(-\infty) = P_2(\infty) = \sqrt{2}D_-^2$ and $P_2(-\infty) = P_1(\infty) = \sqrt{2}D_+^2$. The result means the exact CPI from any initial state $|\psi(-\infty)\rangle$ to the corresponding final state $|\psi(\infty)\rangle$ [19, 25, 29]. Thus for a set of fixed parameters obeying Eq. (8), the final states are controlled by imposing appropriate initial conditions. The CPI is shown for the initial time $t_0 = -100$ and five different initial states, as in Fig. 4(b), where the analytical results (the circular points) from Eq. (9) agree with the numerical results (the curves) from Eq. (3) very well.

We have studied how to design simple combined modulations of amplitude and frequency to construct the exact analytical solutions of simple forms for a two-level (or equivalent double-well) system, and to perform the new transparent synchronous and asynchronous control of quantum transition (or tunnel) analytically and numerically. It is shown that although we can obtain many

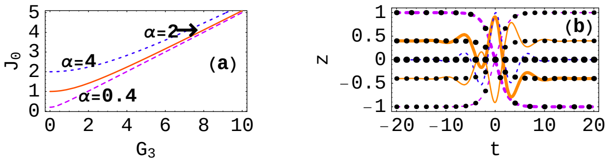

FIG. 4: (Color online) (a) Plot of the parameter G_3 versus J_0 for $\alpha = 4$ (dotted curve), $\alpha = 2$ (solid curve) and $\alpha = 0.4$ (dashed curve). (b) Time evolutions of the population imbalance showing the CPI for the parameters $J_0 = 0.5$, $\alpha = 0.4$, $G_3 = 0.9165$ and the initial conditions $P_1(-100) = 1$ (thick dashed curve), $P_1(-100) = 0.7$ (thick solid curve), $P_1(-100) = 0.5$ (dotted curve), $P_1(-100) = 0.3$ (thin dolid curve), $P_1(-100) = 0$ (thin dashed curve) and the corresponding phases $\arg C_j(-100)$ given by Eq. (9).

analytically exact solutions of complicated forms, most of the physical interesting phenomena could be conveniently described only by several simple solutions which are the coherent superpositions of two special solutions respectively. In the synchronous case, the amplitude modulation is proportional to the frequency modulation such that for any form of the modulated function we show the transparent control from CCPO to IDT by adjusting the proportionality constant and imposing appropriate initial conditions. Thus we can control the system to an arbitrary stationary state, through the IDT. The example of periodic modulation gives the exact solutions being the Floquet and non-Floquet states. For a square sech-shaped modulated function we find the exact CDT and ACPT between the asymptotic stationary states by selecting the suitable ratio of the driving strength and the inverse pulse width. In the asynchronous case, the sech-shaped amplitude and tanh-shaped frequency modulations are adopted and the controlled CPI is illustrated. The results can be applied to many physical contexts, such as the qubit control, two state collision problems and the artificial two-level electronic system, and so on. Specifically, the simplicity of the combined modulations and exact solutions bring experimental convenience for direct observation as physical realizations of qubits and quantum logic gates by using the current experimental setups.

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