Superradiance transition in a system with a single qubit and a single oscillator

S. Ashhab

Phys. Rev. A **87**, 013826 — Published 23 January 2013

DOI: 10.1103/PhysRevA.87.013826

Supperradiance transition in a system with a single qubit and a single oscillator

S. Ashhab[1, 2]

[1] *Advanced Science Institute, RIKEN, Wako-shi, Saitama 351-0198, Japan*
[2] *Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA*

We consider the phase-transition-like behaviour in the Rabi model containing a single two-level system, or qubit, and a single harmonic oscillator. The system experiences a sudden transition from an uncorrelated state to an increasingly correlated one as the qubit-oscillator coupling strength is varied and increased past a critical point. This singular behaviour occurs in the limit where the oscillator's frequency is much lower than the qubit's frequency; away from this limit one obtains a finite-width transition region. By analyzing the energy-level structure, the value of the oscillator field and its squeezing and the qubit-oscillator correlation, we gain insight into the nature of the transition and the associated critical behaviour.

I. INTRODUCTION

The interaction between light and matter, and more generally between harmonic oscillators and few-level quantum systems, is ubiquitous in nature. In spite of the simplicity in its basic mathematical description, it results in a wide variety of phenomena, some of which have been analyzed in detail over the past few decades [1].

One of the interesting phenomena involving light-matter interaction is superradiance. The study of this phenomenon started with the idea that an ensemble containing a large number of atoms can exhibit quantum-coherent collective behaviour in its absorption and emission of photons [2]. This observation gave rise to the Dicke model, where a large number of atoms interact with a single (harmonic-oscillator) mode of the electromagnetic field. It was later realized that the Dicke model exhibits a phase transition, both thermal and quantum, between a state with negligible light-matter correlations and one with strong correlations [3–6]. In the case of the quantum phase transition, the correlations appear when the coupling strength between the two subsystems exceeds a certain critical value.

The prediction of the superradiance phase transition in the Dicke model has resulted in enormous interest, including a debate that continues to this day [7] on whether such a phase transition could occur for a system with the usual electric coupling between light and matter.

Studies on the superradiance phase transition in the Dicke model typically consider the thermodynamic limit, where the number of atoms approaches infinity with the effective coupling strength between the collective atomic mode and the electromagnetic mode kept independent of atom number. This approach to realizing the phase transition is naturally motivated by the fact that the interaction between natural atoms and optical-frequency cavities is weak compared to the bare atomic and cavity frequencies.

Recently, the realization of qubit-oscillator systems using superconducting qubit circuits has made it possible to achieve the so-called ultrastrong-coupling regime, where the coupling strength between a single qubit and a single oscillator is comparable to the bare frequencies of the two constituents [8–12]. This ability relaxes the requirement of using large atomic ensembles in order to study strong-coupling effects; a single (artificial) atom suffices. Since the artificial atom in these studies is effectively a two-level system, we shall sometimes refer to it as the qubit.

It has been noted in a number of recent studies that the single-qubit-single-oscillator problem exhibits similar phase-transition-like behaviour in the limit where the ratio of the resonator frequency to the qubit frequency tends to zero [13–16]. Here we examine this limit closely and analyze the associated transition. We do so by analyzing the behaviour of several physical quantities in the transition region. These include the energy-level structure, the average value of the field in the cavity, the squeezing in the oscillator and the qubit-oscillator entanglement. It should be emphasized that the limit considered here is clearly distinct from the thermodynamic limit with a large number of qubits. We shall therefore not use the term "phase transition" in this paper. It is quite interesting that several ground-state properties exhibit essentially the same behaviour in the two distinct limits. The correspondence between the two limits is not complete, however, as evidenced by the fact that no thermal phase transition occurs in the simple system considered here (see Sec. V).

II. MODEL HAMILTONIAN

The system that we consider here is composed of a single qubit coupled to a single harmonic oscillator. The coupling contains only one term, and this term is linear in the oscillator variables. The Hamiltonian describing this quantum system is given by:

$$\hat{H} = \hat{H}_{\rm q} + \hat{H}_{\rm ho} + \hat{H}_{\rm int}, \tag{1}$$

where

$$\begin{aligned} \hat{H}_{\rm q} &= -\frac{\Delta}{2}\hat{\sigma}_x - \frac{\epsilon}{2}\hat{\sigma}_z \\ \hat{H}_{\rm ho} &= \hbar\omega_0 \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\omega_0 \end{aligned} \tag{2}$$

$$\hat{H}_{\rm int} \;=\; \lambda\left(\hat{a}+\hat{a}^{\dagger}\right)\hat{\sigma}_z,$$

$\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the usual Pauli matrices (with $\hat{\sigma}_z\left|\uparrow\right\rangle = \left|\uparrow\right\rangle, \hat{\sigma}_z\left|\downarrow\right\rangle = -\left|\downarrow\right\rangle$), and $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators of the harmonic oscillator. The parameters Δ and ϵ are the so-called gap and bias of the qubit, ω_0 is the oscillator's characteristic frequency, and λ is the qubit-oscillator coupling strength.

For purposes of the present study, we focus on the case with the qubit biased at its symmetry point ($\epsilon = 0$), where the Hamiltonian can simply be expressed as:

$$\hat{H} = -\frac{\Delta}{2}\hat{\sigma}_x + \hbar\omega_0\hat{a}^{\dagger}\hat{a} + \lambda\left(\hat{a}+\hat{a}^{\dagger}\right)\hat{\sigma}_z. \qquad (3)$$

It should also be noted that in writing this Hamiltonian [and also in Eq. (2)] we ignore the so-called A^2-term that is at the heart of the superradiance-phase-transition controversy [7].

In the absence of coupling, i.e. when $\lambda = 0$, the ground state of the system is given by

$$\left|{\rm GS}\right\rangle_{\lambda=0} = \frac{\left|\uparrow\right\rangle + \left|\downarrow\right\rangle}{\sqrt{2}} \otimes \left|{\rm vac}\right\rangle. \qquad (4)$$

For very strong coupling, the quantitative definition of which will become clear below, the ground state is highly correlated and (to a good approximation) given by

$$\left|{\rm GS}\right\rangle_{({\rm large}\;\lambda)} = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle\otimes\left|\alpha\right\rangle + \left|\downarrow\right\rangle\otimes\left|-\alpha\right\rangle\right), \qquad (5)$$

where $\left|\pm\alpha\right\rangle$ are coherent states with opposite values of the oscillator variable that couples to the qubit, i.e. the field operator $(\hat{a}+\hat{a}^{\dagger})/2$.

III. TRANSITION POINT AND CRITICAL BEHAVIOUR

Using the results from the thermodynamic limit, i.e. the limit where the number of atoms is large, the critical coupling strength separating the uncorrelated and correlated ground states is expected to occur at the point

$$\lambda_{\rm c} = \frac{\sqrt{\hbar\omega_0\Delta}}{2}. \qquad (6)$$

We shall show through the behaviour of various quantities that similar behaviour is obtained in the single-atom case in the limit $\hbar\omega_0/\Delta \rightarrow 0$. It should be emphasized that, even though one might worry about the possibility that $\lambda_{\rm c}$ might vanish or diverge in this limit, this apparent complication disappears if one treats $\lambda_{\rm c}$ as a reference point for measuring the coupling strength. If one then considers the behaviour of the system as the parameter $\lambda/\lambda_{\rm c}$ is varied across the point $\lambda/\lambda_{\rm c} = 1$, no complications related to the behaviour of $\lambda_{\rm c}$ arise.

The tendency towards singular behaviour (in the dependence of various physical quantities on λ) in the limit $\hbar\omega_0/\Delta \rightarrow 0$ is illustrated in Figs. 1-5. In these figures, the entanglement, spin-field correlation function, low-lying energy levels (measured from the ground state) and the oscillator's squeezing parameter are plotted as functions of the coupling strength. It is clear from Figs. 2 and 3 that when $\hbar\omega_0/\Delta \leq 10^{-3}$ both the entanglement (which is quantified through the von-Neumann entropy $S = {\rm Tr}\{\rho_{\rm q}\log_2\rho_{\rm q}\}$ with $\rho_{\rm q}$ being the qubit's reduced density matrix) and the correlation function $C = \langle\sigma_z {\rm sign}(a+a^{\dagger})\rangle$ rise sharply upon crossing the crit-

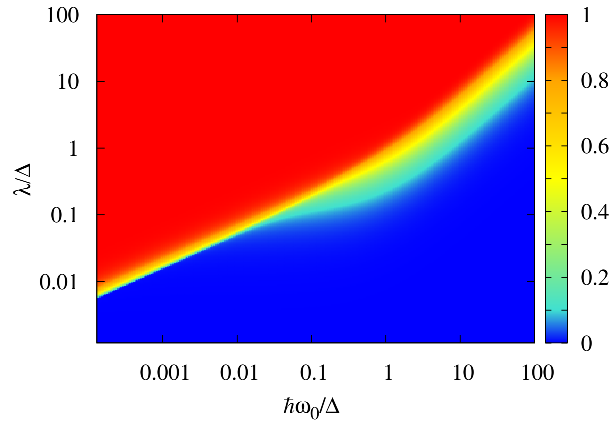

FIG. 1: (Color online) The von-Neumann entropy S as a function of the oscillator frequency $\hbar\omega_0$ and the coupling strength λ, both measured in comparison to the qubit frequency Δ. One can see clearly that moving in the vertical direction the rise in entropy is sharp in the regime $\hbar\omega_0/\Delta \ll 1$, whereas it is smooth when $\hbar\omega_0/\Delta$ is comparable to or larger than 0.1.

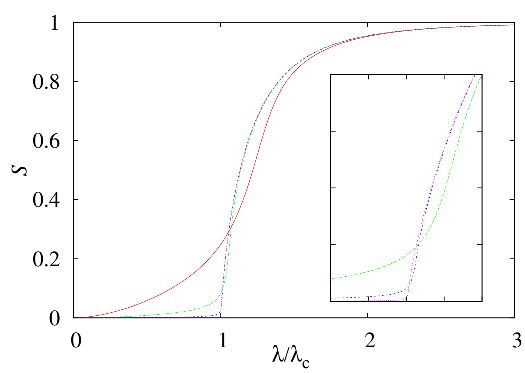

FIG. 2: (Color online) The von-Neumann entropy S as a function of coupling strength λ [measured in comparison to the critical coupling strength given by Eq. (6)] for various values of the ratio $\hbar\omega_0/\Delta$. In particular, we use the values 10^{-1} (red solid line), 10^{-2} (green dashed line), 10^{-3} (blue short-dashed line) and 10^{-4} (purple dotted line). The inset shows a magnified plot in the region around the critical point: the ranges of the x and y axes are [0.9,1.1] and [0,0.4], respectively. As the ratio $\hbar\omega_0/\Delta$ decreases, the onset of entropy becomes increasingly sudden.

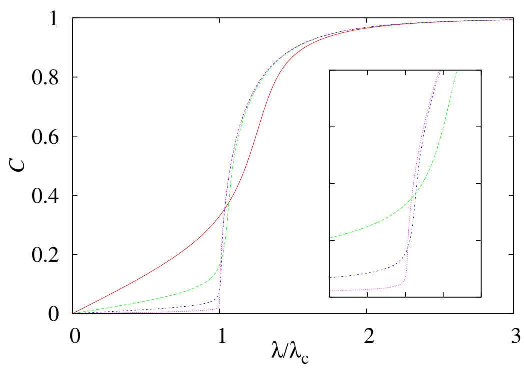

FIG. 3: (Color online) The correlation function $C = \langle \sigma_z \mathrm{sign}(a + a^\dagger) \rangle$ as a function of coupling strength λ [measured in comparison to the critical coupling strength given by Eq. (6)] for various values of the ratio $\hbar\omega_0/\Delta$: 10^{-1} (red solid line), 10^{-2} (green dashed line), 10^{-3} (blue short-dashed line) and 10^{-4} (purple dotted line). The inset shows a magnified plot in the region around the critical point: the ranges of the x and y axes are [0.9,1.1] and [0,0.4], respectively. The correlation function C exhibits behaviour similar to that of the entropy, which is shown in Fig. 2.

ical point [17]. The low-lying energy levels, shown in Fig. 4, approach each other to form a large group of almost degenerate energy levels at the critical point before they separate again into pairs of asymptotically degenerate energy levels. This approach is not complete, however, even when $\hbar\omega_0/\Delta = 10^{-3}$; for this value the energy level spacing in the closest-approach region is roughly ten times smaller than the energy level spacing at $\lambda = 0$. The squeezing parameter is defined by the width of the momentum distribution relative to that in the case of an isolated oscillator. For consistency with Ref. [13], we define it as:

$$s_p + 1 = \frac{\langle \hat{p}^2 \rangle}{\langle \hat{p}^2 \rangle|_{\lambda=0}}, \tag{7}$$

where $\hat{p}$ is the oscillator's momentum operator, which is proportional to $i(\hat{a}^\dagger - \hat{a})$ in our definition of the operators. The squeezing parameter mirrors the behaviour of the low-lying energy levels. In particular we can see from Fig. 5 that only when $\hbar\omega_0/\Delta$ reaches the value 10^{-5} does the squeezing become almost singular at the critical point.

We now look more closely at the critical exponents around the critical point. Analytical expressions describing the critical behaviour of some quantities can be obtained using a semiclassical calculation [13]. In particular, this approximation gives the result that just below the critical point, the energy-level separation has the functional dependence:

$$E_n - E_{n-1} = \sqrt{2}\hbar\omega_0 \left(1 - \frac{\lambda}{\lambda_c}\right)^{1/2}, \tag{8}$$

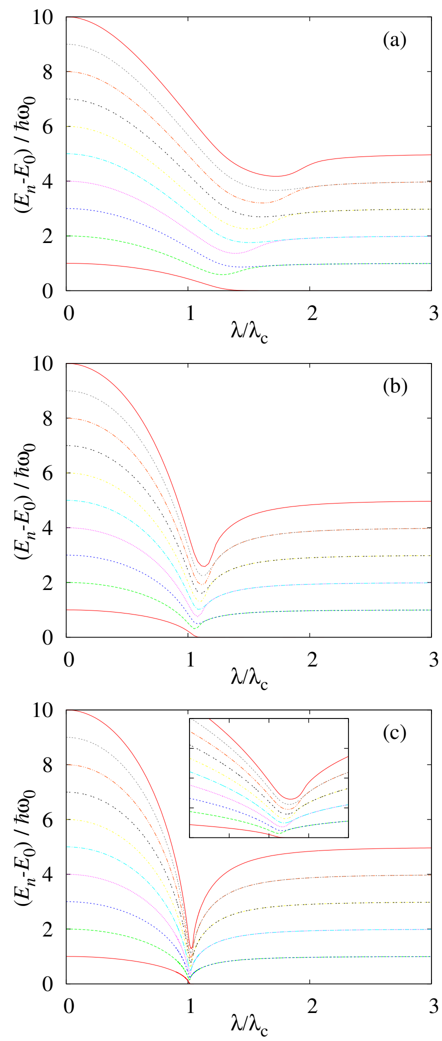

FIG. 4: (Color online) The energy levels of the first ten excited states relative to the ground state energy as functions of the coupling strength λ [measured in comparison to the critical coupling strength given by Eq. (6)]. In (a) we take $\hbar\omega_0/\Delta = 10^{-1}$, in (b) we take $\hbar\omega_0/\Delta = 10^{-2}$, and in (c) we take $\hbar\omega_0/\Delta = 10^{-3}$. The energy levels become increasingly dense as the coupling strength approaches the critical value, and they separate again (forming pairs) after the coupling strength exceeds the critical value. The inset in (c) shows a magnified plot in the region around the critical point: the ranges of the x and y axes are [0.9,1.1] and [0,4], respectively.

while just above the critical point it is given by

$$E_n - E_{n-2} = 2\hbar\omega_0 \left(\frac{\lambda}{\lambda_c} - 1\right)^{1/2}. \tag{9}$$

The squeezing parameter should exhibit the same behaviour. The numerical results shown in Figs. 4 and 5 approach this functional dependence on both sides of the critical point as we decrease the ratio $\hbar\omega_0/\Delta$. Under the same semiclassical approximation, one can also analytically calculate the average value of the field, i.e. $\langle (\hat{a} + \hat{a}^\dagger)/2 \rangle$ [In the superradiance region one calcu-

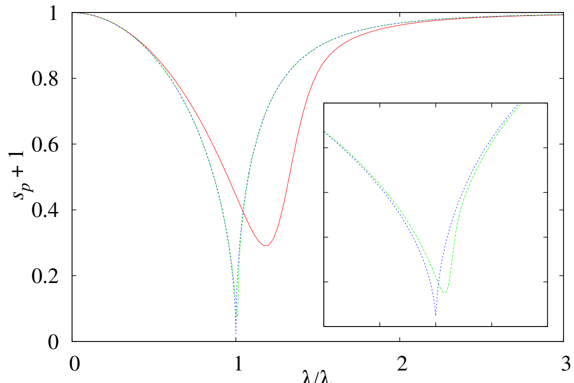

FIG. 5: (Color online) The squeezing quantifier $s_p + 1$ as a function of the coupling strength λ [measured in comparison to the critical coupling strength given by Eq. (6)]. The different lines correspond to $\hbar\omega_0/\Delta = 10^{-1}$ (solid red line), 10^{-3} (dashed green line) and 10^{-5} (short-dashed blue line). The inset shows a magnified plot in the region around the critical point: the ranges of the x and y axes are [0.9,1.1] and [0,0.5], respectively. The behaviour of the squeezing parameter mirrors that of the low-lying excited states shown in Fig. 4.

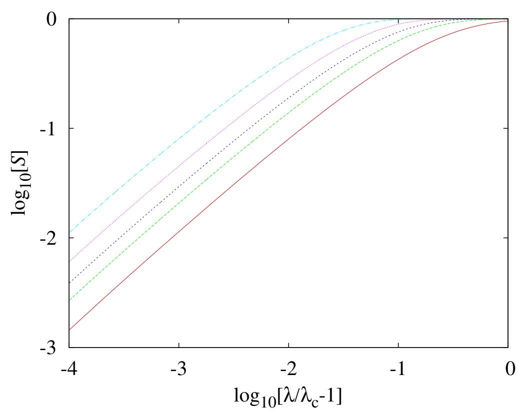

FIG. 6: (Color online) The logarithm of the von-Neumann entropy S as a function of the logarithm of the quantity $(\lambda/\lambda_c) - 1$, which measures the distance of the coupling strength from the critical value. The red solid line corresponds to the single-qubit case, whereas the other lines correspond to the multi-qubit case: $N = 2$ (green dashed line), 3 (blue short-dashed line), 5 (purple dotted line) and 10 (dash-dotted cyan line). All the lines correspond to $\hbar\omega_0/\Delta = 10^{-7}$. The slope of all lines is approximately 0.92 when $(\lambda/\lambda_c) - 1 = 10^{-4}$. The ratio of the entropy in the multi-qubit case to that in the single-qubit case approaches N for all the lines as we approach the critical point.

lates the value in one of the two branches of the wave function, i.e. one calculates the value of α in Eq. (5)]. The average value of the field vanishes below the critical point, and it has the form

$$\alpha = \frac{\Delta}{4\lambda}\left[\left(\frac{\lambda}{\lambda_c}\right)^4 - 1\right]^{1/2} \tag{10}$$

above the critical point. Its dependence on the coupling strength just above the critical point can alternatively be

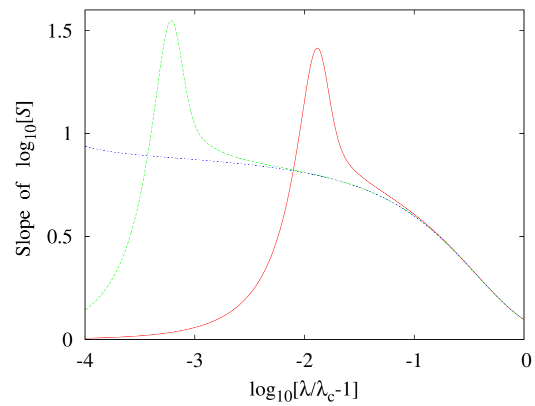

FIG. 7: (Color online) The slope of the logarithm of the von-Neumann entropy S as a function of the logarithm of the quantity $(\lambda/\lambda_c) - 1$, which measures the distance of the coupling strength from the critical value. The different lines correspond to $\hbar\omega_0/\Delta = 10^{-3}$ (solid red line), 10^{-5} (dashed green line) and 10^{-7} (short-dashed blue line). The slope seems to be approaching the value 1 as we approach the critical point, but at some point, determined by the ratio $\hbar\omega_0/\Delta$, the slope has a peak and drops to zero. The deviation from the simple asymptotic behaviour is related to the fact that S has a non-zero value at $\lambda/\lambda_c = 1$, as can be seen in Fig. 2.

expressed as

$$\alpha = \frac{\Delta}{2\lambda_c}\left(\frac{\lambda}{\lambda_c} - 1\right)^{1/2}. \tag{11}$$

In Fig. 6 we plot the von-Neumann entropy as a function of coupling strength on a log-log scale (above the critical point). The slope of the curve for the smallest values of λ is approximately 0.92. This value suggests that the true critical exponent might be unity. One difficulty in calculating the asymptotic value of the slope is the fact that for any finite value of $\hbar\omega_0/\Delta$, the von-Neumann entropy deviates from the behaviour shown in Fig. 6 if one comes too close to the critical point. This deviation can be seen in Fig. 2 and is illustrated more clearly in Fig. 7.

IV. MULTI-QUBIT CASE

Let us now consider the case with a finite number N of qubits [18]. In this model, the qubits are usually assumed to have the same single-qubit energies Δ and the same coupling strength to the oscillator, which is usually defined as $\lambda/\sqrt{N}$. The Hamiltonian in this case is given by:

$$\begin{aligned} \hat{H} &= -\sum_{j=1}^{N}\frac{\Delta}{2}\hat{\sigma}_x^{(j)} + \hbar\omega_0\hat{a}^\dagger\hat{a} + \sum_{j=1}^{N}\frac{\lambda}{\sqrt{N}}\left(\hat{a}+\hat{a}^\dagger\right)\hat{\sigma}_z^{(j)} \\ &= -\Delta\hat{J}_x + \hbar\omega_0\hat{a}^\dagger\hat{a} + 2\frac{\lambda}{\sqrt{N}}\left(\hat{a}+\hat{a}^\dagger\right)\hat{J}_z, \end{aligned} \tag{12}$$

where we have defined the total spin operators $\hat{J}_\alpha = \sum \hat{\sigma}_\alpha/2$. In the limit $\hbar\omega_0/\Delta \to 0$, all the results concerning the low-energy spectrum of the resonator remain unchanged; one could say that the reduction of the coupling strength by the factor $\sqrt{N}$ is compensated by the strengthening of the spin raising and lowering operators by the same factor because of the collective behaviour of the qubits. In particular, the transition occurs at the critical coupling strength given by Eq. (6). Because the qubits now have a larger total spin (when compared to the single-qubit case), spin states that are separated by small angles can be drastically different (i.e. have a small overlap). In particular, the overlap for N qubits is given by $\cos^{2N}(\theta/2)$. By expanding this function to second order around $\theta = 0$, one can see that for small values of θ the relevant overlap is lower than unity by an amount that is proportional to N. This dependence translates into the dependence of the qubit-oscillator entanglement on the coupling strength just above the critical point. The entanglement therefore rises more sharply in the multi-qubit case (with the increase being by a factor N), as demonstrated in Fig. 6.

V. FINITE-TEMPERATURE BEHAVIOUR

Equation (5) gives the ground state deep in the superradiance region. The first-excited state has the same form, but with a minus sign instead of the plus sign. The energy separation between these two states decreases exponentially with increasing λ. As a result, an infinitesimally small temperature would be sufficient to destroy the coherence between the two branches of the wave function in thermodynamic equilibrium. Nevertheless, the correlation function C exhibits essentially the same behaviour for the two states. Furthermore, all low-lying energy levels have a qualitatively similar correlation between the state of the qubit and the field in the oscillator (even though the entanglement might be lost). One can therefore ask whether a finite-temperature phase transition would still occur between a region of correlated qubit-oscillator states and a region with no correlation.

The energy level structure in the single-qubit case is simple in principle. In the limit $\hbar\omega_0/\Delta \to 0$, one can say that the energy levels form two sets, one corresponding to each qubit state. Each one of these sets has a structure that is similar to that of a harmonic oscillator with some modifications that are not central in the present context. In particular the density of states has a weak dependence on energy, a situation that cannot support a thermal phase transition. If the temperature is increased while all other system parameters are kept fixed, qubit-oscillator correlations (which are finite only above the critical point) gradually decrease and vanish asymptotically in the high-temperature limit. No singular point is encountered along the way. This result implies that the transition point is independent of temperature. In other words, it remains at the value given by Eq. (6) for all temperatures. If, for example, one is investigating the dependence of the correlation function C on the coupling strength (as plotted in Fig. 3), the only change that occurs as we increase the temperature is that the qubit-oscillator correlations change more slowly when the coupling strength is varied.

VI. RELATION TO PHASE TRANSITION IN THE THERMODYNAMIC LIMIT

As we have mentioned above, the phase transition in the limit $\hbar\omega_0/\Delta \to 0$ is distinct from that encountered in the thermodynamic limit $N \to \infty$. Given the similarities between the two phase transitions, one can ask whether it is possible to identify a single, unified condition for the realization of singular behaviour. For example, one candidate for this unified condition could be $\hbar\omega_0/(N\Delta) \to 0$.

If such a unified condition existed, we would expect that for any large value of N there is a proportionately large value of $\hbar\omega_0/\Delta$ above which the sharp transition is replaced by a smooth crossover. However, the presence of a phase transition in the thermodynamic limit is independent of the ratio $\hbar\omega_0/\Delta$, including the zero and infinite limits. In particular, if we consider the Dicke model with an arbitrary value of N and first take the limit $\hbar\omega_0/\Delta \to \infty$ (meaning that this is the most dominant infinite limit in the problem), the system effectively reduces to the Lipkin-Meshkov-Glick model, which exhibits singular behaviour in the limit $N \to \infty$ (see e.g. Refs. [16, 20]). We therefore conclude that the two limits $\hbar\omega_0/\Delta \to 0$ and $N \to \infty$ cannot be unified in a nontrivial manner.

In fact, the consideration of the thermodynamic limit provides at least a partial explanaton for why singular behaviour is obtained only in the limit $\hbar\omega_0/\Delta \to 0$ in the single-qubit case. In the limit $N \to \infty$, the phase transition occurs independently of the ratio $\hbar\omega_0/\Delta$. However, depending on the value of this ratio, the phase transition region involves larger changes in the lower-frequency subsystem, i.e. either the collective state of the qubits or the state of the oscillator. In the limit $\hbar\omega_0/\Delta \to 0$, the state of the qubits only slightly deviates from the ground state when the transition point is crossed, and it is plausible that the singular behaviour would persist even when the ensemble of qubits is replaced by a single qubit. In the limit $\hbar\omega_0/\Delta \to \infty$, the oscillator stays close to its ground state while the state of the qubit ensemble undergoes large changes upon crossing the transition point, a behaviour that clearly cannot translate straightforwardly to the single-qubit case.

VII. CONCLUSION

We have analyzed the transition from an uncorrelated composite system to superradiance behaviour in the single-qubit-single-oscillator Rabi model. We have shown that as the ratio of the oscillator's frequency to the qubit's frequency approaches zero, various physical quantities exhibit singular dependence that closely resembles that encountered in the study of the superradiance phase transition in the thermodynamic limit of the Dicke model.

The qubit-oscillator entanglement and appropriate qubit-oscillator correlation functions remain very small below the transition point but increase rapidly as soon as the coupling strength exceeds a certain critical value. The low-lying energy levels (almost) collapse to a single highly degenerate ground-state manifold at the transition point. The amount of squeezing also peaks in a singular manner at the critical point.

The energy level separations and the degree of squeezing scale as $|\lambda/\lambda_c - 1|^{1/2}$ on both sides of the critical point, while the qubit-oscillator entanglement rises as $|\lambda/\lambda_c - 1|^{\alpha}$ above the critical point, with the exponent α being slightly below unity.

In spite of the similarities in the behaviour of this system with the behaviour of the Dicke model in the thermodynamic limit, the analogy is not complete, as evidenced by the absence of a thermal phase transition in the single-qubit-single-oscillator system.

The Rabi model with arbitrary coupling strength remains an active area of research. Recent studies have addressed questions related to the integrability of the model [19], various approximations and exact solutions [15, 21], dynamics and dissipation [22], proposals of potentially robust designs for quantum bits [23] and novel strongly correlated many-polariton states [24]. The present work deals with this ubiquitous physical model, and we expect that the results presented here will help improve our understanding of the basic properties of the model.

Acknowledgments

We would like to thank P. Forn-Díaz, J. R. Johansson and N. W. Lambert for useful discussions. This work was supported in part by ARO, NSF Grant No. 0726909, JSPS-RFBR Contract No. 09-02-92114, Grant-in-Aid for Scientific Research (S), MEXT Kakenhi on Quantum Cybernetics, and the JSPS via its FIRST program.

[1] See e.g. C. C. Gerry and P. L. Knight, Introductory Quantum Optics (Cambridge University Press, 2005); D. F. Walls and G. J. Milburn, Quantum Optics (Springer, Berlin, 1994); M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge University Press, 1997).
[2] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[3] K. Hepp, and E. H. Lieb, E. H. Ann. Phys. **76**, 360 (1973).
[4] Y. K. Wang and F. T. Hioe, Phys. Rev. A **7**, 831 (1973).
[5] C. Emary and T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003); N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
[6] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, 1999).
[7] K. Rzazewski, K. Wodkiewicz, and W. Zacowicz, Phys. Rev. Lett. **35**, 432 (1975); I. Bialynicki-Birula and K. Rzazewski, Phys. Rev. A **19**, 301 (1979); P. Nataf and C. Ciuti, Nat. Commun. **1**, 72 (2010); O. Viehmann, J. von Delft, and F. Marquardt, Phys. Rev. Lett. 107, 113602 (2011); D. Z. Xu, Y. B. Gao, and C. P. Sun, arXiv:1204.2602; A. Vukics and P. Domokos, arXiv:1206.0752.
[8] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. Garcia-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, and R. Gross, Nature Phys. **6**, 772 (2010).
[9] P. Forn-Diaz, J. Lisenfeld, D. Marcos, J. J. Garcia-Ripoll, E. Solano, C. J. P. M. Harmans, J. E. Mooij, Phys. Rev. Lett. **105**, 237001 (2010); A. Fedorov, A. K. Feofanov, P. Macha, P. Forn-Diaz, C. J. P. M. Harmans, and J. E. Mooij, Phys. Rev. Lett. **105**, 060503 (2010).
[10] M. H. Devoret. S. Girvin and R. Schoelkopf, Ann. Phys. **16**, 767 (2007).
[11] For recent reviews on superconducting qubit circuits, see e.g. J. Q. You and F. Nori, Phys. Today **58** (11), 42 (2005); Nature **474**, 589 (2011); G. Wendin and V. Shumeiko, in Handbook of Theoretical and Computational Nanotechnology, ed. M. Rieth and W. Schommers (ASP, Los Angeles, 2006); R. J. Schoelkopf and S. M. Girvin, Nature, **451**, 664 (2008); J. Clarke and F. K. Wilhelm, Nature **453**, 1031 (2008); T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, Nature (London) **464**, 45 (2010); I. Buluta, S. Ashhab, and F. Nori, Rep. Prog. Phys. **74**, 104401 (2011).
[12] For studies on the experimental realization of the superradiance in other systems, see e.g. D. Nagy, G. Konya, G. Szirmai, and P. Domokos, Phys. Rev. Lett. **104**, 130401 (2010); K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature **464**, 1301 (2010); G. Scalari, C. Maissen, D. Turcinkova, D. Hagenmüller, S. De Liberato, C. Ciuti, C. Reichl, D. Schuh, W. Wegscheider, M. Beck, and J. Faist, Science **335**, 1323 (2012).
[13] S. Ashhab and F. Nori, Phys. Rev. A **81**, 042311 (2010).
[14] G. Levine and V. N. Muthukumar, Phys. Rev. B **69**, 113203 (2004); A. P. Hines, C. M. Dawson, R. H. McKenzie, and G. J. Milburn, Phys. Rev. A **70**, 022303 (2004).
[15] M.-J. Hwang and M. S. Choi, Phys. Rev. A. **82**, 025802 (2010).
[16] L. Bakemeier, A. Alvermann, and H. Fehske, Phys. Rev. A **85**, 043821 (2012).
[17] The entropy S and the correlation function C exhibit qualitatively similar features. Even though (for the same value of $\hbar\omega_0/\Delta$) the features are less pronounced in the case of C as compared to S, they are more robust. In particular, even if one considers a case where the first-excited state becomes substantially populated, the entanglement can be drastically reduced [since for example

an equal mixture of the ground and first-excited state is equivalent to a mixture of the two terms in Eq. (5)], while the correlation function C is only slightly modified. In other words, an experiment might fail to observe the results plotted in Fig. 2 but still observe those plotted in Fig. 3 (up to some small imperfections). The reason behind this robustness is the fact that all low-lying energy eigenstates possess similar correlations between the spin and the cavity field.

[18] It should perhaps be mentioned here that finite size effects have been considered in the context of how the system approaches phase-transition behaviour as the number of atoms increases and approaches infinity; See e.g. G. Liberti, R. L. Zaffino, F. Piperno, and F. Plastina, Phys. Rev. A **73**, 032346 (2006); O. Castanos, E. Nahmad-Achar, R. Lopez-Pena, and J. G. Hirsch, Phys. Rev. A **83**, 051601 (2011); G. Konya, D. Nagy, G. Szirmai, and P. Domokos, Phys. Rev. A **86**, 013641 (2012); M. Aparicio Alcalde, M. Bucher, C. Emary, and T. Brandes, Phys. Rev. E **86**, 012101 (2012); P. Nataf, M. Dogan, and K. Le Hur, arXiv:1204.3065v1.

[19] D. Braak, Phys. Rev. Lett. **107**, 100401 (2011); E. Solano, Physics **4**, 68 (2011); Q. H. Chen, C. Wang, S. He, T. Liu, and K.-L. Wang, Phys. Rev. A **86**, 023822 (2012); K. Ziegler, arXiv:1204.3865; see also G. Benivegna and A. Messina, Phys. Rev. A **35**, 3313 (1987); G. Benivegna, A. Messina, and E. Paladino, J. Phys. A **29**, 2485 (1996).

[20] D. I. Tsomokos, S. Ashhab, and F. Nori, New J. Phys. **10**, 113020 (2008).

[21] J. Hausinger and M. Grifoni, New J. Phys. **10**, 115015 (2008); J. Casanova, G. Romero, I. Lizuain, J. J. Garcia-Ripoll, and E. Solano, Phys. Rev. Lett. **105**, 263603 (2010); J. Hausinger and M. Grifoni, Phys. Rev. A **82**, 062320 (2010); *ibid.* **83**, 030301 (2011); Y. W. Zhang, G. Chen, L. X. Yu, Q. F. Liang, J. Q. Liang, and S. T. Jia, Phys. Rev. A **83**, 065802 (2011); Q. H. Chen, T. Liu, Y. Y. Zhang, K. L. Wang, Europhys. Lett. **96**, 14003 (2011); V. V. Albert, G. D. Scholes, and P. Brumer, Phys. Rev. A **84**, 042110 (2011); S. Agarwal, S. M. Hashemi Rafsanjani, and J. H. Eberly, Phys. Rev. A **85**, 043815 (2012); L. Yu, S. Zhu, Q. Liang, G. Chen, and S. Jia, Phys. Rev. A **86**, 015803 (2012); L.-H. Du, X.-F. Zhou, Z.-W. Zhou, X. Zhou, and G.-C. Guo, Phys. Rev. A **86**, 014303 (2012); S. He, Q.-H. Chen, X.-Z. Ren, T. Liu, K.-L. Wang, Phys. Rev. A **86**, 033837 (2012); Y.-Y. Zhang and Q.-H. Chen, arXiv:1210.5326v1; E. Nahmad-Achar, O. Castanos, R. Lopez-Pena, J. G. Hirsch, arXiv:1211.6692v1.

[22] D. Zueco, G. M. Reuther, S. Kohler, and P. Hänggi, Phys. Rev. A **80**, 033846 (2009); X. Cao, J. Q. You, H. Zheng, and F. Nori, New J. Phys. **13**, 073002 (2011); E. del Valle and F. P. Laussy, Phys. Rev. Lett. **105**, 233601 (2010); Phys. Rev. A **84**, 043816 (2011); F. Beaudoin, J. M. Gambetta, and A. Blais, Phys. Rev. A **84**, 043832 (2011); J. Song, Y. Xia, X. D. Sun, Y. Zhang, B. Liu, and H. S. Song, Eur. Phys. J. D **66**, 90 (2012); Y. M. Wang, B. Liu, J. L. Lian, and J. Q. Liang, Opt. Express **20**, 9 (2012); G. M. Reuther, P. Hänggi, and S. Kohler, Phys. Rev. A **85**, 062123 (2012); A. Ridolfo, M. Leib, S. Savasta, and M. J. Hartmann, Phys. Rev. Lett. **109**, 193602 (2012); F. A. Wolf, F. Vallone, G. Romero, M. Kollar, E. Solano, and D. Braak, arXiv:1211.6469v1; A. Ridolfo, S. Savasta, and M. J. Hartmann, arXiv:1212.1280v1; F. Altintas and R. Eryigit, arXiv:1212.4071v1; C. K. Law, arXiv:1212.5864v1; A. L. Grimsmo and S. Parkins, arXiv:1212.6063v1.

[23] P. Nataf and C. Ciuti, Phys. Rev. Lett. **107**, 190402 (2011).

[24] M. Schiró, M. Bordyuh, B. Öztop, and H. E. Türeci, Phys. Rev. Lett. **109**, 053601 (2012).