Thermal entanglement between non-nearest-neighbor spins on fractal lattices

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Thermal entanglement between non-nearest-neighbor spins on fractal lattices

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Abstract

We investigate the thermal entanglement between two end sites in spin chains and on fractal lattices by taking the negativity as a measure and using the decimation renormalization-group method. The effects of the temperature T, the anisotropy parameter Δ and the size of system L on the entanglement are examined detailedly. It is found that the entanglement decreases monotonically with increasing T and vanishes beyond a critical value T_c. Our results also show that with increasing Δ from $-\infty$ to zero the entanglement first increases to the maximum and then decreases sharply to zero. Different from the cases of spin chains and Koch curves, the entanglement on the diamond-type hierarchical (DH) lattices presents some interesting behaviors. As the sizes of the DH lattices L become large, the entanglement is rather robust and there exists sizable entanglement between long-distance end sites. This result indicates that different fractal structures can result in various entanglement properties.

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I. INTRODUCTION

It is well known that the entanglement is the intriguing nonlocal correlation phenomenon to test the fundament of the quantum mechanics [1–4], as well as the key resource to realize the quantum information processing (QIP) such as quantum cryptography and quantum communication [5–7]. Since the entanglement is fragile to environment and the decoherence occurs easily for the entangled systems, many efforts are devoted to the research of the stable entanglements for realistic systems at finite temperature. This problem has been studied intensively under the name "thermal entanglement", and it is theoretically and experimentally demonstrated that there exists thermal entanglement in solid systems even at high temperature [8–10].

Solid spin systems, which can perfectly simulate various realistic systems such as ultracold atoms and quantum dots [11], have become naturally important candidates for QIP. The entanglement properties of the solid spin systems have also received considerable attention in condensed-matter physics, because these works can provide a deeper understanding of the quantum correlation in many-body systems, for example, the superconduction and quantum Hall effect [12].

The thermal entanglements on solid spin systems have been investigated extensively by several measure methods containing negativity, concurrence, entanglement of formation [8, 9, 13–21]. Most of these previous works focused on the pairwise thermal entanglement between nearest, next-nearest or next-to-next-nearest neighbor sites in one-dimensional (1-D) spin chains and other simple lattices. There are few studies about the entanglement properties on the low symmetry systems including fractal lattices which can simulate some systems such as random magnets, surfaces, and the like. This motivates us to propose a question whether there exists entanglement between two non-nearest-neighbor end sites on fractal lattices, and if there exists, how the entanglement evolves as the size of system becomes large. It is very difficult or impossible to obtain exact results of the entanglement on many-body complicated lattices especially on fractal lattices.

Recently, the entanglement properties of many-body systems at zero temperature have been studied by using the renormalization-group (RG) methods such as the density matrix RG [22, 23], the Kadanoff's block RG [24, 25]. These works enlighten people on studying thermal entanglement on complicated lattices. In this paper, we study the thermal entanglement between two end sites in the spin chains, Koch curves and on the diamond-type hierarchical (DH) lattices by taking the negativity as measure method and applying the decimation RG method.

The organization of this paper is as follows. In Sec. II we briefly introduce the spin systems on different fractal lattices and how to calculate the entanglement of end sites by using the RG method. The entanglement properties in spin chains and on fractal lattices are studied in Sec. III and Sec. IV, respectively. The summary is given in Sec. V.

II. MODEL AND METHOD

Let us consider a L-site anisotropic ferromagnetic Heisenberg spin system in thermal equilibrium with the effective Hamiltonian

$$-\beta H = \sum_{\langle i,j \rangle} K \left[(1-\Delta) \left(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right) + \sigma_i^z \sigma_j^z \right], \tag{1}$$

where σ_i^α $(\alpha = x, y, z)$ denote the Pauli operators at site i. $K \equiv \beta J \equiv J/k_B T$, J is the exchange coupling parameter, k_B is the Boltzmann constant, and T is the absolute temperature. For simplicity, we set $J/k_B = 1$. Δ is the anisotropy parameter, for $\Delta = 0$ and $\Delta = 1$ the isotropic Heisenberg and Ising systems are obtained, respectively. The sum is over all the nearest-neighbor spin pairs on this system. The equilibrium state of this spin system at certain temperature can be described by the canonical density operator $\rho = e^{-\beta H}/Z$, where $Z = \mathrm{Tr} e^{-\beta H}$ is the partition function of the system.

For a dimer containing two spins in the equilibrium state ρ_{12}, the thermal entanglement can be measured by using the negativity [36] which is based on the partial transpose method [37] and is defined as

$$N(\rho_{12}) = 2 \sum_i |\mu_i|, \tag{2}$$

where μ_i is the negative eigenvalue of $\rho_{12}^{T_1}$, T_1 denotes the partial transpose with respect to the first subsystem. Here, the factor 2 can make our results more clear (in order that $N = 1$ for the Bell state) and will not influence the correctness of the physical picture.

With the definition of negativity, we will focus on how to obtain the entanglement of two end sites in spin chains and on both fractal lattices (which were given in the previous Ref [26–28]) by implementing the decimation RG method [29–35]. In this RG method, the transformations can be done by tracing out the internal spins step by step (for simplicity, the procedure of the transformation can be shown in Fig. 1). By comparing the Hamiltonians after and before the transformation, the recurrence relations between the new parameters and the original parameters can be obtained. Through combining the recurrence relations and the negativity of dimer, we can work out the negativity of two distant end sites in spin chains and on fractal lattices.

III. HEISENBERG SPIN CHAIN

We first study how the entanglement between two end sites in spin chain containing L sites varies with temperature T. The results (shown in Fig. 2 (a)) exhibit the similar feature that the entanglement decreases monotonically with increasing T (with the unit J/k_B, the same below) and vanishes beyond the critical temperature T_c. At $T = 0$, the system is in the entangled ground state and the entanglement decreases as T increases due to the mixture of the unentangled excited state with the ground state. At $T = T_c$, the system is mainly governed by the unentangled excited state, and therefore the entanglement naturally vanishes. It is also clear to see that with the increase of L the maximum of entanglement (at $T = 0$) decreases sharply and the critical temperature T_c becomes low. Different from the maximally entangled Bell ground state for the dimer with $L = 2$:

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \tag{3}$$

the ground states for the chains with $L > 2$ are related to Δ and degenerate states which are not maximally entangled. For example, the ground states for the chain with $L = 3$ at $\Delta < 0$ are twofold degenerate:

$$|\psi_1\rangle = C\left(|\uparrow\uparrow\downarrow\rangle + m\,|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right), \tag{4}$$

$$|\psi_2\rangle = C\left(|\uparrow\downarrow\downarrow\rangle + m\,|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right), \tag{5}$$

where $C = \frac{1}{\sqrt{m^2+2}}$ is the normalization constant, and

$$m = \frac{\sqrt{8\left(1-\Delta\right)^2 + 1} - 1}{2\left(1-\Delta\right)}. \tag{6}$$

There exists an energy gap E_{gap} between the entangled ground state and the unentangled excited one which is dependent on the energy level of the system. Because of the competition between the energy gap $E_{\rm gap}$ and the thermal excitation energy $E_{\rm T}$, the chains with $L > 2$ can easily overcome $E_{\rm gap}$ and enter the unentangled excited states. It leads to that T_c decreases with the increase of L. This phenomenon reflects that thermal fluctuation can suppress easily quantum effect in this system.

The influence of the anisotropic parameter Δ on the entanglement in the spin chains at $T = 0.01$ is shown in Fig. 2 (b). As can be seen, there is no thermal entanglement between two end spins when $0 < \Delta \leq 1$. It means that the entanglement is absent when the coupling along z-direction is stronger than the coupling along the x- and y-orientation. With increasing Δ at $\Delta < 0$, the dimer with $L = 2$ first keeps in the maximally entangled Bell ground state $|\varphi_1\rangle$, but for the chains with $L > 2$ the entanglement first increases with increasing Δ. As Δ approaches the isotropic point ($\Delta = 0$), the entanglement for all cases decreases sharply to zero (shown in the inset of Fig. 2 (b)). For the chains with $L > 2$, the entanglement first increases with increasing Δ because the ground states are related to Δ (for the $L = 3$ case, $|\psi_1\rangle$ and $|\psi_2\rangle$). As Δ is close to the isotropic point, the energy gap $E_{\rm gap}$ between the ground and the unentangled excited states becomes so small that the system can jump to the unentangled excited state. Therefore there exists a sharp decease to zero for entanglement when Δ is close to zero. This result also accords with that the entanglement is always absent in an isotropic Heisenberg ferromagnetic chain in Ref [8]. From above results, we can see that the entanglement on the chains is fragile and the entanglement vanishes when $L \geq 17$.

IV. FRACTAL LATTICES

We turn to studying the entanglement in Koch curves with fractal dimension $d_f = 1.26$ and plot the numerical results of negativity versus T and Δ in Fig. 3. Compared with the result in the spin chain ($L = 5$) at $\Delta = -0.2$, it has similar properties that the entanglement decreases with increasing T, the maximum of entanglement and the corresponding T_c are approximately equal. But the variation of the entanglement versus Δ for Koch curves and spin chains are different, i.e., for Koch curves the range of Δ where the entanglement can survive is smaller and the corresponding maximum of entanglement is lager. The entanglement decreases quickly as L increases and disappears when $L \geq 17$. We can deduce this result from the similar energy level structure and open boundary conditions of these two systems.

For two kinds of DH lattices with fractal dimensions $d_f = 2$ (lattice A, for simplicity) and $d_f = 2.32$ (lattice B, for simplicity), we first discuss the dependence of the entanglement on T with $\Delta = -0.2$ (shown in Fig. 4). It is found that the entanglement is the maximum at $T = 0$, and decreases with increasing T and vanishes beyond the critical temperature T_c. However, some interesting phenomena are also observed that the entanglement on both lattices decreases more slowly with increasing L and two end sites are entangled even though the size of the system becomes very large ($L = 1564$). For the lattice B, as L increases, the maximum of entanglement at $T = 0$ decreases and there exists a crossing point on the entanglement curves at $T \approx 0.49$ (shown in Fig. 4 (b)). With L becoming large, the entanglement decreases more slowly and the critical temperature T_c does not decrease but increases and finally tends to the stable value. This result indicates that the entanglement on the DH lattice is much robust against the decoherence caused by temperature. For both DH lattices, the energy gap E_{gap} between the entangled ground state and the unentangled excited state is so large that the system can jump to the unentangled excited state only at higher temperature. The behavior of the critical temperature T_c with the increase of the size of system L is shown in Fig. 5. It is clear to see that T_c of both DH lattices (Fig. 5 (c) and (d)) decrease slowly or increase when L becomes large in contrast with that in spin chain (Fig. 5 (a)) and Koch curve (Fig. 5 (b)). The fractal structure can affect the energy level structures of systems, which can lead to the different entanglement properties.

The variation of the entanglement with Δ at $T = 0.01$ is also obtained. For the lattice A, Fig. 6 (a) presents that the entanglement firstly increases as Δ increases, and then it quickly decreases to zero when Δ reaches to the isotropic point. The entanglement decreases very slowly when L becomes very large and there is no cross point when Δ reaches zero. For the lattice B, Fig. 6 (b) shows that the entanglement also exhibits stable and it changes slowly when Δ is not very close to zero. The entanglement mainly remains robust with the increase of L. It is well known that the indirect interaction between sites which do not connect directly may lead to long-distance correlation for them. For the DH lattices, there exist stronger indirect interactions between both end spins than that in spin chains and Koch curves, which causes the subsistence of entanglement for large L. We can also observe that an "entanglement cross point" occurs at $\Delta \approx -0.045$ in the inset of Fig. 6 (b). At a certain temperature, the thermal excitation energy of the system is certain. Only when Δ is very close to zero, the energy gap between the entangled ground state and the unentangled excited state becomes so small that the system can enter the unentangled state. It also indicates that different fractal lattices have various entanglement properties due to difference of energy level structures. The spin systems with short-ranged interaction on both DH lattices do not show a fast decay of entanglement with the size of the system. The spin systems with robust long-distance entanglement can be used to realize quantum communication channels for teleportation and transfer state. This special topological feature can be useful in developing the quantum networks. Some works about the quantum complex networks can prove that long-distance entanglement between arbitrary nodes can exists on 2-D lattices [38, 39].

In this paper, we have applied the decimation RG method. In the procedure of decimation transformation, some approximations, i. e., ignoring the non-commutative term of Hamiltonians, are adopted, which can produce some errors. This approximation and the influence of the error in this method have been detailedly discussed in Ref [31]. It is found that the error is reduced with the increase of the temperature.

V. SUMMARY

We have investigated the thermal entanglement between two end spins in Heisenberg chains, Koch curves and on DH lattices by using the decimation RG method. By considering the energy level and the competition between the thermal excitation energy and the energy gap, the effects of the temperature and the anisotropy parameter on thermal entanglement are discussed. We have also found that the entanglement on some special lattices may exhibit interesting properties when the size of system becomes large. The phenomenon of the "entanglement cross point" indicates that the special fractal structure does influence on the entanglement between non-nearest-neighbor sites. The entanglement on both DH lattices is quite robust and the long-distance entanglement exists between end sites, but the entanglement in spin chain and the Koch curve is fragile. This special DH lattices can be used to design quantum communications channels for teleportation and transfer state.

VI. ACKNOWLEDGMENTS

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Figure captions:

Fig. 1. (Color online) The procedure of the RG transformation: (a) 1-D spin chain; (b) Koch curve; (c) DH lattice with $d_f = 2$; (d) DH lattice with $d_f = 2.32$.

Fig. 2. (Color online) The entanglement between two end sites in Heisenberg chain with L-spin: (a) the entanglement versus temperature T (with the unit J/k_B, the same below) at $\Delta = -0.2$; (b) the entanglement versus anisotropy parameter Δ at $T = 0.01$.

Fig. 3. (Color online) The entanglement of end sites in Koch curve with L-spin: (a) the entanglement versus temperature T at $\Delta = -0.2$; (b) the entanglement versus Δ at $T = 0.01$.

Fig. 4. (Color online) The entanglement of end sites on the DH lattice versus T at $\Delta = -0.2$: (a) lattice A with $d_f = 2$; (b) lattice B with $d_f = 2.32$. The entanglement curves for different L cases have a cross point at $T \approx 0.49$.

Fig. 5. (Color online) The critical temperature T_c versus the system size L at $\Delta = -0.2$: (a) 1-D spin chain; (b) Koch curve; (c) DH lattice with $d_f = 2$; (d) DH lattice with $d_f = 2.32$.

Fig. 6. (Color online) The entanglement on the DH lattice versus Δ at $T = 0.01$: (a) lattice A with $d_f = 2$; (b) lattice B with $d_f = 2.32$. The inset shows that the entanglement curves for different L cases have a cross point at $\Delta \approx -0.045$.

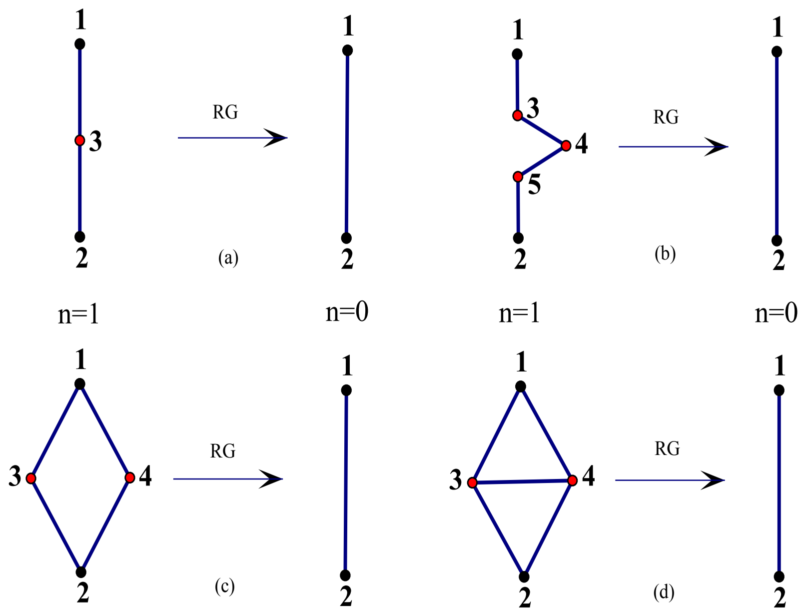

Figure 1 XXXXXXXX 05Dec2012

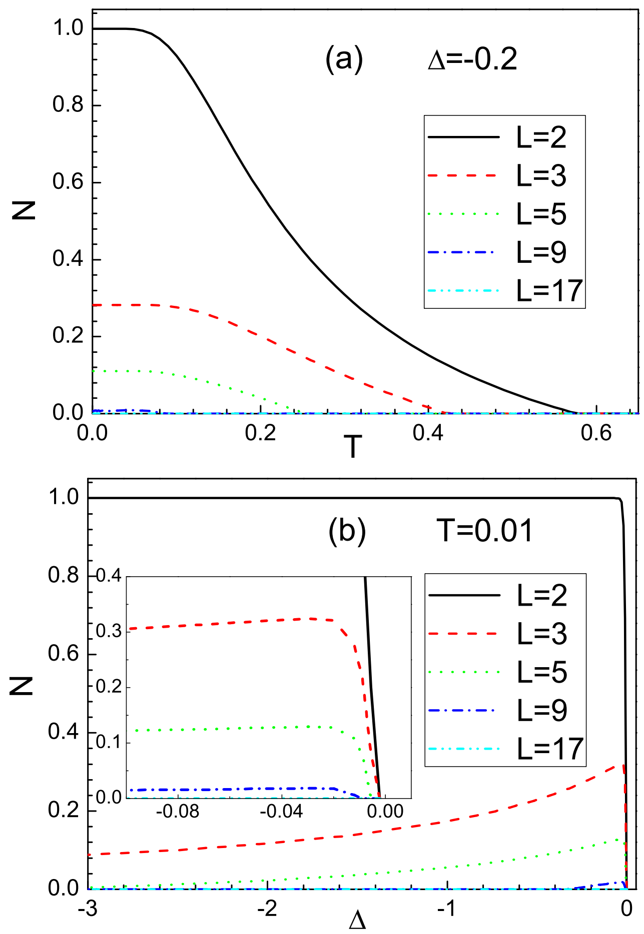

Figure 2 XXXXXXXX 05Dec2012

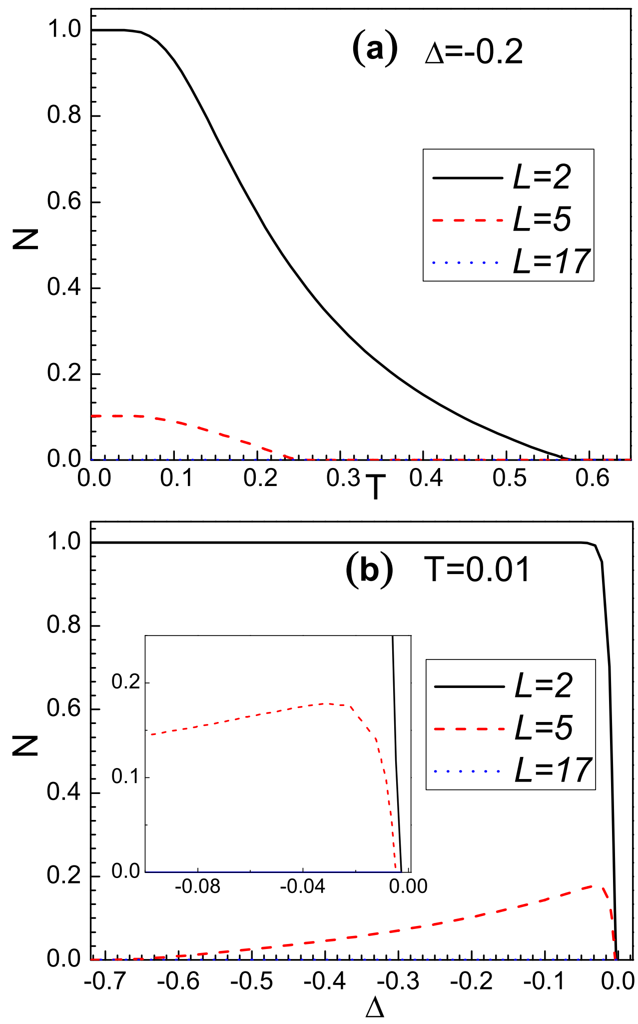

Figure 3 XXXXXXXX 05Dec2012

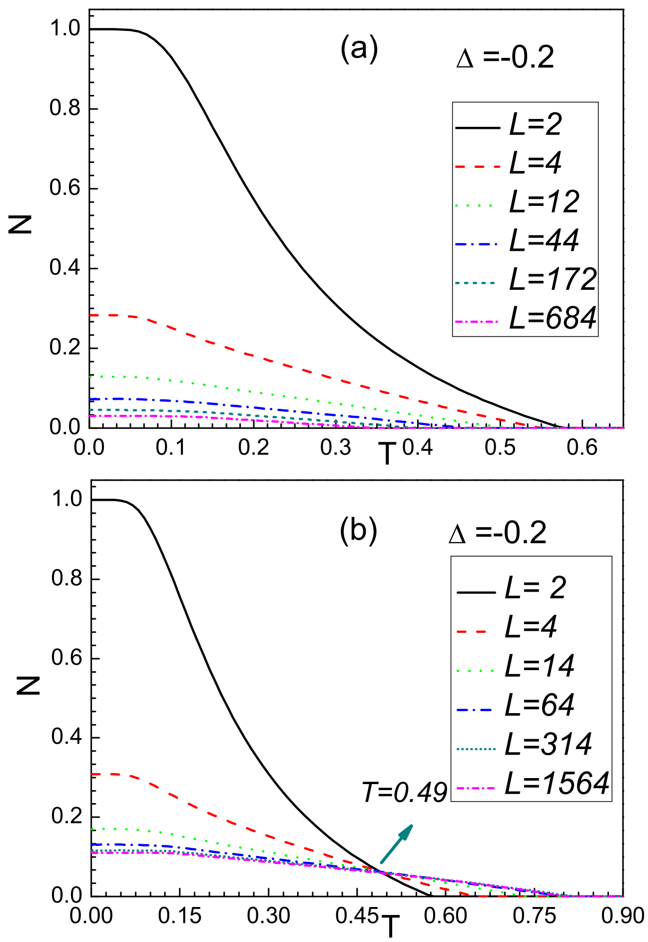

Figure 4 XXXXXXXX 05Dec2012

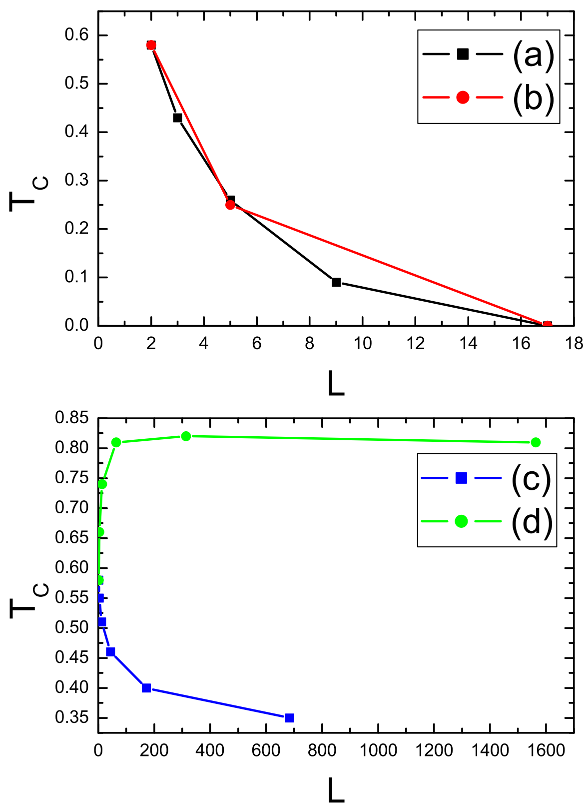

Figure 5 XXXXXXXXX 05Dec2012

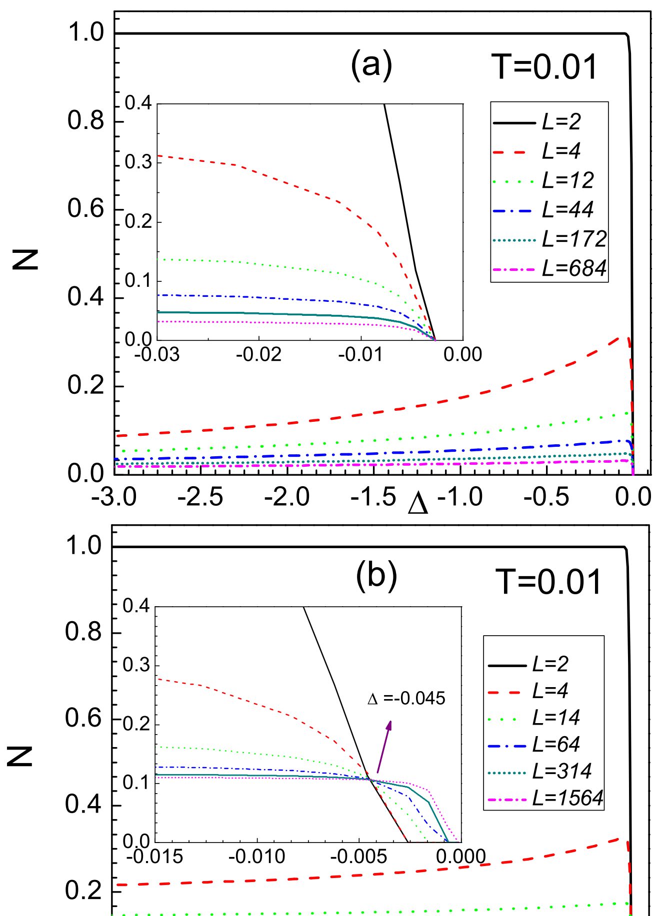

Figure 6 XXXXXXXX 05Dec2012