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Bright Solitons in Spin-Orbit Coupled Bose-Einstein Condensates

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We study bright solitons in a Bose-Einstein condensate with a spin-orbit coupling that has been realized experimentally. Both stationary bright solitons and moving bright solitons are found. The stationary bright solitons are the ground states and possess well-defined spin parity, a symmetry involving both spatial and spin degrees of freedom; these solitons are real-valued but not positive-definite and the number of their nodes depends on the strength of spin-orbit coupling. For the moving bright solitons, their shapes are found to change with velocity due to the lack of Galilean invariance in the system.

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INTRODUCTION

Solitons are one of the most interesting topics in nonlinear systems. The most fascinating and well-known feature of this localized wave packet is that it can propagate without changing its shape as a result of the balance between nonlinearity and dispersion [1]. The achievement of Bose-Einstein condensation in a dilute atomic gas has offered a clean and parameter-controllable platform to study the properties of solitons [2]. In a Bose-Einstein condensate (BEC), the nonlinearity originates from the atomic interactions and is manifest by the nonlinear term in the Gross-Pitaevskii equation (GPE), which is the mean-field description of BEC [3]. With attractive and repulsive interatomic interactions, the GPE can have bright and dark solitons solutions, respectively. Such dark or bright solitons in BECs have been studied extensively both theoretically [4–10] and experimentally [11–18].

The developments with two component BECs have further enriched the investigation of solitons in matter waves. The two component BECs not only introduce more tunable parameters, for example, the interaction between the two species, but also bring in novel nonlinear structures which have no counterparts in the scalar BEC, such as dark-bright solitons (one component is dark soliton while the other is bright) [19–22], dark-dark solitons [23], bright-bright solitons [24–26], and domain walls [27–29].

Recently in a landmark experiment, the Spielman group at NIST have engineered a synthetic spin-orbit coupling (SOC) for a BEC [30]. In the experiment, two Raman laser beams are used to couple a two-component BECs. The momentum transfer between lasers and atoms leads to synthetic spin-orbit coupling [31–40]. This kind of spin-orbit coupling has subsequently been realized for neutral atoms in other laboratories [41–44]. These experimental breakthroughs [30, 44, 45] have stimulated extensive theoretical investigation of the properties of spin-

orbit coupled BEC [46–67], which includes some early studies on solitons. For example, bright soliton solutions were found analytically for spin-orbit coupled BEC by neglecting the kinetic energy [68]. Dark solitons for such a system were studied in a one dimensional ring [69]. In this work we conduct a systematic study of bright solitons for a BEC with attractive interactions and the experimentally realized SOC [30, 42–44]. By solving the GPE both analytically and numerically, we find that these solitons possess a number of novel properties due to the SOC.

In particular, we find that the stationary bright solitons that are the ground state of the system have nodes in their wavefunction. For a conventional BEC without SOC, its ground state must be nodeless thanks to the “no-node” theorem for the ground state of a bosonic system [70]. Furthermore, these solitons are found to have well-defined spin parity, a symmetry that involves both spatial and spin degrees of freedom and can exist in systems with SOC.

We have also found solutions for moving bright solitons. They have a very interesting feature that their shapes change dramatically with increasing velocity. For a conventional BEC, the shape of soliton does not change with velocity due to the Galilean invariance of the system. In other nonlinear systems, such as KdV systems, the shape of a soliton changes only in height and width with velocity [71]. In stark contrast, bright solitons in a BEC with SOC can change shape dramatically from nodeless to having many nodes with varying velocity. The new feature arises because of the lack of Galilean invariance due to SOC [62]. It is worthwhile noting that a similar model was proposed a long time ago in the context of nonlinear birefringent fiber [72]. This shows that our results will find applications in nonlinear optics.

MODEL EQUATION

A BEC with the experimentally realized SOC is described by the following GPE

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}(p_x + \hbar\kappa\sigma_y)^2 + \hbar\Delta\sigma_z - g\Psi^\dagger \cdot \Psi \right] \Psi, \quad (1)$$

where the spinor wavefunction $\Psi = (\Psi_1, \Psi_2)^T$, and $\Psi^\dagger \cdot \Psi = |\Psi_1|^2 + |\Psi_2|^2$ with Ψ_1 for up-spin and Ψ_2 for down-spin. The nonlinear coefficient $-g < 0$ is for attractive interatomic interactions, and we have taken $g_{11} = g_{22} = g_{12}$ for simplicity. The SOC is realized experimentally by two counter-propagating Raman lasers that couple two hyperfine ground states Ψ_1 and Ψ_2 . The strength of SOC κ depends on the relative incident angle of the Raman beams and can be changed [65]. The Rabi frequency Δ can be tuned easily by modifying the intensity of the Raman beams. σ are Pauli matrices. A bias homogeneous magnetic field is applied along the y direction. We consider the case that the radial trapping frequency is large and, therefore, the system is effectively one dimensional [14, 15].

For numerical simulation, we rewrite Eq. (1) in a dimensionless form by scaling energy with $\hbar\Delta$ and length with $\sqrt{\hbar/m\Delta}$. The dimensionless GPE is

$$i\partial_t\Phi = \left[-\frac{1}{2}\partial_x^2 + i\alpha\sigma_y\partial_x + \sigma_z - \gamma\Phi^\dagger \cdot \Phi \right] \Phi. \quad (2)$$

The dimensionless parameters $\alpha = -\kappa\sqrt{\hbar\Delta/m}$ and $\gamma = Ng\sqrt{m/(\hbar\Delta)}/\hbar$ with N being the total number of atoms. The dimensionless wavefunctions Φ satisfy $\int dx(|\Phi_1|^2 + |\Phi_2|^2) = 1$. The SOC term $i\alpha\sigma_y\partial_x$ in Eq. (2) indicates that spin σ_y only couples the momentum in the x direction. The energy functional of our system is

$$E = \int dx \left[\frac{1}{2}|\partial_x\Phi_1|^2 + \frac{1}{2}|\partial_x\Phi_2|^2 + |\Phi_1|^2 - |\Phi_2|^2 + \alpha\Phi_1^*\partial_x\Phi_2 - \alpha\Phi_2^*\partial_x\Phi_1 - \frac{\gamma}{2}(|\Phi_1|^4 + |\Phi_2|^4 + 2|\Phi_1|^2|\Phi_2|^2) \right]. \quad (3)$$

STATIONARY BRIGHT SOLITONS

We focus on the simplest stationary bright solitons, which are the ground states of the system. To find these solitons, we solve Eq. (2) by using imaginary time evolution method. Two typical bright solitons are shown in Fig. 1. One interesting feature is immediately noticed. There are “nodes” in these ground state bright solitons. This is very different from the conventional BEC, where there are no nodes in this kind of ground state soliton as demanded by the “no-node” theorem for the ground state of a boson system. Our results confirm that this

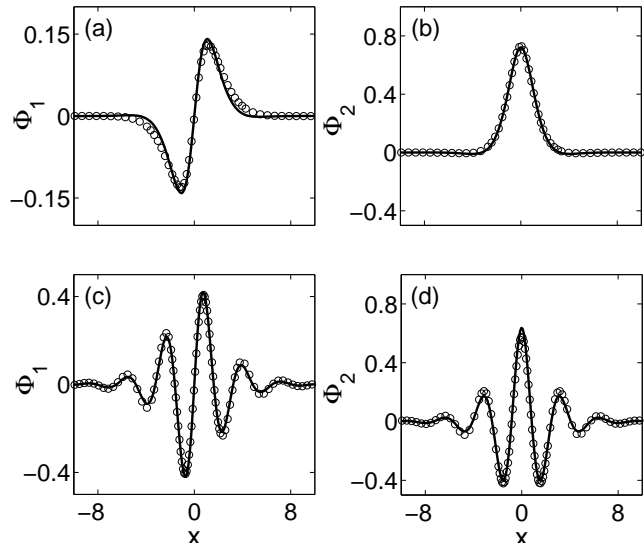


FIG. 1: Stationary bright solitons at $\gamma = 1.0$. The solid lines are numerical results and the circles are from the variational method. In (a) and (b), $\alpha = 1.0$. In (c) and (d), $\alpha = 2.0$.

“no-node” theorem does not hold for systems with SOC [70].

There can exist a unique symmetry for systems with SOC, spin parity, which involves both spatial and spin degrees of freedom. The operator for spin parity is defined as

$$\mathcal{P} = P\sigma_z, \quad (4)$$

where P is the parity operator. It is easy to verify that our system is invariant under the action of spin parity \mathcal{P} . By direct observation, one can see that the bright solitons shown in Fig. 1 satisfy

$$\mathcal{P} \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = - \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix}, \quad (5)$$

as the up component Φ_1 has odd parity while the other component Φ_2 is even. Therefore, these bright solitons have spin parity -1 . In fact, all the ground state bright solitons that we have found have spin parity -1 . That the eigenvalue of \mathcal{P} for these solitons is -1 and not 1 can be understood in the following manner. When the strength of SOC α decreases to zero, the up component Ψ_1 shrinks to zero and only the down component survives. Since the system becomes a conventional BEC without SOC, the “no-node” theorem demands that the surviving down component have even symmetry. As the SOC is turned up continuously and slowly, the symmetry of the second component should remain and the spin parity has to be -1 .

For a more detailed analysis of these bright solitons, we attempt to find an analytical approximation for the

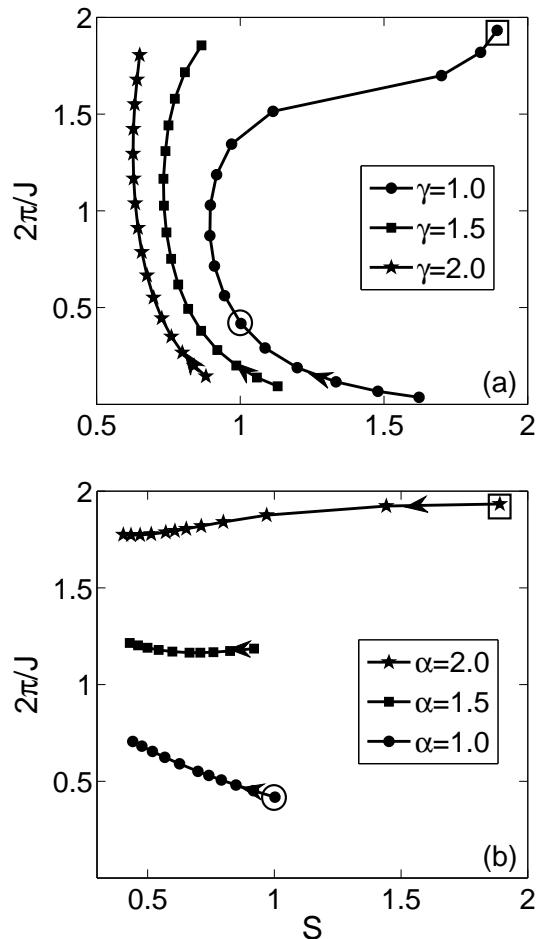


FIG. 2: The relation between the number of soliton nodes ($2\pi/J$) and the soliton width (S). (a) Circle, square, and star are for $\gamma = 1.0, 1.5, 2.0$, respectively. Here α increases from 0.5 to 2.0 (the arrow direction). (b) circle, square, and star are for $\alpha = 1.0, 1.5, 2.0$. Here γ increases from 1.0 to 4.0 (the arrow direction). The open circle and square correspond to the bright solitons in Fig. 1(a, b) and Fig. 1(c, d) respectively.

wavefunctions using the variational method. Motivated by the features of the stationary bright solitons shown in Fig. 1, we propose the following trial wave functions for these solitons,

$$\Phi = \begin{pmatrix} A \sin(2\pi x/J) \\ B \cos(2\pi x/J) \end{pmatrix} \text{sech}(x/S). \quad (6)$$

The parameters A, B, J , and S are determined by minimizing the energy functional in Eq. (3) with the constraining normalization. The results of the trial wave functions are compared with the numerical results in Fig. 1 where it can be seen that they are in excellent agreement.

It is clear from Eq. (6) that the parameter $2\pi/J$ can

be regarded roughly as the number of nodes in the bright solitons while S is for the overall width of the soliton. Both of them depend on the SOC strength α and the interaction strength γ . In Fig. 2 we have plotted the relation between $2\pi/J$ and S , demonstrating how the number of nodes is related to the soliton width, for different values of α and γ . As shown in the figure, for solitons with the same number of nodes, they are wider for smaller interaction strength γ (Fig. 2(a)); the solitons with the same width have more nodes for larger SOC strength α (Fig. 2(b)).

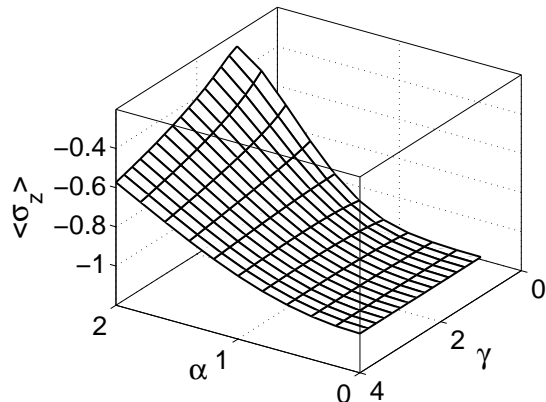


FIG. 3: (color online) The spin polarization ($\langle \sigma_z \rangle$) as a function of α and γ .

It has been reported that there exists quantum phase transition for the ground states in the spin-orbit coupled system with repulsive interaction [65]. It is interesting to check whether such a phase transition exists for the case of attractive interaction. For this purpose, we have computed the spin polarization $\langle \sigma_z \rangle = \int dx (\Phi_1^2 - \Phi_2^2)$ for these bright solitons and the results are plotted in Fig. 3. We see that for a given γ , it changes smoothly with the SOC strength α . For the cases of repulsive interaction and no interaction, the spin polarization is found to change sharply with α , indicating a quantum phase transition [65]. The smooth behavior of Fig. 3 suggests there is no quantum phase transition.

MOVING BRIGHT SOLITONS

After the study of stationary bright solitons, we turn our attention to moving bright solitons. For a conventional BEC without SOC, it is straightforward to find a moving bright soliton from a stationary soliton: if the wave function Φ_s describes a stationary soliton, then $\exp(ivx)\Phi_s(x-vt)$ is the wave function, up to a trivial phase, for a soliton moving at speed v . This is due to the invariance of the system under Galilean transformations.

However, Galilean invariance is violated for a spin-orbit coupled BEC [62]. To see this explicitly, we assume moving solitons having the following form,

$$\Phi_M(x, t) = \Phi_v(x - vt, t) \exp(ivx - i\frac{1}{2}v^2t), \quad (7)$$

where Φ_v is a localized function. Substitution of $\Phi_M(x, t)$ into Eq. (2) yields

$$i\partial_t \Phi_v = \left[-\frac{1}{2}\partial_x^2 + \alpha\sigma_y(i\partial_x - v) + \sigma_z - \gamma\Phi_v^\dagger \cdot \Phi_v \right] \Phi_v. \quad (8)$$

Compared to Eq. (2), this dynamical equation has an additional term $\alpha v\sigma_y$, indicating the violation of Galilean invariance. This violation means that it is no longer a trivial task to find a moving bright soliton for a BEC with SOC.

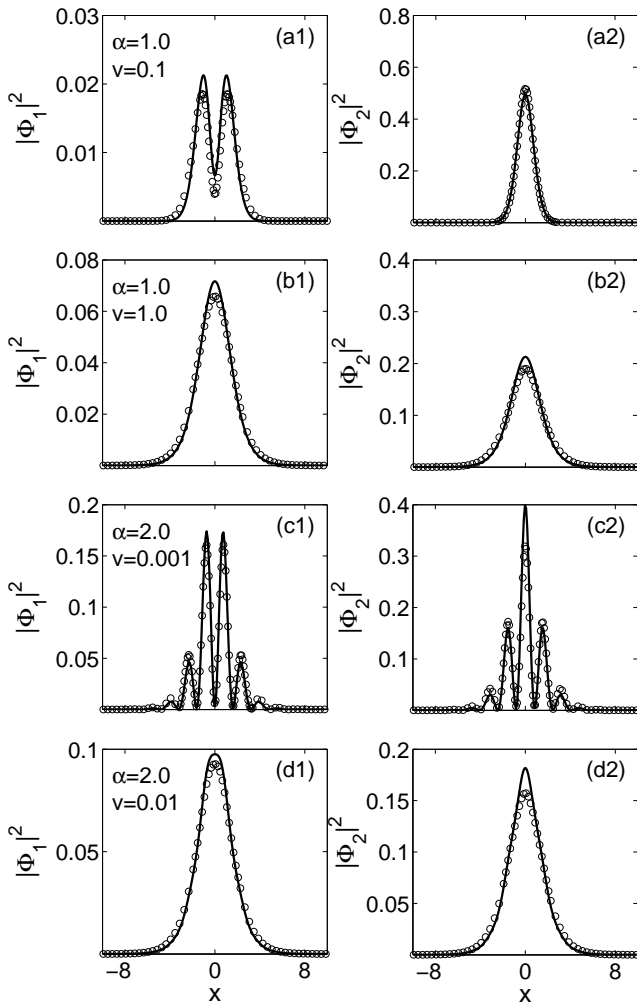


FIG. 4: Moving bright soliton profiles $\Phi_v(x)$ from the numerical calculation (solid line) and the variational method (circles) with Eq. (9). $\gamma = 1.0$. In (a) and (b), $\alpha = 1.0$. In (c) and (d) $\alpha = 2.0$.

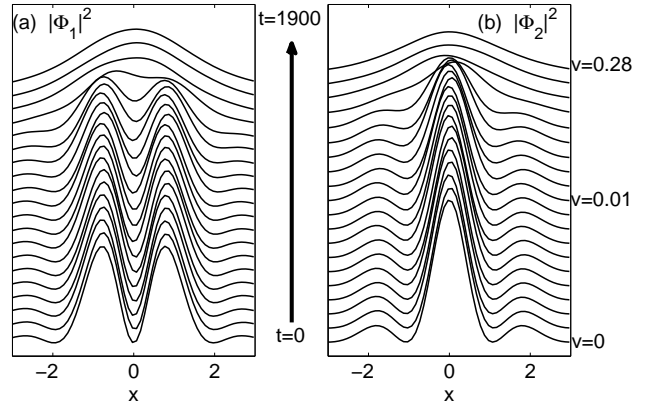


FIG. 5: Dynamical evolution of a bright soliton under the influence of a small linear potential $V(x) = \eta x$. $\alpha = 1.7$, $\gamma = 1.0$, $\eta = 0.001$. For better comparison of shapes, the centers of the soliton densities have all been shifted to the zero point. The velocity of the soliton is labeled at the right of (b).

To find moving bright solitons, we numerically solve Eq. (8) using the imaginary time evolution method. Two typical moving bright solitons are shown in Fig. 4, where we see clearly that the shapes of moving bright solitons change with their velocities. As seen in Fig. 4(a) and Fig. 4(b), when the velocity v is changed from 0.1 to 1, the density of the up component changes from having two peaks to having only one. At a larger SOC strength, such as $\alpha = 2$ in Fig. 4(c) and 4(d), a small change of velocity leads to a dramatic change in the soliton profiles.

Similar to the stationary soliton, these moving bright solitons can also be found with the variational method by minimizing the energy functional with the following trial wave function,

$$\Phi_v(x) = \left(\begin{array}{c} A \left[\sin \frac{2\pi x}{J} + \rho_1 i \cos \frac{2\pi x}{J} \right] \\ B \left[\cos \frac{2\pi x}{J} + \rho_2 i \sin \frac{2\pi x}{J} \right] \end{array} \right) \text{sech} \frac{x}{S}, \quad (9)$$

with two new parameters ρ_1 and ρ_2 . When $\rho_1 = \rho_2 = 0$, we recover the stationary soliton in Eq. (6). The solutions obtained with the variational method are plotted in Fig. 4 and they agree well with the numerical results. It is clear from the trial wave function that the moving bright soliton has no well-defined spin parity \mathcal{P} .

These moving bright solitons are adiabatically linked to the stationary bright solitons. To see this, we slowly accelerate the stationary bright soliton by adding a small linear potential in Eq. (2) integrating with a stationary bright soliton as the initial condition. The dynamical evolution of this soliton is shown in Fig. 5, where we see clearly how a stationary soliton is developed into a moving soliton with its shape changing constantly. Note that the centers of solitons in Fig. 5 have all been shifted to the zero point for better comparison between shapes.

We note that there are other solitons, which also change their shapes with velocity, for example, dark soli-

tons in a BEC and bright solitons in the KdV system [71]. It is also caused by the lack of the Galilean invariance in the system. For the dark soliton, the constant background provides a preferred reference frame and breaks the Galilean invariance. In the KdV system, the violation is caused by the non-quadratic linear dispersion. However, in these systems, the change of shape with velocity is not as dramatic: there are only changes in the height and width of the solitons. With a spin-orbit coupled BEC, the number of peaks in the solitons can change with a slight change of velocity.

CONCLUSION

We have systematically studied both stationary and moving bright solitons in a spin-orbit coupled BEC. These bright solitons have a number of novel features caused by spin-orbit coupling. For example, the existence of nodes and spin parity in the stationary bright solitons; the change of shape with velocity in the moving bright solitons. Although there are multiple peaks in the soliton profiles, the bright solitons that we have found are single solitons. It would be very interesting to seek out multiple solitons solutions for this spin-orbit coupled system. These bright solitons should be able to be observed in experiment. One can apply the same Raman laser setup to generate the synthetic spin-orbit coupling for an optical dipole trapped ${}^7\text{Li}$ condensate where the interatomic interaction is attractive by nature.

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Note added: As we were preparing this manuscript, we noticed a newly posted preprint arXiv:1211.0199 on bright solitons in a BEC with SOC [73]. Some of their results overlap with ours.

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