Channel-state duality

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On channel-state duality

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The channel-state duality refers to the correspondence between quantum channels and bipartite states, and is extremely useful and fruitful in quantum information theory. It is often called the Jamiołkowski isomorphism, the Jamiołkowski-Choi isomorphism, or the Choi-Jamiołkowski isomorphism. We trace the original roots of this duality from a historic perspective, clarify the somewhat misleading term of "isomorphism", reveal the underlying subtle nature, highlight the seemingly similar but actually different features, of the original correspondences à la Pillis, Jamiołkowski, and Choi, that lead to the duality. We further illustrate some fundamental properties and diverse applications.

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I. INTRODUCTION

The celebrated channel-state duality, usually manifested in the form of the Jamiołkowski-Choi isomorphism, refers to the statement that any channel (i.e., quantum operation, or equivalently, linear, completely positive, trace-preserving map) from the state space of an input quantum system to that of an output system corresponds to a bipartite state of the tensor product of the two relevant systems [1–6]. This correspondence links dynamics to kinematics, and is not merely mathematical, but also has fundamental physical meaning, profound consequences, and a plethora of applications.

To phrase it more explicitly, let H^a be a finite dimensional Hilbert space and $L(H^a)$ be the Hilbert space of all linear operators on H^a equipped with the Hilbert-Schmidt inner product $\langle A_1|A_2\rangle := \mathrm{tr}A_1^\dagger A_2$. Let H^b be another finite dimensional Hilbert space (which may or may not be identical to H^a). For any linear map $\mathbb{X} : L(H^a) \to L(H^b)$ sending operators on H^a to operators on H^b, consider the following two correspondences

$$\mathbb{X} \;\to\; \sigma_{\mathbb{X}} := \dagger\otimes\mathbb{X}(|\phi\rangle\langle\phi|) = \sum_{ij} e_{ij}^\dagger \otimes \mathbb{X}(e_{ij}), \qquad (1)$$

$$\mathbb{X} \;\to\; \rho_{\mathbb{X}} := \mathbb{1}\otimes\mathbb{X}(|\phi\rangle\langle\phi|) = \sum_{ij} e_{ij} \otimes \mathbb{X}(e_{ij}). \qquad (2)$$

Here $\dagger$ is the adjoint operation, and $\mathbb{1}$ is the identity channel, associated with H^a, that is, $\dagger(A) = A^\dagger$, $\mathbb{1}(A) = A$ for $A \in L(H^a)$, and $|\phi\rangle := \sum_i |i\rangle\otimes|i\rangle$ is the canonical maximally entangled (unnormalized) state in $H^a\otimes H^a$ with $\{|i\rangle\}$ an orthonormal base of H^a, $e_{ij} := |i\rangle\langle j|$. It should be emphasized that in the defining Eq. (1), $\dagger\otimes\mathbb{X}$ is only a symbolic notation, and should be understood only acting on the decomposition form $|\phi\rangle\langle\phi| = \sum_{ij} e_{ij}\otimes e_{ij}$, rather than on any general combination of tensor product of operators. Otherwise, it is not well defined. This stands in sharp contrast to $\mathbb{1}\otimes\mathbb{X}$ or the partial transpose $\mathrm{t}\otimes\mathbb{1}$, which are always well defined. The subtle point here can be readily illustrated by a simple example: For $A \in L(H^a)$, $B \in L(H^b)$, while $iA\otimes B = A\otimes B$, but $\dagger\otimes\mathbb{1}(iA\otimes B) = -iA\otimes B$, which is not equal to $\dagger\otimes\mathbb{1}(A\otimes iB) = iA\otimes B$ in general. This is apparently due to the fact that the partial complex conjugation depends on the decompositions of operators. In this work, we will take the convention to regard any non-negative operator as a quantum state, which is very convenient for our discussion without any loss of generality.

While both the correspondences defined by Eqs. (1) and (2) are rather simple isomorphisms between the entire space of linear maps from $L(H^a)$ to $L(H^b)$ and the entire space of bipartite operators on $H^a\otimes H^b$, it is a highly non-trivial and physically relevant issue to restrict the above two correspondences to certain subsets (e.g., positive maps, completely positive maps, and more specifically, channels) of the space of general linear maps, and furthermore identify the corresponding image sets. This study was pioneered by Pillis [7], Jamiołkowski [8], and Choi [9], whose results are summarized in Table I.

TABLE I: "Isomorphisms" à la Pillis, Jamiołkowski, and Choi

Who	Which isomorphism
Pillis	$\mathbb{X} \to \sigma_{\mathbb{X}}$ (general map $\to$ general operator)
Jamiołkowski	$\mathbb{X} \to \sigma_{\mathbb{X}}$ (positive map $\to$ 1-positive operator)
Choi	$\mathbb{X} \to \rho_{\mathbb{X}}$ (CP map $\to$ positive operator)

A careful study of the original and seminal papers of Pillis [7], Jamiołkowski [8], and Choi [9] shows that the correspondence defined by Eq. (1) was introduced and studied by Pillis [7] and Jamiołkowski [8], while the correspondence defined by Eq. (2), which is usually referred to as the Jamiołkowski-Choi isomorphism, was employed by Choi in the equivalent form of the so called Choi matrix $(\mathbb{X}(e_{ij}))_{ij}$, in his elegant proof of the operator-sum representation of channels [9]. This operator-sum representation in turn has its motivation and origin in Refs. [10, 11]. Thus, the original source leading to the Jamiołkowski-Choi isomorphism constitutes actually two different lines: the Pillis-Jamiołkowski isomorphism, as defined by Eq. (1), and the Choi isomorphism, as defined by Eq. (2). Since people have always lumped them together, the term

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"Jamiołkowski-Choi isomorphism" now is so popular, we will follow this convention, and call Eq. (2) the Jamiołkowski-Choi isomorphism.

It is remarkable that the channel-state duality can only be established via Eq. (2), and *cannot* via Eq. (1)! Moreover, it should be emphasized that although the Jamiołkowski-Choi isomorphism is an injection in the sense that a channel corresponds to a unique bipartite state, the converse is not true: The correspondence is not onto! There are many bipartite states which cannot be represented as $\rho_{\mathbb{X}}$. This is evident since $\mathrm{tr}_b \rho_{\mathbb{X}} = \mathbf{1}$ (identity operator). Thus strictly speaking, the term of "Jamiołkowski-Choi isomorphism" is somewhat a slight misnomer in two senses:

(i) Firstly, the original correspondence, Eq. (1), considered by Jamiołkowski, cannot be a correspondence between channels and states, it is just a correspondence between positive maps and certain bipartite operators (actually 1-positive operators, not necessarily bipartite states).

(ii) Secondly, the correspondence is never an "isomorphism" between channels and states, it is only a linear injection between these sets, although it is trivially an isomorphism between the entire spaces of general linear maps and general bipartite operators.

Apart from the above mathematical consideration of the Jamiołkowski-Choi isomorphism, there is also a physical interpretation: If one considers the system as the reduced part of a purified ambient system, with the channel acting only on the reduced system, then the correlations between the output system and the ancillary can be exploited to characterize the channel.

This paper is devoted to analyzing and synthesizing the two correspondences defined by Eqs. (1) and (2). By tracing carefully the original works leading to the channel-state duality [7–9], which is usually identified as the Jamiołkowski-Choi isomorphism [1–6], we clarify the subtle differences between various related isomorphisms, reveal their remarkable properties, investigate their intrinsic relations, and illustrate their interesting applications. More specifically, in Sec. II, we characterize and reveal some fundamental properties of the Jamiołkowski-Choi isomorphism and related isomorphism based on different initial states. We demonstrate, in some sense, the base independence of Eq. (1) and the base dependence of Eq. (2). In Sec. III, we reveal an intrinsic link between channel-state duality and certain operator transforms, and thus extend the horizon of channel-state duality to the widely studied paradigm of coherent states and associated integral transforms. In Sec. IV, we review a unified picture for the different isomorphisms in a hierarchial structure of positivity [6]. We further present some applications of the channel-state duality by translating the results from states into channels, and vice versa, in Sections V and VI. Finally, we conclude with some discussion in Sec. VII.

II. BASIS (IN)DEPENDENCE

The correspondences $\mathbb{X} \to \sigma_{\mathbb{X}}$ and $\mathbb{X} \to \rho_{\mathbb{X}}$, as defined by Eqs. (1) and (2), respectively, are based on the canonical maximally entangled state $|\phi\rangle = \sum_i |i\rangle \otimes |i\rangle \in H^a \otimes H^a$. For another maximally entangled state $|\psi\rangle$, the correspondences will be different in general. Now the natural question arises: What are the relationships between the correspondences based on different maximally entangled states? Given the fundamental importance and wide applications of the Jamiołkowski-Choi isomorphism, it is desirable to investigate this issue.

The general framework is as follows. Based on the fiducial state $|\phi\rangle$, any maximally entangled state on $H^a \otimes H^a$ can be represented as

$$|\phi_{U\otimes V}\rangle = U \otimes V|\phi\rangle = \sum_i U|i\rangle \otimes V|i\rangle,$$

where U and V are unitary operators on H^a. In analogy to Eqs. (1) and (2), we define

$$\mathbb{X} \to \sigma_{\mathbb{X},U\otimes V} := \dagger \otimes \mathbb{X}(|\phi_{U\otimes V}\rangle\langle\phi_{U\otimes V}|), \quad (3)$$
$$\mathbb{X} \to \rho_{\mathbb{X},U\otimes V} := \mathbb{1} \otimes \mathbb{X}(|\phi_{U\otimes V}\rangle\langle\phi_{U\otimes V}|), \quad (4)$$

then

$$\sigma_{\mathbb{X},U\otimes V} = (UV^\dagger \otimes \mathbf{1})\sigma_{\mathbb{X}}(UV^\dagger \otimes \mathbf{1})^\dagger, \quad (5)$$
$$\rho_{\mathbb{X},U\otimes V} = (UV^t \otimes \mathbf{1})\rho_{\mathbb{X}}(UV^t \otimes \mathbf{1})^\dagger. \quad (6)$$

Here $\mathbf{1}$ denotes the identity operator (recall that in contrast, $\mathbb{1}$ denotes the identity map on operator spaces). Noting the subtle but important difference between the above two equations: One involves the adjoint $V^\dagger$, while the other involves the transpose V^t (with respect to the computational base $\{|i\rangle\}$). In particular,

$$\sigma_{\mathbb{X},U\otimes U} = \sigma_{\mathbb{X}} \quad \text{for any unitary } U,$$

but

$$\rho_{\mathbb{X},U\otimes U} = \rho_{\mathbb{X}} \quad \text{only for real unitary } U,$$

and

$$\rho_{\mathbb{X},\bar{U}\otimes U} = \rho_{\mathbb{X}} \quad \text{for any unitary } U.$$

To establish Eq. (5), first note that

$$\begin{aligned}\sigma_{\mathbb{X},U\otimes V} &= \sum_{ij} (U|i\rangle\langle j|U^\dagger)^\dagger \otimes \mathbb{X}(V|i\rangle\langle j|V^\dagger) \\ &= \sum_{ij} U(|i\rangle\langle j|)^\dagger U^\dagger \otimes \mathbb{X}(V|i\rangle\langle j|V^\dagger) \\ &= \sum_{ij} UV^\dagger (V|i\rangle\langle j|V^\dagger)^\dagger VU^\dagger \otimes \mathbb{X}(V|i\rangle\langle j|V^\dagger) \\ &= (UV^\dagger \otimes \mathbf{1})\sigma_{\mathbb{X},V\otimes V}(UV^\dagger \otimes \mathbf{1})^\dagger.\end{aligned}$$

Thus it suffices to show that

$$\sigma_{\mathbb{X},V\otimes V} = \sigma_{\mathbb{X}},$$

or equivalently, $\sigma_{\mathbb{X},V\otimes V}$ is independent of the unitary operator V. Indeed, consider the Hilbert space $L(H^a) \otimes L(H^b)$ equipped with the Hilbert-Schmidt inner product, then for any operator $A^\dagger \otimes B \in L(H^a) \otimes L(H^b)$, we have

$$\begin{aligned}
&\langle A^\dagger \otimes B|\sigma_{\mathbb{X},V\otimes V}\rangle \\
=& \sum_{ij}\langle A^\dagger|V|j\rangle\langle i|V^\dagger\rangle \cdot \langle B|\mathbb{X}(V|i\rangle\langle j|V^\dagger)\rangle \\
=& \sum_{ij}\langle A^\dagger|V|j\rangle\langle i|V^\dagger\rangle \cdot \langle \mathbb{X}^\dagger(B)|V|i\rangle\langle j|V^\dagger\rangle \\
=& \sum_{ij}\mathrm{tr}(AV|j\rangle\langle i|V^\dagger) \cdot \mathrm{tr}((\mathbb{X}^\dagger(B))^\dagger V|i\rangle\langle j|V^\dagger) \\
=& \sum_{ij}\langle i|V^\dagger AV|j\rangle\langle j|V^\dagger(\mathbb{X}^\dagger(B))^\dagger V|i\rangle \\
=& \mathrm{tr}A(\mathbb{X}^\dagger(B))^\dagger \\
=& \langle \mathbb{X}^\dagger(B)|A\rangle \\
=& \langle B|\mathbb{X}(A)\rangle,
\end{aligned}$$

which is independent of V. In summary,

$$\langle A^\dagger \otimes B|\sigma_{\mathbb{X},V\otimes V}\rangle = \langle B|\mathbb{X}(A)\rangle, \tag{7}$$

which provides an alternative characterization of the correspondence defined by Eq. (3) specified to the special case $U = V$. In particular, since A and B are arbitrary operators, we conclude that $\sigma_{\mathbb{X},V\otimes V}$ is independent of V, and thus $\sigma_{\mathbb{X},V\otimes V} = \sigma_{\mathbb{X},\mathbf{1}\otimes\mathbf{1}} = \sigma_{\mathbb{X}}$.

Similarly, Eq. (6) follows from

$$\begin{aligned}
\rho_{\mathbb{X},U\otimes V} &= \sum_{ij} U|i\rangle\langle j|U^\dagger \otimes \mathbb{X}(V|i\rangle\langle j|V^\dagger) \\
&= \sum_{ij} U\bar{V}^\dagger\bar{V}|i\rangle\langle j|\bar{V}^\dagger\bar{V}U^\dagger \otimes \mathbb{X}(V|i\rangle\langle j|V^\dagger) \\
&= (U\bar{V}^\dagger \otimes \mathbf{1})\rho_{\mathbb{X},\bar{V}\otimes V}(U\bar{V}^\dagger \otimes \mathbf{1})^\dagger \\
&= (UV^t \otimes \mathbf{1})\rho_{\mathbb{X},\bar{V}\otimes V}(UV^t \otimes \mathbf{1})^\dagger,
\end{aligned}$$

and

$$\rho_{\mathbb{X},\bar{V}\otimes V} = \rho_{\mathbb{X}},$$

which in turn is implied by $(\bar{V} \otimes V)|\phi\rangle = |\phi\rangle$. In fact, the maximally entangled state $|\phi\rangle$ is in the so-called class of isotropic states ς (i.e., combination of the identity operator and the canonical maximally entangled states) which have the following invariant property [12]: $(\bar{V} \otimes V)\varsigma(\bar{V} \otimes V)^\dagger = \varsigma$. More generally, we have

$$(A_1 \otimes A_2)|\phi\rangle = (A_1A_2^t \otimes \mathbf{1})|\phi\rangle = (\mathbf{1} \otimes A_2A_1^t)|\phi\rangle$$

for any operators $A_1, A_2 \in L(H^a)$. To prove this, simply check that the inner products of all these expressions with $|j\rangle \otimes |k\rangle$ leads to the same quantity $\langle j|A_1A_2^t|k\rangle$.

In analogy to Eq. (7), we have

$$\langle A^\dagger \otimes B|\rho_{\mathbb{X},\bar{V}\otimes V}\rangle = \langle B|\mathbb{X}(A^t)\rangle,$$

which also readily implies that $\rho_{\mathbb{X},\bar{V}\otimes V} = \rho_{\mathbb{X},\mathbf{1}\otimes\mathbf{1}} = \rho_{\mathbb{X}}$.

Inspired by Eqs. (3) and (4), we may further define the following correspondence

$$\mathbb{X} \to \tau_{\mathbb{X},U\otimes V} := \mathrm{t} \otimes \mathbb{X}(|\phi_{U\otimes V}\rangle\langle\phi_{U\otimes V}|). \tag{8}$$

Here t means the operation of transpose, with respect to the canonical base $\{|i\rangle\}$, on the first system H^a. Clearly,

$$\tau_{\mathbb{X},U\otimes V} = \sigma_{\mathbb{X},\bar{U}\otimes V} \tag{9}$$

and in particular,

$$\tau_{\mathbb{X}} = \sigma_{\mathbb{X}}, \tag{10}$$

although in general, $\tau_{\mathbb{X},U\otimes V} \neq \sigma_{\mathbb{X},U\otimes V}$.

In view of Eqs. (9) and (10), we have a very simple method to relate the correspondences $\mathbb{X} \to \sigma_{\mathbb{X}}$ and $\mathbb{X} \to \tau_{\mathbb{X}}$. However, they are not so simply related to, and are actually very different from, the correspondence $\mathbb{X} \to \rho_{\mathbb{X}}$, as will be further illustrated late.

Combining Eqs. (5) and (9), we obtain

$$\tau_{\mathbb{X},U\otimes V} = (\bar{U}V^\dagger \otimes \mathbf{1})\tau_{\mathbb{X}}(\bar{U}V^\dagger \otimes \mathbf{1})^\dagger. \tag{11}$$

The above equation also follows readily from Eq. (6) by taking partial transpose with respect to the first system since

$$\tau_{\mathbb{X},U\otimes V} = (\mathrm{t} \otimes \mathbb{1})\rho_{\mathbb{X},U\otimes V}.$$

However, it should be emphasized that Eq. (11) *cannot* be naively derived from Eq. (5) simply by taking formal adjoint with respect to the first system, i.e., employing the operation $\dagger\otimes\mathbb{1}$, because this latter operation is not well defined on different tensor product decompositions of the operators, as already illustrated in Sec. I.

In view of the formal similarity and symmetry among the correspondences defined via Eqs. (1), (2) and (8), one might be tempted to think that these correspondences have essentially the same properties. However, they are radically different as long as positivity is involved, as will be illustrated in Sec. III. In particular, the correspondence defined by Eq. (1) is reminiscent (and indeed is the origin) of the PPT (partial positive transpose) criterion for entanglement [13, 14], and motivates us to study the closely related partial transpose correspondence defined by Eq. (8). Moreover, each correspondence have their own appealing features.

The characterizations and transformation properties of the various correspondences are summarized in Tables II and III, respectively.

TABLE II: Characterization of $\sigma_{\mathbb{X}}$, $\rho_{\mathbb{X}}$ and $\tau_{\mathbb{X}}$

Correspondence	Characterization
$\mathbb{X} \to \sigma_{\mathbb{X}}$	$\langle A^\dagger \otimes B\|\sigma_{\mathbb{X}}\rangle = \langle B\|\mathbb{X}(A)\rangle$
$\mathbb{X} \to \rho_{\mathbb{X}}$	$\langle A^\dagger \otimes B\|\rho_{\mathbb{X}}\rangle = \langle B\|\mathbb{X}(A^t)\rangle$
$\mathbb{X} \to \tau_{\mathbb{X}}$	$\langle A^\dagger \otimes B\|\tau_{\mathbb{X}}\rangle = \langle B\|\mathbb{X}(A)\rangle$

TABLE III: Base (in)dependence of $\sigma_{\mathbb{X}}$, $\rho_{\mathbb{X}}$ and $\tau_{\mathbb{X}}$

Correspondence	Covariant transformation
$\mathbb{X} \to \sigma_{\mathbb{X}}$	$\sigma_{\mathbb{X},U\otimes V} = (UV^\dagger \otimes \mathbf{1})\sigma_{\mathbb{X}}(UV^\dagger \otimes \mathbf{1})^\dagger$
$\mathbb{X} \to \rho_{\mathbb{X}}$	$\rho_{\mathbb{X},U\otimes V} = (UV^t \otimes \mathbf{1})\rho_{\mathbb{X}}(UV^t \otimes \mathbf{1})^\dagger$
$\mathbb{X} \to \tau_{\mathbb{X}}$	$\tau_{\mathbb{X},U\otimes V} = (\bar{U}V^\dagger \otimes \mathbf{1})\tau_{\mathbb{X}}(\bar{U}V^\dagger \otimes \mathbf{1})^\dagger$

III. INVERSE AND OPERATOR TRANSFORMS

The inverse of the correspondence defined by Eq. (1) is $\sigma \to \mathbb{X}_\sigma$ (here σ is a bipartite operator, not necessarily a bipartite state), with the linear map (not necessary a channel) $\mathbb{X}_\sigma : L(H^a) \to L(H^b)$ being defined by the partial inner product over $L(H^a)$ as

$$\mathbb{X}_\sigma(A) := \langle A^\dagger|\sigma\rangle_a \equiv \mathrm{tr}_a(A \otimes \mathbf{1})\sigma, \quad A \in L(H^a). \tag{12}$$

In contrast, the inverse of the correspondence defined by Eq. (2) is $\rho \to \mathbb{X}_\rho$, with $\mathbb{X}_\rho : L(H^a) \to L(H^b)$ being defined by

$$\mathbb{X}_\rho(A) := \langle A^{t\dagger}|\rho\rangle_a \equiv \mathrm{tr}_a(A^t \otimes \mathbf{1})\rho, \quad A \in L(H^a). \tag{13}$$

Indeed, for $\sigma \in L(H^a \otimes H^b)$,

$$\begin{aligned} \sigma_{\mathbb{X}_\sigma} &= (\dagger \otimes \mathbb{X}_\sigma)(|\phi\rangle\langle\phi|) \\ &= \sum_{ij} e_{ij}^\dagger \otimes \mathrm{tr}_a \sigma(e_{ij} \otimes \mathbf{1}) \\ &= \sigma, \end{aligned}$$

and for $A \in L(H^a)$,

$$\begin{aligned} \mathbb{X}_{\sigma_\mathbb{X}}(A) &= \mathrm{tr}_a\Big(\sum_{ij} A e_{ij}^\dagger \otimes \mathbb{X}(e_{ij})\Big) \\ &= \sum_{ij} \mathrm{tr}(A e_{ij}^\dagger) \cdot \mathbb{X}(e_{ij}) \\ &= \mathbb{X}\Big(\sum_{ij} \mathrm{tr}(A e_{ij}^\dagger) \cdot e_{ij}\Big) \\ &= \mathbb{X}(A). \end{aligned}$$

Similarly, we can verify that

$$\rho_{\mathbb{X}_\rho} = \rho, \quad \mathbb{X}_{\rho_\mathbb{X}} = \mathbb{X}$$

for $\rho \in L(H^a \otimes H^b)$ and $\mathbb{X} \in L(L(H^a), L(H^b))$.

Finally, since $\tau_\mathbb{X} = \sigma_\mathbb{X}$, the inverse of $\mathbb{X} \to \tau_\mathbb{X}$ is the same as that of $\mathbb{X} \to \sigma_\mathbb{X}$, and is thus given by $\tau \to \mathbb{X}_\tau$ with

$$\mathbb{X}_\tau(A) = \mathrm{tr}_a(A \otimes \mathbf{1})\tau, \quad A \in L(H^a).$$

For convenience, the inverses of the various correspondences are listed in Table IV.

TABLE IV: Inverse of $\sigma_\mathbb{X}, \rho_\mathbb{X}$ and $\tau_\mathbb{X}$ (noting that $\sigma_\mathbb{X} = \tau_\mathbb{X}$)

Correspondence	Inverse
$\mathbb{X} \to \sigma_\mathbb{X}$	$\sigma \to \mathbb{X}_\sigma$ with $\mathbb{X}_\sigma(A) = \mathrm{tr}_a(A \otimes \mathbf{1})\sigma$
$\mathbb{X} \to \rho_\mathbb{X}$	$\rho \to \mathbb{X}_\rho$ with $\mathbb{X}_\rho(A) = \mathrm{tr}_a(A^t \otimes \mathbf{1})\rho$
$\mathbb{X} \to \tau_\mathbb{X}$	$\tau \to \mathbb{X}_\tau$ with $\mathbb{X}_\tau(A) = \mathrm{tr}_a(A \otimes \mathbf{1})\tau$

The inverse correspondences actually give rise to state-induced channels. In particular, the channel defined by Eq. (12) may be interpreted as a quantum generalization of the so called coherent state transform. To see this, let us first recall the coherent state transform (Segal-Bargmann transform) [15–17], originally introduced by Segal and Bargmann [15], is the isometry $B : L^2(R) \to H^2(C)$ given by

$$Bf(x) := \int_R b(z,x) f(x) dx, \quad f \in L^2(R) \tag{14}$$

where $b(z,x) = \pi^{-1/4} e^{-z^2/2 + \sqrt{2}zx - x^2/2}$, $x \in R, z \in C$, $H^2(C)$ is the Bargmann space of analytical function on the complex plane C, square integrable with respect to $\pi^{-1} e^{-|z|^2} dxdy$ with $z = x + iy$. The states $|z\rangle = b(z,\cdot) \in L^2(R)$ are the so-called coherent states in $L^2(R)$ satisfying the completeness relation $\int_C |z\rangle\langle z| dxdy/\pi = \mathbf{1}$.

Now, we point out a remarkable analogy between the state-induced channel, as defined by Eq. (12), which is the inverse of the channel-state correspondence $\mathbb{X} \to \sigma_\mathbb{X}$ and the celebrated coherent state transform, as defined by Eq. (14). The analogy is summarized in Table V. If we make a formal correspondence between the left column and the right column, we see indeed that the channel-state duality is in some sense an operator generalization of the coherent state transform. In this sense, we may regard the channel-duality as a kind of coherent operator transform.

TABLE V: State-induced channel vs. coherent state transform

$\mathbb{X}_\sigma : L(H^a) \to L(H^b)$	$B : L^2(R) \to H^2(C)$
$\mathbb{X}_\sigma(A) = \mathrm{tr}_a(A \otimes \mathbf{1})\sigma$	$Bf(x) = \int_R b(z,x) f(x) dx$
$A \in L(H^a)$	$f \in L^2(R)$
$\sigma \in L(H^a) \otimes L(H^b)$	$b \in L^2(R) \otimes H^2(C)$
tr_a	$\int_R$

Moreover, we emphasize that in defining $\sigma_\mathbb{X}, \rho_\mathbb{X}$ and $\tau_\mathbb{X}$, we may replace $|\phi\rangle$ by other state $\sum_j |z_j\rangle \otimes |z_j\rangle$, or in continuous variable case, $\int |z\rangle \otimes |z\rangle d\mu(z)$, satisfying the completeness relation $\sum_j |z_j\rangle\langle z_j| = \mathbf{1}$ or $\int |z\rangle\langle z| d\mu(z) = \mathbf{1}$. In particular, it may happen that $\{|z_j\rangle\}$ is a over-complete family of coherent states [18–21], rather than an orthogonal base. Thus it is also desirable to study the channel-state duality in terms of superposition of coherent states. This will be pursued elsewhere.

IV. A UNIFIED PICTURE

Both Eqs. (1) and (2) lead to some isomorphisms between linear maps and bipartite operators. However, only the correspondence defined by Eq. (2) restricts to an isomorphism between channels and certain bipartite states, while the correspondence defined by Eq. (1) can never lead to such an isomorphism, and thus cannot establish the channel-state duality!

To put the various isomorphisms in a unified picture [6], we introduce the following notation. Let $L_k(L(H^a), L(H^b))$ be the set of linear map $\mathbb{X}$ from $L(H^a)$ to $L(H^b)$ which is k-positive in the sense that $\mathbb{1}_k \otimes \mathbb{X}$ is a positive map on $L(C^k) \otimes L(H^a)$, $k = 0, 1, \cdots, d, \cdots$, with $L_0(L(H^a), L(H^b)) = L(L(H^a), L(H^b))$. It is well know that whenever $k \geq d = \dim H^a$, $L_k(L(H^a), L(H^b))$ coincides with the set of completely positive maps. Let $L_k(H^a \otimes H^b)$ be the set of operators ξ on $H^a \otimes H^b$ such that

$\langle\psi|\xi|\psi\rangle \geq 0$ for any vector $|\psi\rangle \in H^a \otimes H^b$ with Schmidt rank k or less [22, 23], then clearly, $L_d(H^a \otimes H^b) := S(H^a \otimes H^b)$ is the set of (unnormalized) bipartite states on $H^a \otimes H^b$. Elements of $L_k(H^a \otimes H^b)$ are called k-positive operators. By convention, $L_0(H^a \otimes H^b) := L(H^a \otimes H^b)$. Now we can make precise the various isomorphisms established by Pillis [7], Jamiołkowski [8], and Choi [9].

(i) In the pioneering work [7], Pillis first explicitly introduced and showed that $\mathbb{X} \to \sigma_{\mathbb{X}}$ is an isomorphism between $L_0(L(H^a), L(H^b))$ and $L_0(H^a \otimes H^b)$ in his study of Hermiticity-preserving maps.

(ii) As an innovative and significant advance, Jamiołkowski first proved that $\mathbb{X} \to \sigma_{\mathbb{X}}$ is an isomorphism between $L_1(L(H^a), L(H^b))$ and $L_1(H^a \otimes H^b)$ [8]. He only addressed the issue of positive maps rather than completely positive maps.

(iii) Finally, Choi employed the matrix $(\mathbb{X}(e_{ij}))_{ij}$ which is equivalent to $\rho_{\mathbb{X}}$, and established elegantly that $\mathbb{X} \to \rho_{\mathbb{X}}$ is an isomorphism between $L_d(L(H^a), L(H^b))$ and $L_d(H^a \otimes H^b)$. Recall that $d = \dim H^a$.

The results established by Pillis and Jamiołkowski for the correspondence $\mathbb{X} \to \sigma_{\mathbb{X}}$ also carry over to the correspondence $\mathbb{X} \to \rho_{\mathbb{X}}$. Furthermore, the results can be unified in the following sense [4, 6]: The correspondence $\mathbb{X} \to \rho_{\mathbb{X}}$ is an isomorphism between $L_k(L(H^a), L(H^b))$ and $L_k(H^a \otimes H^b)$ for $k = 0, 1, 2, \cdots, d$. However, it is amazing and remarkable that this is not true for the map $\mathbb{X} \to \sigma_{\mathbb{X}}$. This latter correspondence may send a completely positive map to a negative operator on $H^a \otimes H^b$. In particular, $\sigma_{\mathbb{1}}$ is not a positive operator!

More generally, the various isomorphisms derived from the correspondences $\mathbb{X} \to \sigma_{\mathbb{X}}$ and $\mathbb{X} \to \rho_{\mathbb{X}}$ are illustrated in Tables VI and VII, respectively.

TABLE VI: Correspondence $\mathbb{X} \to \sigma_{\mathbb{X}}$ à la Pillis and Jamiołkowski

Map $\mathbb{X} : L(H^a) \to L(H^b)$	*Operator* $\sigma_{\mathbb{X}} : H^a \otimes H^b \to H^a \otimes H^b$
Linear map	*Linear operator*
Hermiticity-preserving map	*Hermitian operator*
1-Positive map	1- *positive operator*
Completely positive map	*? (Not necessary positive operator)*

TABLE VII: Correspondence $\mathbb{X} \to \rho_{\mathbb{X}}$ à la Choi

Map $\mathbb{X} : L(H^a) \to L(H^b)$	*Operator* $\rho_{\mathbb{X}} : H^a \otimes H^b \to H^a \otimes H^b$
Linear map	*Linear operator*
Hermiticity-preserving map	*Hermitian operator*
k-Positive map	*k-positive operator*
Completely positive map	*Positive operator*

We make some explanations for Table VII.

Firstly, any linear map $\mathbb{X} \in L(L(H^a), L(H^b))$ may be represented as

$$\mathbb{X}(A) = \sum_j X_j A Y_j, \qquad A \in L(H^a)$$

where $X_j \in L(H^a, H^b)$ and $Y_j \in L(H^b, H^a)$. The corresponding operator $\rho_{\mathbb{X}}$ is a general linear operator on $H^a \otimes H^b$.

Secondly, when $\mathbb{X}$ is Hermiticity-preserving in the sense that $(\mathbb{X}(A))^\dagger = \mathbb{X}(A^\dagger)$, then $\rho_{\mathbb{X}}$ is a Hermitian operator on $H^a \otimes H^b$ in the sense that $\rho_{\mathbb{X}}^\dagger = \rho_{\mathbb{X}}$. In this case, $\mathbb{X}$ can always be represented as

$$\mathbb{X}(A) = \sum_j \epsilon_j X_j A X_j^\dagger$$

with $\epsilon_j = 1, -1$.

Thirdly, $\mathbb{X}$ is k-positive in the sense that $\mathbb{1}_k \otimes \mathbb{X}$ is a positive map associated with $C^k \otimes H^a$ (thus the conventional positivity is the same as 1-positivity) if and only if $\rho_{\mathbb{X}}$ is k-positive in the sense that $\langle\psi|\rho_{\mathbb{X}}|\psi\rangle \geq 0$ for any vector $|\psi\rangle$ with Schmidt rank less than or equal to k. A vector has Schmidt rank 1 if and only if it is a product state (separable), it follows that this condition reduces to that first established by Jamiołkowski [8]. Usually, it is difficult to determine whether a map is positive or not.

Lastly, $\mathbb{X}$ is completely positive (a channel) in the sense that $\mathbb{1}_k \otimes \mathbb{X}$ is a positive map for any k (it suffices for $k = d$, the dimension of H^a), if and only if $\rho_{\mathbb{X}}$ is a positive operator (a positive operator with unital partial trace over H^b) on $H^a \otimes H^b$. In this case, $\mathbb{X}$ can always be represented as

$$\mathbb{X}(A) = \sum_j X_j A X_j^\dagger.$$

Since any Hermitian operator can be represented as the difference of two positive operators, it follows that every Hermiticity-preserving map can be written as the difference of two completely positive maps. To emphasize, the channel-state duality is a particular instance of the last scenario: A channel from $L(H^a)$ to $L(H^b)$ corresponds to a bipartite state on $H^a \otimes H^b$ with unital partial trace over H^b.

V. COMPOSING STATES VIA CHANNELS

There is a natural composition law for channels, which may be translated to states via the channel-state duality [1, 5]. Accordingly, the correlations in bipartite states may be endowed with a product structure.

Consider the relations between $\rho_{\mathbb{XY}}$ with $\rho_{\mathbb{X}}$ and $\rho_{\mathbb{Y}}$, we define

$$\rho_{\mathbb{X}} \circ \rho_{\mathbb{Y}} := \rho_{\mathbb{XY}}.$$

More explicitly,

$$\rho_{\mathbb{XY}} = \sum_{ij} e_{ij} \otimes \mathbb{XY}(e_{ij})$$

for $\rho_{\mathbb{X}} = \sum_{ij} e_{ij} \otimes \mathbb{X}(e_{ij})$, $\rho_{\mathbb{Y}} = \sum_{ij} e_{ij} \otimes \mathbb{Y}(e_{ij})$. This composition (product) of certain bipartite states has the following interesting properties.

(i) It is not commutative. More precisely,

$$\rho_{[\mathbb{X},\mathbb{Y}]} = \rho_{\mathbb{X}} \circ \rho_{\mathbb{Y}} - \rho_{\mathbb{Y}} \circ \rho_{\mathbb{X}}.$$

Thus, $\rho_{\mathbb{X}} \circ \rho_{\mathbb{Y}} = \rho_{\mathbb{Y}} \circ \rho_{\mathbb{X}}$ if and only if $\mathbb{XY} = \mathbb{YX}$.

(ii) The maximally entangled state $|\phi\rangle\langle\phi| = \rho_{\mathbb{1}}$ is a unit for this product since

$$\rho_{\mathbb{1}} \circ \rho_{\mathbb{X}} = \rho_{\mathbb{X}} \circ \rho_{\mathbb{1}} = \rho_{\mathbb{X}}.$$

(iii) The quantum mutual information is decreasing under the composition in the sense that

$$I(\rho_{\mathbb{X}} \circ \rho_{\mathbb{Y}}) \leq \min\{I(\rho_{\mathbb{X}}), I(\rho_{\mathbb{Y}})\}.$$

Here $I(\cdot)$ denotes the quantum mutual information of the associated normalized state [24–26]. This follows from the monotonicity of the quantum mutual information under local channels [24, 25]. Moreover, $I(\rho_{\mathbb{X}} \circ \rho_{\mathbb{Y}}) = I(\rho_{\mathbb{X}})$ if and only if $\mathbb{Y}$ is reversible, and $I(\rho_{\mathbb{X}} \circ \rho_{\mathbb{Y}}) = I(\rho_{\mathbb{Y}})$ if and only if $\mathbb{X}$ is reversible. The extent of reversibility of the channel $\mathbb{X}$ is related to, and can be quantified by, the correlations in the corresponding state $\rho_{\mathbb{X}}$. In particular, we may define an index for the reversibility of $\mathbb{X}$ as

$$R(\mathbb{X}) := \frac{I(\rho_{\mathbb{X}})}{I(\rho_{\mathbb{1}})}.$$

Thus, $\mathbb{X}$ is reversible if and only if $R(\mathbb{X}) = 1$. In contrast, if $\mathbb{X}$ is a complete decoupling channel in the sense that $\rho_{\mathbb{X}}$ is a product state, then $R(\mathbb{X}) = 0$.

Conversely, one may also induce certain composition structures of channels from those of states such as the Hadamard (Schur) product [2]. Their implications and applications remain further investigations.

VI. CLASSIFYING AND COMPARING CHANNELS VIA STATES

Since bipartite states are well classified according to correlations therein [27–29], we may classify channels by translating the classifications in bipartite states with the mediation of the channel-state duality, and investigate the conditions for channels in order to generate corresponding states with certain correlation structures. We illustrate this general idea by a variety of examples and applications.

(i) A bipartite state ρ is a product state if $\rho = \rho^a \otimes \rho^b$, otherwise it is correlated. Now the questions arises as which channel corresponds to a product state, and which corresponds to a correlated one, under the correspondence $\mathbb{X} \to \rho_{\mathbb{X}}$. We call a channel $\mathbb{X}$ a completely decoupling channel if $\rho_{\mathbb{X}}$ is a product state. From Eq. (2), we readily conclude that a channel is completely decoupling if and only if $\mathbb{X}$ is fully degenerate in the sense that it sends any operator to a fixed one: $\mathbb{X}(A) = B_0$ for any $A \in L(H^a)$ and some fixed $B_0 \in L(H^b)$.

(ii) In the entanglement and separability paradigm [27], a bipartite state ρ is called separable if it can be represented as

$$\rho = \sum_j \rho_j^a \otimes \rho_j^b,$$

where ρ_j^a and ρ_j^b are (unnormalized) states for H^a and H^b, respectively. Otherwise it is called entangled (recall that we have taken the convention to regard any non-negative operator as a state). According to the elegant results of Horodecki *et al.* [30], a channel $\mathbb{X}$ leads to a separable state if and only if it is an entanglement breaking channel in the sense that [30–32]

$$\mathbb{X}(A) = \sum_j \mathrm{tr}(AE_j^a) \cdot \rho_j^b, \quad A \in L(H^a),$$

i.e., a measurement-and-preparation map. Here $\{E_j^a\}$ is a POVM on H^a, and ρ_j^b are positive operators on H^b.

(iii) In the classical and quantum scenario of correlations [29, 33], a bipartite state is called quantum-classical (correlated) if it can be represented as

$$\rho = \sum_j \rho_j^a \otimes |j\rangle_b\langle j|.$$

Here ρ_j^a are (unnormalized) states of H^a, and $\{|j\rangle_b\}$ is an orthonormal base for H^b. Otherwise it is called quantum correlated. According to Refs. [34, 35], a channel is quantum-classical if and only if it is a measurement map in the sense that

$$\mathbb{X}(A) = \sum_j \mathrm{tr}(AE_j^a) \cdot \Pi_j^b, \quad A \in L(H^a).$$

Here $\{E_j^a\}$ is a POVM on H^a, and $\{\Pi_j^b\}$ are orthogonal positive operators on H^b.

(iv) Translating the symmetric characterization of the Werner states and isotropic states [12, 27], we may define the corresponding Werner channels and isotropic channels. Recall that a bipartite state ω on $H^a \otimes H^b$ with $H^b = H^a$ is called a Werner state if it is invariant in the sense that $(U \otimes U)\omega(U \otimes U)^\dagger = \omega$. In contrast, a state ς is called an isotropic state if $(\bar{U} \otimes U)\varsigma(\bar{U} \otimes U)^\dagger = \varsigma$. Explicitly, a Werner state and an isotropic state are, respectively, of the form [12, 27, 28]

$$\begin{aligned} \omega &= \frac{d-x}{d^3-d}\mathbf{1} + \frac{dx-1}{d^3-d}F, \qquad x \in [-1, 1] \\ \varsigma &= \frac{d-y}{d^3-d}\mathbf{1} + \frac{dy-1}{d^3-d}|\phi\rangle\langle\phi|, \quad y \in [0, d]. \end{aligned}$$

Here $d = \dim H^a$, $F := \sum_{ij} |i\rangle\langle j| \otimes |j\rangle\langle i|$, $|\phi\rangle := \sum_i |i\rangle \otimes |i\rangle$, and $\mathbf{1}$ is the identity operator on $H^a \otimes H^a$.

Let the channel $\mathbb{P}$ be the von Neumann measurement along the base $\{|i\rangle\}$, i.e., $\mathbb{P}(A) = \sum_i \langle i|A|i\rangle|i\rangle\langle i|$. It is interesting to note that

$$\sigma_{\mathbb{P}} = \rho_{\mathbb{P}} = \mathbf{1}, \quad F = \sigma_{\mathbb{1}} = \rho_{\mathrm{t}}, \quad |\phi\rangle\langle\phi| = \rho_{\mathbb{1}}.$$

Thus

$$\begin{aligned} \omega &= \frac{d-x}{d^3-d}\rho_{\mathbb{P}} + \frac{dx-1}{d^3-d}\rho_{\mathrm{t}}, \quad x \in [-1, 1] \\ \varsigma &= \frac{d-y}{d^3-d}\rho_{\mathbb{P}} + \frac{dy-1}{d^3-d}\rho_{\mathbb{1}}, \qquad y \in [0, d] \end{aligned}$$

which recast the Werner state and the isotropic states in a symmetrical position. Consequently, we may call

$$\begin{aligned} \mathbb{X}_\omega &= \frac{d-x}{d^3-d}\mathbb{P} + \frac{dx-1}{d^3-d}\mathrm{t} \\ \mathbb{X}_\varsigma &= \frac{d-y}{d^3-d}\mathbb{P} + \frac{dy-1}{d^3-d}\mathbb{1} \end{aligned}$$

the Werner channel and the isotropic channel, respectively. These can also be derived directly from the inversion formulas (12) and (13) .

(v) For the random unitary channel

$$\mathbb{X}(A) = \sum_i p_i U_i A U_i^\dagger, \quad A \in L(H^a)$$

the corresponding states are

$$\sigma_{\mathbb{X}} = \sum_i p_i \sigma_{\mathbb{1},\mathbf{1}\otimes U_i}, \quad \rho_{\mathbb{X}} = \sum_i p_i \rho_{\mathbb{1},\mathbf{1}\otimes U_i}.$$

In particular, for the Pauli channel

$$\mathbb{X}(A) = \sum_{j=0}^{3} p_j \sigma_j A \sigma_j, \quad A \in L(H^a)$$

the corresponding state

$$\rho_{\mathbb{X}} = p_0|\phi^+\rangle\langle\phi^+| + p_1|\psi^+\rangle\langle\psi^+| + p_2|\psi^-\rangle\langle\psi^-| + p_3|\phi^-\rangle\langle\phi^-|$$

is essentially a Bell-diagonal state, which can be alternatively cast in the form [36]

$$\rho_{\mathbb{X}} = \sum_{j=0}^{3} c_j \sigma_j \otimes \sigma_j$$

with $c_0 = 1/4, c_1 = (p_0 + p_1 - p_2 - p_3)/4$, $c_2 = (-p_0 + p_1 - p_2 + p_3)/4$, $c_3 = (p_0 - p_1 - p_2 + p_3)/4$. Here $\{p_j\}$ is a probability distribution, $\sigma_0 = \mathbf{1}$ and σ_j are the Pauli matrices, $|\phi^\pm\rangle = |0\rangle\otimes|0\rangle \pm |1\rangle\otimes|1\rangle, |\psi^\pm\rangle = |0\rangle\otimes|1\rangle \pm |1\rangle\otimes|0\rangle$. In fact, the Pauli channels and the Bell-diagonal states are in one-to-one correspondence.

(vi) Two channels $\mathbb{X}$ and $\mathbb{Y}$ are unitary equivalent if $\mathbb{Y} = \mathbb{U}_1 \circ \mathbb{X} \circ \mathbb{U}_2$. Here the unitary channels $\mathbb{U}_j$ is defined as $\mathbb{U}_j(A) := U_j A U_j^\dagger$ with U_j a unitary operator. In view of Eq. (5), $\mathbb{X}$ and $\mathbb{Y}$ are unitary equivalent if and only if $\sigma_{\mathbb{X}}$ and $\sigma_{\mathbb{Y}}$ are *locally* unitary equivalent as bipartite states. This follows from the equivalence between $\mathbb{Y} = \mathbb{U}_1 \circ \mathbb{X} \circ \mathbb{U}_2$ and

$$\begin{aligned}
\sigma_{\mathbb{Y}} &= \sum_{ij} e_{ij}^\dagger \otimes U_1 \mathbb{X}(U_2 e_{ij} U_2^\dagger) U_1^\dagger \\
&= (\mathbf{1} \otimes U_1)\sigma_{\mathbb{X},\mathbf{1}\otimes U_2}(\mathbf{1} \otimes U_1)^\dagger \\
&= (\mathbf{1} \otimes U_1)(U_2^\dagger \otimes \mathbf{1})\sigma_{\mathbb{X}}(U_2^\dagger \otimes \mathbf{1})^\dagger(\mathbf{1} \otimes U_1)^\dagger \\
&= (U_2^\dagger \otimes U_1)\sigma_{\mathbb{X}}(U_2^\dagger \otimes U_1)^\dagger.
\end{aligned}$$

Similarly, $\mathbb{X}$ and $\mathbb{Y}$ are unitary equivalent if and only if $\rho_{\mathbb{X}}$ and $\rho_{\mathbb{Y}}$ are *locally* unitary equivalent as bipartite states. In general, it is quite difficult to determine whether two channels are unitary equivalent or not. Now, thanks to the recent results in Ref. [37], we can determine unitary equivalence of any two channels easily by checking several standard invariants of the derived bipartite states via the channel-state duality.

(vii) The Jamiołkowski-Choi isomorphism provides a convenient method for analyzing the decoherent properties of channels via correlations. Given two channels $\mathbb{X}$ and $\mathbb{Y}$ acting on the same space, it is desirable to tell whether there exists a channel $\mathbb{T}$ such that $\mathbb{Y} = \mathbb{T} \circ \mathbb{X}$. If there is the case, we say that $\mathbb{Y}$ is a coarse graining of $\mathbb{X}$, and write symbolically $\mathbb{Y} \geq \mathbb{X}$, which gives a natural partial order for the set of channels. This is reminiscent of the "cleanness" of POVMs studied in Ref. [38]. An extremely important issue is to determine whether $\mathbb{Y}$ is a coarse graining of $\mathbb{X}$. In general, this is a quite difficult problem. However, via the channel-state duality, we can obtain some convenient necessary conditions for checking this. More precisely, if $\mathbb{Y} = \mathbb{T} \circ \mathbb{X}$, then by the monotonicity of the quantum mutual information,

$$I(\rho_{\mathbb{Y}}) = I(\rho_{\mathbb{T}\circ\mathbb{X}}) \leq I(\rho_{\mathbb{X}}).$$

Consequently, whenever the above inequality is violated, then $\mathbb{Y}$ cannot be the coarse graining of $\mathbb{X}$. Recall that a quantum dynamics $\mathcal{X} = \{\mathbb{X}_t\}$ is Markovian if the channels $\mathbb{X}_t$ is always a coarse graining of $\mathbb{X}_s$ whenever $s < t$. Thus $I(\rho_{\mathbb{X}_t})$ is a decreasing function of t. This observation has a remarkable application in characterizing and quantifying non-Markovian dynamics. Following Ref. [39], a measure of non-Markovianity of a dynamical processes $\mathcal{X} = \{\mathbb{X}_t\}$ on the system H^a may be defined as

$$\mathcal{N}_0(\mathcal{X}) := \int_{\frac{d}{dt}I(\rho_{\mathbb{X}_t})>0} \frac{d}{dt} I(\rho_{\mathbb{X}_t}) dt,$$

which synthesizes the degree of violation of coarse graining effect of Markovian dynamics.

Moreover, the natural question arises as how does the non-Markovianity measure depend on the choice of the initial maximally entangled state $|\phi\rangle$? In view of Eq. (6) and the fact that quantum mutual information cannot be changed by local unitary operations, we have $I(\rho_{\mathbb{X},U\otimes V}) = I(\rho_{\mathbb{X}})$ for any unitary operators U and V. Accordingly, $I(\rho_{\mathbb{X}_t}) = I(\rho_{\mathbb{X}_t,U\otimes V})$, and thus $\mathcal{N}_0(\mathcal{X})$ is indeed an intrinsic quantity independent of the choice of the initial maximally entangled state. Similarly, some further implications of Eq. (6) are that the various decoherence measures introduced in [40] are also intrinsic quantities independent of the choice of the initial maximally entangled state.

VII. DISCUSSION

Although the channel-state duality has been extensively studied and widely used, we have revealed some subtle points concerning the structures and properties of the original, closely related but quite different correspondences which lead to the duality. The first was introduced by Pillis [7] and further investigated by Jamiołkowski [8], while the second was suggested by Choi [9]. The celebrated Jamiołkowski-Choi isomorphism refers to the second one, and is usually the precise formulation of the so called channel-state duality. We have pointed out that the Jamiołkowski-Choi isomorphism is neither the original map studied by Jamiołkowski, nor an isomorphism between channels and bipartite states in a strict sense.

The channel-state duality relies on the choice of a reference, which is a maximally entangled state. We have characterized completely the relationship between different dualities based on different reference states.

By exploiting the channel-state duality, we have illustrated some applications and implications. By translating results for channels, we obtain corresponding properties for bipartite states, and vice verse. Richer results stem from the channel-state duality with profound consequences, which need further exploitation.

We have only treated finite dimensional cases. It will also be important to study the infinite dimensional cases and continuous variable cases. Some significant advance has been made by Holevo and others [31, 32]. Finally, we emphasized the channel state duality can also be established via (non-orthogonal) coherent states, rather than orthonormal states, as long as the completeness relation is satisfied.

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