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Competitions between quantum correlations in the quantum-memory-assisted entropic uncertainty relation

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With the aid of a quantum memory, the uncertainty about the measurement outcomes of two incompatible observables of a quantum system can be reduced. We investigate this measurement uncertainty bound by considering an additional quantum system connected with both the quantum memory and the measured quantum system. We find that the reduction of the uncertainty bound induced by a quantum memory, on the other hand, implies its increasing for a third participant. We also show that the properties of the uncertainty bound can be viewed from perspectives of both quantum and classical correlations, in particular, the behavior of the uncertainty bound is a result of competitions of various correlations between different parties.

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I. INTRODUCTION

The Heisenberg uncertainty principle [1] is one of the most remarkable features of quantum theory which differs quantum world essentially from the classical world. It sets limits on the precise prediction of the outcomes of two incompatible quantum measurements Q and R on a particle, and is expressed in various forms [2–4]. However, Berta *et al*. [5] showed recently that the uncertainty bound imposed by the Heisenberg principle could actually be violated with the aid of a quantum memory B that is entangled with the particle A to be measured. This quantum-memory-assisted entropic uncertainty relation reads [5]

$$S(Q|B) + S(R|B) \geqslant \log_2 \frac{1}{c} + S(A|B), \tag{1}$$

the equivalent form of which was previously conjectured by Renes and Boileau [6]. Here, $S(A|B)$ is the conditional von Neumann entropy of the density operator ρ_{AB}, $S(A|B) = S(\rho_{AB}) - S(\rho_B)$. In the left-hand side (LHS) of the inequality, $S(X|B)$ is that of the postmeasurement state $\rho_{XB} = \sum_k (\Pi_k^X \otimes \mathbb{I}) \rho_{AB} (\Pi_k^X \otimes \mathbb{I})$ which represents uncertainty of the measurement outcomes of $X = \{Q, R\}$ conditioned on the prior information stored in B, where $\Pi_k^X = |\Psi_k^X\rangle\langle\Psi_k^X|$ with $|\Psi_k^X\rangle$ being the eigenstates of X, and $c = \max_{k,l} |\langle\Psi_k^Q|\Psi_l^R\rangle|^2$ with $1/c$ quantifies the complementarity of Q and R.

This generalized entropic uncertainty relation has been confirmed in all-optical experiments [4, 7]. Meanwhile, the related relations expressed by other entropic quantities such as the Rényi entropy which is important in physical models [8], are also exploited [9, 10]. Since its fundamental role, this quantum-memory-assisted entropic uncertainty relation can be studied from various viewpoints [11–13], and can be applied in other quantum information processions [14–16].

Uncertainty relation of Eq. (1) differs from its original one [3] by an additional term $S(A|B)$. It is clear that the bound of the entropic uncertainty, the right-hand side of inequality (1) named as uncertainty bound (UB) hereafter, is reduced whenever $S(A|B) < 0$. It is remarkable that the quantity of conditional entropy $S(A|B)$ has many important implications in quantum information processing. Its negativity means the inseparability [17] and gives the lower bound of the one-way distillable entanglement for ρ_{AB} [18]. It quantifies partial quantum information [19] and can be related with quantum correlation measures [12, 20–24].

In this Letter, we go one step further from bipartite state ρ_{AB} to consider its purification $|\Psi\rangle_{ABC}$ or a tripartite state ρ_{ABC}, i.e., a third party C is entangled with both the particle A and the memory B. Some fundamental and interesting phenomena are found: For example, there exists correlative capacities which indicate the uncertainty reduction of UB because of B implies its increasing for other parties; The changing of UB is induced by competitions of various quantum correlations between different pairs. These results have important conceptual implications and shed new light on the foundations of quantum mechanics.

II. CORRELATION CAPACITIES

We begin with a simple yet meaningful observation. For any three-partite system ABC with density matrix ρ_{ABC}, we have

$$S(A|B) + S(A|C) \geqslant 0, \tag{2}$$

which can be proved directly by the strong subadditivity inequality: $S(\rho_B) + S(\rho_C) \leqslant S(\rho_{AB}) + S(\rho_{AC})$ [25]. Eq. (2) indicates that whenever $S(A|B) < 0$, we always have $S(A|C) > 0$. Therefore, the reduction of the UB on A with quantum information stored in B excludes its reduction by quantum information stored in C. Since the reduction of the UB originates from the quantum correlations established between the measured particle and the quantum memory [5], this observation may be interpreted as a fact that particle A reaches its potential correlative capacities with the quantum memory

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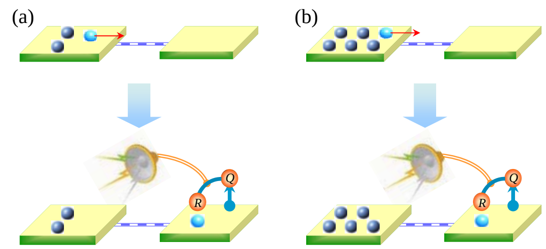

FIG. 1: (Color online) Schematic representation of the "uncertainty game" with three (a) and six (b) players.

B in the sense that any other quantum memory except B always gives increasing UB of measurement uncertainty on A.

Inequality (2) also has important physical implications. To be convinced, let us consider a variant of the imaginary "uncertainty game" presented in [5]: three players Alice (A), Bob (B), and Charlie (C) share a quantum state ρ_{ABC}, the form of which is known only to Bob and Charlie. They begin the game by preagreeing on two measurements Q and R. Alice then measures either Q or R randomly on the particle A, and informs Bob and Charlie of her measurement choice but not the outcome. What we want to determine here is whether Bob and Charlie (communication between them is forbidden) can predict the outcomes of Alice both with improved precision (see Fig. 1 for an illustration). As for fixed Q and R, the UB of measurement is determined only by the conditional entropy $S(A|X)$ for ρ_{AX} of the observed particle A and the quantum memories $X=\{B,C\}$, and Eq. (2) excludes the possibility for $S(A|B)$ and $S(A|C)$ taking the negative values simultaneously, the prediction precision of Bob and Charlie cannot be improved simultaneously in this game, i.e., the improvement of Bob's prediction precision implies the degradation for that of Charlie's, and vice versa. Particularly, for pure ρ_{AB}, Eq. (2) simplifies to $S(A|B)+S(A|C)=0$, which tells us that the more precisely the measurement outcome is predicted by one participant, the less precisely that will be predicted by another one. In some sense, one may say that this implies another kind of uncertainty relation, because it sets limits on Bob and Charlie's (under the constraint of no communication between them) ability to predict correctly the measurement outcomes of Alice simultaneously, that is to say, the certainty of prediction for one participant implies the uncertainty of prediction for another participant

The arguments above can also be easily generalized to the N-player case, i.e., we have

$$\sum_{i=1}^{N-1} S(A|X_i) \geqslant 0, \tag{3}$$

where the particles $X_i=\{B,C,D,\cdots\}$ belong to Bob, Charlie and Daniel, *et al.*, respectively. This inequality can be proved directly by combination of the strong subadditivity of the von Neumann entropy (Theorem 11.14 of [25]) and the subadditivity of the conditional entropy (Theorem 11.16 of [25]), and it implies that even for the multi-player case, the precision of predictions about Alice's measurement outcomes cannot be improved simultaneously for all of the participants.

Also we would like to remark here that the generalized "uncertainty game" illustrated in Fig. 1 can be immediately tested by similar all-optical setups as those in Refs. [4, 7].

III. COMPETITION OF QUANTUM DISCORDS

Next let us consider some quantum correlations and begin with quantum discord [20]. The measure of the classical correlation takes the form

$$J(B|A)=S(\rho_B)-\min_{\{E_k^A\}} S(B|\{E_k^A\}), \tag{4}$$

where $S(B|\{E_k^A\})=\sum_k p_k S(\rho_{B|E_k^A})$, with $\rho_{B|E_k^A}=\mathrm{Tr}_A(E_k^A\rho_{AB})/p_k$ being the nonselective postmeasurement state of B after the positive operator valued measure (POVM) on A, and $p_k=\mathrm{Tr}(E_k^A\rho_{AB})$ is the probability for obtaining the outcome k. $J(B|A)$ is usually interpreted as the maximum information gained about B with the measurement outcome of A. The quantum discord is then defined by the discrepancy between quantum mutual information $I(A:B)=S(\rho_A)+S(\rho_B)-S(\rho_{AB})$ and $J(B|A)$ as

$$D(B|A)=I(A:B)-J(B|A). \tag{5}$$

The quantum discord can therefore be interpreted as the minimal loss of correlations due to the POVM $\{E_k^A\}$. It survives for states with quantumness of correlation and vanishes for states with only classical correlation. It attracts much attention recently because of its fundamental role in quantum information processing [26–29]. Here, we demonstrate a new perspective of quantum discord in the uncertainty principle of quantum mechanics.

Assume $|\Psi\rangle_{ABC}$ being the purification of the bipartite state ρ_{AB}, we first have the following Proposition.

Proposition 1. When the UB on A is reduced with the aid of a quantum memory B, then both $D(B|A)$, $J(B|A)$ and the entanglement of formation (EoF), $E_f(\rho_{AB})$, are larger than those between A and its purifying system C.

Proof. By using the Koashi-Winter equality for $|\Psi\rangle_{ABC}$ [30], we obtain

$$\begin{aligned} E_f(\rho_{BC})+J(B|A)&=S(\rho_B),\\ E_f(\rho_{CB})+J(C|A)&=S(\rho_C), \end{aligned} \tag{6}$$

where $E_f(\rho_{BC})$ is the EoF for ρ_{BC}, defined as $E_f(\rho_{BC})=\min_{\{p_i,|\psi_i\rangle_{BC}\}}\sum_i p_i S(\mathrm{Tr}_C|\psi_i\rangle_{BC}\langle\psi_i|)$ [31], and the minimum is taken over all pure state decompositions $\rho_{BC}=\sum_i p_i|\psi_i\rangle_{BC}\langle\psi_i|$. Since $E_f(\rho_{BC})=E_f(\rho_{CB})$, Eq. (6) yields $J(B|A)-J(C|A)=S(\rho_B)-S(\rho_C)=-S(A|B)>0$, and therefore $J(B|A)>J(C|A)$. Furthermore, by combining Eqs. (5) and (6), we obtain an equivalent form of the Koashi-Winter equalities

$$\begin{aligned} D(B|A)+S(B|A)&=E_f(\rho_{BC}),\\ D(C|A)+S(C|A)&=E_f(\rho_{CB}), \end{aligned} \tag{7}$$

which gives $D(B|A)-D(C|A)=S(C|A)-S(B|A)=-S(A|B)>0$, and hence $D(B|A)>D(C|A)$. Finally, to prove $E_f(\rho_{AB} > E_f(\rho_{AC})$, we note that the conditional entropy $S(A|B) < 0$ is equivalent to $S(\rho_B) > S(\rho_C)$ for $|\Psi\rangle_{ABC}$. Therefore by using the chain rule [32], we derive

$$S(\rho_B) + E_f(\rho_{CA}) \leqslant S(\rho_C) + E_f(\rho_{AB}), \tag{8}$$

which implies $E_f(\rho_{AB}) - E_f(\rho_{AC}) \geqslant S(\rho_B) - S(\rho_C) = -S(A|B) > 0$, and thus completes the proof. ■

In fact, from Eqs. (4), (5), and $S(\rho_{B|A}) = S(\rho_{C|A}) = E_f(\rho_{BC})$, with $S(\rho_{X|A}) := \min_{\{E_k^A\}} S(X|\{E_k^A\})$ ($X = B$ or C) [30], we can obtain

$$D(B|A) + J(C|A) = D(C|A) + J(B|A) = S(\rho_A). \tag{9}$$

Thus, $D(B|A) > D(C|A)$ and $J(B|A) > J(C|A)$ are in fact equivalent, i.e., the fulfillment of one inequality implies holding of another one. Moreover, we point out here that even for mixed ρ_{ABC}, we still have $J(B|A) > J(C|A)$. This is because for any ρ_{ABC} with the purification $|\Psi\rangle_{ABCD}$, we always have $J(B|A) > J(CD|A) \geqslant J(C|A)$, where the first inequality originates from Proposition 1 (by taking CD as a combined system), and the second one is due to the fact that the classical correlation is nonincreasing under local quantum operations [20].

Eq. (8) also implies that $J(C|B) \geqslant J(B|C)$, which can be convinced by the Koashi-Winter equalities $J(C|B) = S(\rho_C) - E_f(\rho_{CA})$, and $J(B|C) = S(\rho_B) - E_f(\rho_{AB})$. Combining this with $D(A|B) + J(C|B) = S(\rho_B)$ [an equivalent form of Eq. (9)] and $D(A|B) + S(A|B) = E_f(\rho_{AC})$, we further obtain

$$E_f(\rho_{AC}) < D(A|B) \leqslant E_f(\rho_{AB}). \tag{10}$$

This equation indicates that when the UB on A is reduced, the quantum discord $D(A|B)$ is upper bounded by EoF between A and the quantum memory B, and lower bounded by EoF between A and the purifying system C. Furthermore, for pure $|\Psi\rangle_{ABC}$ Eq. (2) turns into $S(A|B) + S(A|C) = 0$, therefore combining this with the Koashi-Winter equalities of the equivalent form of Eq. (7), we have $D(A|B) + D(A|C) = E_f(\rho_{AB}) + E_f(\rho_{AC})$, and hence Eq. (10) also means $E_f(\rho_{AC}) \leqslant D(A|C) < E_f(\rho_{AB})$.

We now discuss the physical mechanism responsible for changing UB. From the proof of Proposition 1 we know that

$$S(A|B) = D(C|A) - D(B|A), \tag{11}$$

therefore $S(A|B)$ is determined by the competition between the quantum discords $D(C|A)$ and $D(B|A)$. This relation has also been noted by Fanchini *et al*. [33]. It explains why the UB is not a monotonic function of the quantum discord between A and the quantum memory B, as while $D(B|A)$ increases, $D(C|A)$ may also increases but with a faster rate, and as a result, this induces the increase of the uncertainty with increasing $D(B|A)$. To be explicitly, we consider the mixed state ρ_{AB} of the following form

$$\rho_{AB} = \sin^2\theta|\Phi\rangle\langle\Phi| + \cos^2\theta|11\rangle\langle 11|, \tag{12}$$

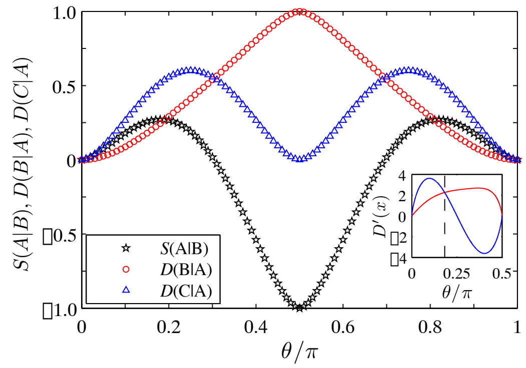

FIG. 2: (Color online) Conditional von Neumann entropy $S(A|B)$, quantum discords $D(B|A)$ and $D(C|A)$ versus θ/π for $|\Psi\rangle_{ABC}$ of Eq. (13) with $\phi = \pi/4$. The insets are derivatives of $D(x)$ with respective to θ/π, with $x = B|A$ (red), $C|A$ (blue), and the vertical dashed line represents constant 0.182.

where $|\Phi\rangle = \cos\phi|01\rangle + \sin\phi|10\rangle$ in the standard basis $\{|0\rangle, |1\rangle\}$. The purification $|\Psi\rangle_{ABC}$ for this state can be written as

$$\begin{aligned}|\Psi\rangle_{ABC} &= \sin\theta\cos\phi|011\rangle + \sin\theta\sin\phi|101\rangle \\ &\quad + \cos\theta|110\rangle,\end{aligned} \tag{13}$$

which is just the generalized *W* state [34].

For the purification $|\Psi\rangle_{ABC}$ of Eq. (13), both the reduced states ρ_{AB} and ρ_{AC} have the X structure, and therefore the discords $D(B|A)$ and $D(C|A)$ can be determined analytically [35]. In Fig. 2 we plot dependence of $S(A|B)$, $D(B|A)$ and $D(C|A)$ on θ/π with $\phi = \pi/4$, i.e., $|\Phi\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. One can see that $S(A|B)$, and thus the UB, increases with increasing values of both $D(B|A)$ and $D(C|A)$ when $\theta/\pi \in [0, 0.182]$, and decreases with decreasing values of both $D(B|A)$ and $D(C|A)$ when $\theta/\pi \in [0.818, 1]$. As illustrated in the inset of Fig. 2 with $\theta/\pi \in [0, 0.5]$, this counterintuitive phenomenon is caused by the faster increasing rate of $D(C|A)$ (the blue line) compared with that of $D(B|A)$ (the red line). Out of the above θ/π regions, either $D(B|A)$ increases more rapidly than that of $D(C|A)$, or $D(B|A)$ increases while $D(C|A)$ decreases, and therefore the UB decreases with increasing $D(B|A)$. So the behavior of UB depends on the competition of quantum discords.

Observation based on one-way unlocalizable quantum discord.—Recently, Xi *et al*. proposed the concept of one-way unlocalizable quantum discord [36], which is in some sense dual to quantum discord [20]. Here, we present some analysis of the quantum-memory-assisted entropic uncertainty relation based on this measure of correlations. By using the same semiological rules as Ref. [36], we denote $E_a(\rho_{XY})$ the entanglement of assistance for ρ_{XY} [37], while $E_u^{\leftarrow}(\rho_{XY})$ the one-way unlocalizable entanglement [38], and $\delta_u^{\leftarrow}(\rho_{XY})$ the one-way unlocalizable quantum discord [36], both for ρ_{XY} with measurements on Y, and $X, Y \in \{A, B, C\}$. Then we have the following result.

Proposition 2. When the UB on A is reduced with the aid of a quantum memory B, then both $E_u^{\leftarrow}(\rho_{BA}) > E_u^{\leftarrow}(\rho_{CA})$

and $\delta_u^{\leftarrow}(\rho_{BA}) > \delta_u^{\leftarrow}(\rho_{CA})$ are always satisfied.

Proof. By using the Buscemi-Gour-Kim equality [38], we have

$$\begin{aligned} E_a(\rho_{BC}) + E_u^{\leftarrow}(\rho_{BA}) = S(\rho_B), \\ E_a(\rho_{CB}) + E_u^{\leftarrow}(\rho_{CA}) = S(\rho_C). \end{aligned} \tag{14}$$

Substraction of the second equality of Eq. (14) from that of the first one gives rise to $E_u^{\leftarrow}(\rho_{BA}) - E_u^{\leftarrow}(\rho_{CA}) = S(\rho_B) - S(\rho_C) = -S(A|B) > 0$, and hence $E_u^{\leftarrow}(\rho_{BA}) > E_u^{\leftarrow}(\rho_{CA})$. Furthermore, combination of the definition of the one-way unlocalizable quantum discord [36] with the Buscemi-Gour-Kim equality [38] implies

$$\begin{aligned} \delta_u^{\leftarrow}(\rho_{BA}) + S(B|A) = E_a(\rho_{BC}), \\ \delta_u^{\leftarrow}(\rho_{CA}) + S(C|A) = E_a(\rho_{CB}). \end{aligned} \tag{15}$$

Then we have $\delta_u^{\leftarrow}(\rho_{BA}) - \delta_u^{\leftarrow}(\rho_{CA}) = S(C|A) - S(B|A) = -S(A|B) > 0$, and therefore $\delta_u^{\leftarrow}(\rho_{BA}) > \delta_u^{\leftarrow}(\rho_{CA})$. ■

This Proposition implies that when the UB on A is reduced using the information stored in a quantum memory B, then both the one-way unlocalizable entanglement and the one-way unlocalizable quantum discord between A and B are always larger than those between A and the purifying system C. This reinforces the interpretation of the potential maximal correlations between A and B as the essential element responsible for the reduction of the measurement uncertainty in Eq. (1).

IV. NEGATIVE CONDITIONAL ENTROPY

As the negativity of the conditional entropy plays such an important role in improving the prediction precision of the uncertainty game, we now present some possible structures of ρ_{AB} ensuring $S(A|B) < 0$. By noting the Araki-Lieb inequality [25]

$$S(\rho_{AB}) \geqslant |S(\rho_A) - S(\rho_B)|, \tag{16}$$

we see that if $S(\rho_B) - S(\rho_A) = S(\rho_{AB})$, then $S(A|B) = -S(\rho_A) \leqslant 0$ due to the non-negativity of the von Neumann entropy. $S(A|B)$ is negative if $S(\rho_A) \neq 0$, i.e., $\rho_A \neq |\mu\rangle\langle\mu|$, with $|\mu\rangle$ being the orthonormal basis of $\mathcal{H}_A$. Recently, a necessary and sufficient equality condition for the inequality (16) was derived in [39]. It states that $S(\rho_B) - S(\rho_A) = S(\rho_{AB})$ if and only if the complex Hilbert space $\mathcal{H}_B$ can be factorized as $\mathcal{H}_B = \mathcal{H}_{B^L} \otimes \mathcal{H}_{B^R}$ such that

$$\rho_{AB} = |\psi\rangle_{AB^L}\langle\psi| \otimes \rho_{B^R}, \tag{17}$$

with $|\psi\rangle_{AB^L} \in \mathcal{H}_A \otimes \mathcal{H}_{B^L}$.

In fact, for state ρ_{AB} of Eq. (17), we have

$$\begin{aligned} S(\rho_{AB}) &= S(|\psi\rangle_{AB^L}\langle\psi|) + S(\rho_{B^R}) \\ &= S(\rho_{B^R}) = S(\rho_B) - S(\rho_{B^L}) \\ &= S(\rho_B) - S(\rho_A), \end{aligned} \tag{18}$$

by using the additivity of the von Neumann entropy [25], and therefore $D(B|A) = D(B_L|A) = S(\rho_A)$. Combination of this with Eq. (9) gives $J(C|A) = 0$. Since quantum correlation cannot exist without classical correlation [20], this further implies $D(C|A) = 0$, and $J(B|A) = S(\rho_A)$, which confirms the arguments presented in Proposition 1, namely, $D(B|A) > D(C|A)$ and $J(B|A) > J(C|A)$ if $S(A|B) < 0$.

As an explicit example, consider a qubit-qudit system with $\rho_{AB} = (|00\rangle + |12\rangle)(\langle 00| + \langle 12|)/4 + (|01\rangle + |13\rangle)(\langle 01| + \langle 13|)/4$ in the standard basis $\{|\mu\nu\rangle\}_{\mu\nu=00}^{13}$. As shown in Ref. [40], this state can be factorized as in Eq. (17) with $|\psi\rangle_{AB^L} = (|00\rangle + |11\rangle)/\sqrt{2}$ and $\rho_{B^R} = \mathbb{I}_{B^R}/2$, and as a result gives the negative conditional entropy $S(A|B) = -S(\rho_A) = -1$.

Finally, note that Eq. (17) is only a sufficient condition for the negativity of $S(A|B)$, and there are bipartite states ρ_{AB} ensuring $S(A|B) < 0$, but cannot be factorized into the form of Eq. (17). An obvious example of such states is the two-qubit Werner state $\rho_{AB} = r|\Psi\rangle\langle\Psi| + (1-r)\mathbb{I}_4/4$ [41], with $|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $r \gtrsim 0.7476$.

V. SUMMARY AND DISCUSSIONS

To summarize, we have established some new physical implications of the quantum-memory-assisted entropic uncertainty relation from the perspective of correlative capacities, which are captured by the concepts of quantum discord, EoF, and the one-way unlocalizable quantum discord. The changing of the uncertainty bound is a result of competitions of various correlations between different players. We showed that whenever the prediction precision is improved compared with that with only classical memory, the observed particle A reaches its potential maximal correlative capacities with the quantum memory B in the sense that their correlations (both quantum and classical) are always larger than those between A and the purifying system C. We hope these results may shed some new light on exploring the physical implications of the entropic uncertainty principle, especially from the perspective of quantum correlations.

As a concluding remark, we would also like to point out here that the resulting certainty on the prediction of the measurement outcomes of two incompatible observables with the aid of a quantum memory may implies another kind of uncertainty. This is convinced by a variant of the "uncertainty game" with more than two players, e.g., the three-player case illustrated in Fig. 1, which shows that the more precisely the measurement outcomes of Alice are predicted by Bob, the less precisely that will be predicted by Charlie, and vice versa.

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