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# Three-Loop Contribution to Hyperfine Splitting in Muonium: Polarization Corrections to Light by Light Scattering Blob

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## Abstract

We calculate corrections of order  $\alpha^3(Z\alpha)E_F$  to hyperfine splitting in muonium generated by the gauge invariant set of diagrams with polarization insertions in the light by light scattering diagrams. This nonrecoil contribution turns out to be  $-2.63$  Hz. The total contribution of all known corrections of order  $\alpha^3(Z\alpha)E_F$  is equal to  $-4.28$  Hz.

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## I. INTRODUCTION

The hyperfine splitting in muonium is one of the intervals best studied both experimentally and theoretically. Theoretical expression for the hyperfine splitting can be calculated in the QED framework in the form of a perturbation theory expansion in  $\alpha$ ,  $Z\alpha$ ,  $m_e/m_\mu$ . Current theoretical uncertainty of this expansion is estimated to be about 70-100 Hz, respective relative error does not exceed  $2.3 \times 10^{-8}$  (see discussions in [1–3]). The experimental error of the best measurements [4, 5] of the muonium HFS is in the interval 16-51 Hz. A new higher accuracy measurement of muonium HFS is now planned at J-PARC, Japan [6]. Combining muonium HFS theory and experiment one can determine the value of  $\alpha^2(m_\mu/m_e)$  with the uncertainty that is dominated by  $2.3 \times 10^{-8}$  relative uncertainty of the HFS theory [3]. This is currently the best way to determine the precise value of the electron-muon mass ratio. Further reduction of the uncertainty of this mass ratio requires improvement of the HFS splitting theory. Main sources of the theoretical uncertainty are due to still unknown three-loop purely radiative contributions, three-loop radiative-recoil contribution, and non-logarithmic recoil contributions (see detailed discussion in [2, 3]). We consider reduction of the theoretical error of HFS splitting in muonium to about 10 Hz as the current goal of the HFS theory.

As a step in this direction we calculate below a three-loop contribution to HFS generated by the light by light scattering diagrams in Figs. 1 and 2 with insertions of one loop polarization in the upper and lower photon lines, respectively.

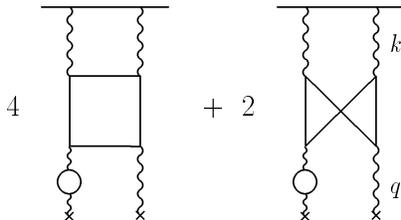


FIG. 1.

## II. CALCULATIONS

We start with the light by light scattering contribution to HFS that was calculated long time ago [7]. It is generated by the diagrams in Fig. 3, where we have not shown explicitly

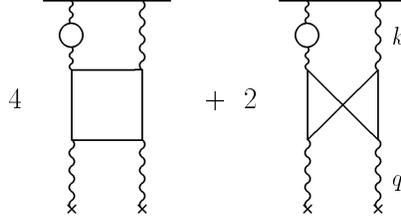


FIG. 2.

three more diagrams with the crossed photon lines. In our calculations below we will follow the general approach developed in [7] and start with the light by light scattering contribution in Fig. 3 (see [7])

$$\Delta E = \frac{\alpha^2(Z\alpha)}{\pi} E_F \frac{3}{64\pi^2} \int \frac{d^4k}{\pi^2 i} \frac{\langle \gamma_\alpha k \gamma_\beta \rangle}{k^4} \left( \frac{1}{k^2 + 2k_0} + \frac{1}{k^2 - 2k_0} \right) \int \frac{d^3q}{4\pi} \frac{\langle \gamma_\mu q \gamma_\nu \rangle}{q^4} S^{\alpha\beta\mu\nu}, \quad (1)$$

where  $k^\mu$  is the four-momentum carried by the upper photon lines,  $q^\mu = (0, \mathbf{q})$  is the spacelike four-momentum carried by the lower photon lines,  $S^{\alpha\beta\mu\nu}$  is the light by light scattering tensor, and all momenta are measured in the electron mass units. The Fermi energy is defined as

$$E_F = \frac{8}{3} (Z\alpha)^4 (1 + a_\mu) \frac{m_e}{m_\mu} \left( \frac{m_r}{m_e} \right)^3 m_e c^2, \quad (2)$$

where  $a_\mu$  is the muon anomalous magnetic moment. The angle brackets in Eq. (1) denote the projection of the  $\gamma$ -matrix structures on the HFS interval (difference between the states with the total spin one and zero).

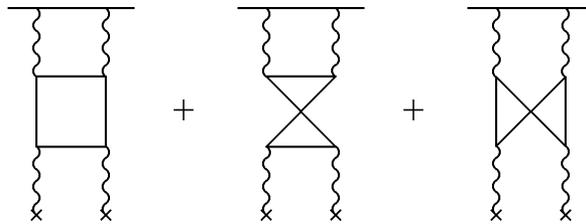


FIG. 3.

With account for three more diagrams with crossed photon lines not shown explicitly in Fig. 3 contributions to HFS of the first two diagrams coincide and we can represent the light by light block as a sum of two contributions corresponding to the first two (ladder) diagrams in Fig. 3 and corresponding to the crossed (last) diagram in Fig. 3

$$S^{\alpha\beta\mu\nu} = \int \frac{d^4p}{\pi^2 i} (2L^{\alpha\beta\mu\nu} + C^{\alpha\beta\mu\nu}), \quad (3)$$

where (we return to dimensionful momenta here)

$$L^{\alpha\beta\mu\nu} = Tr \left[ \gamma_\mu \frac{1}{\not{p} - \not{q} - m} \gamma_\nu \frac{1}{\not{p} - m} \gamma_\beta \frac{1}{\not{p} - \not{k} - m} \gamma_\alpha \frac{1}{\not{p} - m} \right], \quad (4)$$

$$C^{\alpha\beta\mu\nu} = Tr \left[ \gamma_\mu \frac{1}{\not{p} - \not{q} - m} \gamma_\beta \frac{1}{\not{p} - \not{q} - \not{k} - m} \gamma_\nu \frac{1}{\not{p} - \not{k} - m} \gamma_\alpha \frac{1}{\not{p} - m} \right]. \quad (5)$$

Calculating traces we obtain

$$\begin{aligned} L^{\alpha\beta\mu\nu} = & \{ 8D_1^2 g^{\mu\alpha} g^{\nu\beta} + 16D_1 g^{\mu\alpha} [p^\nu q^\beta + k^\nu p^\beta + k^\nu q^\beta - p^\nu p^\beta] - 8D_1 g^{\mu\alpha} g^{\nu\beta} \\ & \times [(p \cdot q) + (p \cdot k) + (k \cdot q)] + 32g^{\mu\alpha} [(k \cdot q)p^\nu p^\beta - (p \cdot q)k^\nu p^\beta - (p \cdot k)p^\nu q^\beta] \\ & + 16(p \cdot k)(p \cdot q)g^{\mu\alpha} g^{\nu\beta} - 32p^\mu p^\alpha k^\nu q^\beta \} \frac{1}{D_1 D_2 D_3 D_4}, \end{aligned} \quad (6)$$

and

$$\begin{aligned} C^{\alpha\beta\mu\nu} = & \{ 8D_1 g^{\mu\alpha} [-3k^\nu q^\beta + k^\nu p^\beta + p^\nu q^\beta] - 8D_2 g^{\mu\alpha} k^\nu p^\beta - 8D_3 g^{\mu\alpha} p^\nu q^\beta \\ & + 16(k \cdot q)g^{\mu\alpha} p^\nu p^\beta - 16(k \cdot q)g^{\mu\alpha} k^\nu p^\beta - 16(k \cdot q)g^{\mu\alpha} p^\nu q^\beta + 16p \cdot (k + q)g^{\mu\alpha} k^\nu q^\beta \} \\ & \times \frac{1}{D_1 D_2 D_3 D_4}, \end{aligned} \quad (7)$$

where

$$D_1 = p^2 - m^2, \quad D_2 = (p - q)^2 - m^2, \quad D_3 = (p - k)^2 - m^2, \quad D_4 = (p - q - k)^2 - m^2. \quad (8)$$

After calculation of the integrals in Eq. (3) we obtain the light by light scattering block in the form

$$S^{\alpha\beta\mu\nu} = 2\mathcal{L}^{\alpha\beta\mu\nu} + \mathcal{C}^{\alpha\beta\mu\nu}, \quad (9)$$

where

$$\begin{aligned}
\mathcal{L}^{\alpha\beta\mu\nu} &= \int \frac{d^4p}{\pi^2 i} L^{\alpha\beta\mu\nu} = 8 \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 d\xi \left\{ -\frac{2y(1-y)}{\Omega(1, y, 1, \xi)} (k \cdot q) g^{\mu\alpha} g^{\nu\beta} \right. \\
&+ \frac{2y^2(1-y)z}{\Omega(1, y, z, \xi)} (k \cdot q) g^{\mu\alpha} g^{\nu\beta} - \frac{2y(2-y+y^2z)}{\Omega(1, y, z, 1)} g^{\mu\alpha} k^\nu q^\beta + \frac{y}{\Omega(1, y, z, 1)} \\
&\times \left[ k^2(1-y) + q^2yz + (k \cdot q)(2-y+yz) \right] g^{\mu\alpha} g^{\nu\beta} + y^2(1-z) \\
&\times \left[ \left( -\frac{3}{\Omega(1, y, z, 1)} + \frac{2}{\Omega^2(1, y, z, 1)} [k^2(1-y)^2 + q^2y^2z^2] \right) [(k \cdot q)g^{\mu\alpha}g^{\nu\beta} - 2g^{\mu\alpha}k^\nu q^\beta] \right. \\
&\left. + \frac{2(1-y)yz}{\Omega^2(1, y, z, 1)} [k^2q^2 + (k \cdot q)^2] g^{\mu\alpha} g^{\nu\beta} - \frac{4(1-y)yz}{\Omega^2(1, y, z, 1)} (k \cdot q) g^{\mu\alpha} k^\nu q^\beta \right] \Bigg\}, \tag{10}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^{\alpha\beta\mu\nu} &= \int \frac{d^4p}{\pi^2 i} C^{\alpha\beta\mu\nu} = 8 \int_0^1 dx \int_0^1 dy \int_0^1 dz \left\{ \left[ \frac{x(1+x)}{\Omega(x, y, 1, 1)} + \frac{x(1-x)}{\Omega(x, y, 0, 1)} \right. \right. \\
&+ \left. \frac{x(1-x)}{\Omega(x, 1, z, 1)} \right] g_{\mu\alpha} k_\nu q_\beta - \frac{x^2y}{\Omega(x, y, z, 1)} (k \cdot q) g_{\mu\alpha} g_{\nu\beta} + \frac{2x^2y}{\Omega^2(x, y, z, 1)} \\
&\times \left[ k^2(1-xy) + q^2(1-x+xyz) + (k \cdot q)(1-xy)(1-x+xyz) \right] g_{\mu\alpha} k_\nu q_\beta \Bigg\}, \tag{11}
\end{aligned}$$

$$\Omega(x, y, z, \xi) = m^2 - k^2xy(1-xy) - q^2x(1-yz)(1-x+xyz) - 2k \cdot qxy[1-x-z(1-xy)]\xi, \tag{12}$$

and  $g^{\mu\nu} = (1, -1, -1, -1)$ .

The  $\gamma$ -matrix structures in Eq. (1) are antisymmetric in  $(\alpha, \beta)$  and  $(\mu, \nu)$ , and we have thrown away all symmetric in  $(\alpha, \beta)$  and  $(\mu, \nu)$  terms in Eq. (10) and Eq. (11). We have also combined terms that coincide after antisymmetrization, and deleted even in  $k$  and  $q$  terms that disappear anyway after substitution in the odd in these momenta integral in Eq. (1). As a result we automatically subtracted symmetric in  $k$  and  $q$  logarithmically divergent contribution in  $S^{\alpha\beta\mu\nu}$ , and the result in Eq. (9) is finite and gauge invariant.

Next we substitute the light by light scattering tensor in Eq. (1), and introduce two new Feynman parameters  $t$  and  $u$  to combine the upper photon propagators, the electron propagator, and the denominator  $\Omega(x, y, z, \xi)$  in the integral representations in Eq. (10) and Eq. (11) of the light by light scattering tensor

$$(1-u) [(1-t)k^2 + t(k^2 - 2mk_0)] + u \left[ \frac{\Omega(x, y, z, \xi)}{-xy(1-xy)} \right] = (k-Q)^2 - \Delta, \tag{13}$$

where  $Q = qd + \tau$ ,  $\Delta = g(-q^2 + a^2)$ ,  $\tau = m(1 - u)t$ , and

$$d = \xi u \left[ z - \frac{1 - x}{1 - xy} \right], \quad g = \frac{u(1 - yz)(1 - x + xyz)}{y(1 - xy)} - d^2, \quad a^2 = \frac{1}{g} \left[ \tau^2 + \frac{m^2 u}{xy(1 - xy)} \right]. \quad (14)$$

After the Wick rotation and integration over  $k$  and  $q$  (we return to dimensionless momenta here) we obtain an expression for the light by light contribution to HFS

$$\Delta E = 2\Delta E_L + \Delta E_C \equiv (2\Delta\epsilon_L + \Delta\epsilon_C) \frac{\alpha^2(Z\alpha)}{\pi} E_F, \quad (15)$$

where

$$\Delta\epsilon_{L(C)} = \sum_i \Delta\epsilon_{L(C)}^{(i)}. \quad (16)$$

The integrals  $\Delta\epsilon_{L(C)}^{(i)}$  arise in calculations of the ladder and crossed diagram contributions and have the general form

$$\Delta\epsilon_{L(C)}^{(i)} = \int_0^\infty dq \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dt \int_0^1 du \int_0^1 d\xi \mathcal{J}_{L(C)}^{(i)}. \quad (17)$$

The integrands  $\mathcal{J}_L^{(i)}$  and  $\mathcal{J}_C^{(i)}$  are collected in Tables I and II, respectively. Notice that not all Feynman parameters arise in all integrands in Tables I and II. Some parameters just do not arise in particular integrals, or take a fixed value, for example,  $x = 1$  in the ladder light by light scattering diagram diagram, see Eq. (10). As a result the expressions for  $\Delta$  in the Tables are simpler than the general expression below Eq. (13).

The third columns in Tables I and II contain separate integrals  $\Delta\epsilon_L^{(i)}$  and  $\Delta\epsilon_C^{(i)}$ , and respective sums in the last lines. The sum of the ladder and crossed diagram contributions in Eq. (15) nicely reproduces the old result [7, 8]

$$\Delta E = -0.472\,514 (1) \frac{\alpha^2(Z\alpha)}{\pi} E_F, \quad (18)$$

for light by light scattering contribution to HFS.

Let us calculate contributions to HFS generated by the diagrams with polarization insertions in Figs. 1 and 2. We use the well known integral representation for the polarization operator (see, e.g., [2])

TABLE I. First Set of Ladder Light by Light Integrals<sup>a</sup>

$i$	$\mathcal{J}_L^{(i)}$	$\Delta\epsilon_L^{(i)}$	$\Delta\epsilon_L^{vp(i)}$
1	$-\frac{8}{\pi^2}(1-t)(1-u)^2 \left(\frac{1}{2\Delta} + \frac{q^2 d^2}{\Delta^2}\right)_{ z=1}$	-0.6255...	-0.914...
2	$\frac{8}{\pi^2}yz(1-t)(1-u)^2 \left(\frac{1}{2\Delta} + \frac{q^2 d^2}{\Delta^2}\right)$	0.1498...	0.215...
3	$\frac{4}{\pi^2} \frac{2-y+y^2z}{1-y} (1-t)(1-u)^2 \left(\frac{1}{\Delta} - \frac{\tau^2}{\Delta^2}\right)_{ \xi=1}$	3.2252...	6.403...
4	$\frac{4}{\pi^2} \frac{(1-u)d}{\Delta}  _{\xi=1}$	0.0697...	0.085...
5	$\frac{4}{\pi^2} q^2 \frac{yz}{1-y} \frac{(1-t)(1-u)^2 d}{\Delta^2}  _{\xi=1}$	0.1628...	0.364...
6	$\frac{4}{\pi^2} \frac{2-y+yz}{1-y} (1-t)(1-u)^2 \left(\frac{1}{2\Delta} + \frac{q^2 d^2}{\Delta^2}\right)_{ \xi=1}$	2.0905...	3.927...
7	$-\frac{12}{\pi^2} \frac{y(1-z)}{1-y} (1-t)(1-u)^2 \left(\frac{3}{2\Delta} + \frac{q^2 d^2 - \tau^2}{\Delta^2}\right)_{ \xi=1}$	-3.8178...	-8.070...
8	$-\frac{8}{\pi^2} (1-z)u(1-u) \left(\frac{3}{2\Delta} + \frac{q^2 d^2 - \tau^2}{\Delta^2}\right)_{ \xi=1}$	-0.3303...	-0.669...
9	$-\frac{8q^2}{\pi^2} \frac{y^2 z^2 (1-z)}{(1-y)^2} (1-t)u(1-u)^2 \left(\frac{3}{2\Delta^2} + \frac{2(q^2 d^2 - \tau^2)}{\Delta^3}\right)_{ \xi=1}$	-0.4842...	-1.078...
10	$-\frac{8q^2}{\pi^2} \frac{yz(1-z)}{1-y} \frac{u(1-u)d}{\Delta^2}  _{\xi=1}$	-0.0193...	-0.036...
11	$-\frac{8q^2}{\pi^2} \frac{yz(1-z)}{1-y} (1-t)u(1-u)^2 d \left(\frac{3}{2\Delta^2} + \frac{2q^2 d^2}{\Delta^3}\right)_{ \xi=1}$	-0.0105...	-0.019...
12	$-\frac{8q^2}{\pi^2} \frac{yz(1-z)}{1-y} (1-t)u(1-u)^2 d \left(\frac{1}{\Delta^2} - \frac{2\tau^2}{\Delta^3}\right)_{ \xi=1}$	-0.0058...	-0.010...
$\sum_i$		0.4045...	0.195...

<sup>a</sup> In this table

$$d = \xi uz, \quad \tau = (1-u)t, \quad g = \frac{uz(1-yz)}{1-y} - d^2, \quad a^2 = \frac{1}{g} \left[ \tau^2 + \frac{u}{y(1-y)} \right], \quad \Delta = g(q^2 + a^2).$$

$$\Pi(q^2) = \frac{\alpha}{\pi} \int_0^1 dv v^2 \left( 1 - \frac{v^2}{3} \right) \frac{q^2}{q^2(1-v^2) + 4}, \quad (19)$$

where the dimensionless momentum  $q$  is Euclidean.

Momentum  $q$  in the integrands in Tables I and II is also Euclidean and to account for the polarization operator insertions in both lower photon lines in Fig. 1 it is sufficient to insert the factor  $2\Pi(q^2)$  in the integrands in Tables I and II. Similarly to Eq. (15) respective contributions to HFS can be written as

$$\Delta E_d = 2\Delta E_L^{vp} + \Delta E_C^{vp} \equiv (2\Delta\epsilon_L^{vp} + \Delta\epsilon_C^{vp}) \frac{\alpha^3(Z\alpha)}{\pi^2} E_F, \quad (20)$$

TABLE II. First Set of Crossed Light by Light Integrals<sup>a</sup>

$i$	$\mathcal{J}_C^{(i)}$	$\Delta\epsilon_C^{(i)}$	$\Delta\epsilon_C^{vp(i)}$
1	$-\frac{2}{\pi^2} \frac{1+x}{y(1-xy)} (1-t)(1-u)^2 \left( \frac{1}{\Delta} - \frac{\tau^2}{\Delta^2} \right)_{ z=1}$	-1.6733...	-3.294...
2	$-\frac{2}{\pi^2} \frac{1-x}{y(1-xy)} (1-t)(1-u)^2 \left( \frac{1}{\Delta} - \frac{\tau^2}{\Delta^2} \right)_{ z=0}$	-0.2729...	-0.405...
3	$-\frac{2}{\pi^2} (1-t)(1-u)^2 \left( \frac{1}{\Delta} - \frac{\tau^2}{\Delta^2} \right)_{ y=1}$	-0.3665...	-0.818...
4	$-\frac{2}{\pi^2} \frac{x}{1-xy} (1-t)(1-u)^2 \left( \frac{1}{\Delta} + \frac{2q^2 d^2}{\Delta^2} \right)$	-0.2997...	-0.431...
5	$-\frac{4}{\pi^2} \frac{u(1-u)}{y(1-xy)} \left( -\frac{1}{\Delta} + \frac{\tau^2}{\Delta^2} \right)$	0.3460...	0.387...
6	$\frac{4}{\pi^2} \frac{1-x+xyz}{y(1-xy)^2} (1-t)u(1-u)^2 \left( \frac{q^2}{\Delta^2} - \frac{2q^2\tau^2}{\Delta^3} \right)$	0.9869...	2.943...
7	$\frac{4}{\pi^2} \frac{1-x+xyz}{y(1-xy)} (1-t)u(1-u)^2 d \left( \frac{q^2}{\Delta^2} - \frac{2q^2\tau^2}{\Delta^3} \right)$	-0.0020...	-0.004...
$\sum_i$		-1.2816...	-1.623...

<sup>a</sup> In this table

$$d = u \left[ z - \frac{1-x}{1-xy} \right], \quad \tau = (1-u)t, \quad g = \frac{u(1-yz)(1-x+xyz)}{y(1-y)} - d^2,$$

$$a^2 = \frac{1}{g} \left[ \tau^2 + \frac{u}{xy(1-xy)} \right], \quad \Delta = g(q^2 + a^2).$$

where

$$\Delta\epsilon_{L(C)}^{vp} = \sum_i \Delta\epsilon_{L(C)}^{vp(i)}. \quad (21)$$

The fourth columns in Tables I and II contain the integrals  $\Delta\epsilon_L^{vp(i)}$  and  $\Delta\epsilon_C^{vp(i)}$ , and their sums in the last row. Collecting these contributions we obtain the total contribution to HFS generated by the diagrams in Fig. 1

$$\Delta E_d = -1.2326(5) \frac{\alpha^3(Z\alpha)}{\pi^2} E_F. \quad (22)$$

Calculation of the contributions to HFS generated by the diagrams in Fig. 2 follows the same general route as for the diagrams with polarization insertions in the lower photons. We again parameterize contribution to HFS generated the light by light scattering diagrams in Fig. 3 exactly like in Eq. (15). However, now it is convenient first to integrate analytically over momentum  $q$  in the integrals in Eq. (17). As a result the separate contributions to HFS

acquire the form

$$\Delta\epsilon_{L(C)}^{(i)} = \int_0^1 dy \int_0^1 dz \int_0^1 dt \int_0^1 du \int_0^1 d\xi \mathcal{K}_{L(C)}^{(i)}. \quad (23)$$

The integrands  $\mathcal{K}_L^{(i)}$  and  $\mathcal{K}_C^{(i)}$  are collected in Tables III and IV.

TABLE III. Second Set of Ladder Light by Light Integrals<sup>a</sup>

$i$	$\mathcal{K}_L^{(i)}$	$\Delta\epsilon_L^{(i)}$	$\Delta\epsilon_L^{vp(i)}$
1	$-\frac{2}{\pi}(1-t)(1-u)^2 \left( \frac{1}{ag} + \frac{d^2}{ag^2} \right)_{ z=1}$	-0.6255...	-0.4631...
2	$\frac{2}{\pi}yz(1-t)(1-u)^2 \left( \frac{1}{ag} + \frac{d^2}{ag^2} \right)$	0.1498...	0.1044...
3	$\frac{1}{\pi} \frac{2-y+y^2z}{1-y} (1-t)(1-u)^2 \left( \frac{2}{ag} - \frac{\tau^2}{a^3g^2} \right)_{ \xi=1}$	3.2252...	2.9251...
4	$\frac{2}{\pi} \frac{(1-u)d}{ag}  _{\xi=1}$	0.0697...	0.1083...
5	$\frac{1}{\pi} \frac{yz}{1-y} \frac{(1-t)(1-u)^2d}{ag^2}  _{\xi=1}$	0.1628...	0.1273...
6	$\frac{1}{\pi} \frac{2-y+yz}{1-y} (1-t)(1-u)^2 \left( \frac{1}{ag} + \frac{d^2}{ag^2} \right)_{ \xi=1}$	2.0905...	1.6319...
7	$-\frac{3}{\pi} \frac{y(1-z)}{1-y} (1-t)(1-u)^2 \left( \frac{3}{ag} + \frac{d^2}{ag^2} - \frac{\tau^2}{a^3g^2} \right)_{ \xi=1}$	-3.8178...	-3.3474...
8	$-\frac{2}{\pi}(1-z)u(1-u) \left( \frac{3}{ag} + \frac{d^2}{ag^2} - \frac{\tau^2}{a^3g^2} \right)_{ \xi=1}$	-0.3303...	-0.5174...
9	$-\frac{1}{\pi} \frac{y^2z^2(1-z)}{(1-y)^2} (1-t)u(1-u)^2 \left( \frac{3}{ag^2} + \frac{3d^2}{ag^3} - \frac{\tau^2}{a^3g^3} \right)_{ \xi=1}$	-0.4842...	-0.4946...
10	$-\frac{2}{\pi} \frac{yz(1-z)}{1-y} \frac{u(1-u)d}{ag^2}  _{\xi=1}$	-0.0193...	-0.0307...
11	$-\frac{3}{\pi} \frac{yz(1-z)}{1-y} (1-t)u(1-u)^2d \left( \frac{1}{ag^2} + \frac{d^2}{ag^3} \right)_{ \xi=1}$	-0.0105...	-0.0168...
12	$-\frac{1}{\pi} \frac{yz(1-z)}{1-y} (1-t)u(1-u)^2d \left( \frac{2}{ag^2} - \frac{\tau^2}{a^3g^3} \right)_{ \xi=1}$	-0.0058...	-0.0092...
$\sum_i$		0.4045...	0.0177...

<sup>a</sup> In this table

$$d = \xi uz, \quad \tau = (1-u)t, \quad g = \frac{uz(1-yz)}{1-y} - d^2, \quad a^2 = \frac{1}{g} \left[ \tau^2 + \frac{u}{y(1-y)} \right].$$

Insertion of the polarization operator in Eq. (19) (with  $q^2 \rightarrow -k^2$ ) in the upper photon lines of the diagrams in Fig. 2 is described by insertion inside the integrands in Tables III and IV of the factor

$$2 \left( \frac{\alpha}{\pi} \right) \int_0^1 dw \int_0^1 dv \frac{v^2}{1-v^2} \left( 1 - \frac{v^2}{3} \right), \quad (24)$$

TABLE IV. Second Set of Crossed Light by Light Integrals<sup>a</sup>

$i$	$\mathcal{K}_C^{(i)}$	$\Delta\epsilon_C^{(i)}$	$\Delta\epsilon_C^{vp(i)}$
1	$-\frac{1}{2\pi}\frac{1+x}{y(1-xy)}(1-t)(1-u)^2\left(\frac{2}{ag}-\frac{\tau^2}{a^3g^2}\right)_{ z=1}$	-1.6733...	-1.5131...
2	$-\frac{1}{2\pi}\frac{1-x}{y(1-xy)}(1-t)(1-u)^2\left(\frac{2}{ag}-\frac{\tau^2}{a^3g^2}\right)_{ z=0}$	-0.2729...	-0.2646...
3	$-\frac{1}{2\pi}(1-t)(1-u)^2\left(\frac{2}{ag}-\frac{\tau^2}{a^3g^2}\right)_{ y=1}$	-0.3665...	-0.3084...
4	$-\frac{1}{\pi}\frac{x}{1-xy}(1-t)(1-u)^2\left(\frac{1}{ag}+\frac{d^2}{ag^2}\right)$	-0.2997...	-0.2088...
5	$\frac{1}{\pi}\frac{u(1-u)}{y(1-xy)}\left(\frac{2}{ag}-\frac{\tau^2}{a^3g^2}\right)$	0.3460...	0.5995...
6	$\frac{1}{2\pi}\frac{1-x+xyz}{y(1-xy)^2}(1-t)u(1-u)^2\left(\frac{2}{ag^2}-\frac{\tau^2}{a^3g^3}\right)$	0.9869...	0.8396...
7	$\frac{1}{2\pi}\frac{1-x+xyz}{y(1-xy)}(1-t)u(1-u)^2d\left(\frac{2}{ag^2}-\frac{\tau^2}{a^3g^3}\right)$	-0.0020...	-0.0038...
$\sum_i$		-1.2816...	-0.8597...

<sup>a</sup> In this table

$$d = u \left[ z - \frac{1-x}{1-xy} \right], \quad \tau = (1-u)t, \quad g = \frac{u(1-yz)(1-x+xyz)}{y(1-y)} - d^2, \quad a^2 = \frac{1}{g} \left[ \tau^2 + \frac{u}{xy(1-xy)} \right].$$

introduction of an additional Feynman parameter  $w$  and the substitution

$$a^2 \rightarrow a^2(w) = a^2 + \frac{4w(1-t)(1-u)}{g(1-v^2)}. \quad (25)$$

The extra Feynman parameter  $w$  does not arise (it is effectively equal one) in the vacuum polarization integrals in the 4th, 8th, and 10th rows in Table III, and in the 5th row in Table IV. For these integrals the substitution reduces to  $a^2 \rightarrow a^2(1)$ .

Similarly to Eq. (20) we represent the contributions to HFS generated by the diagrams in Fig. 2 in the form

$$\Delta E_u = 2\Delta E_L^{vp} + \Delta E_C^{vp} \equiv (2\Delta\epsilon_L^{vp} + \Delta\epsilon_C^{vp}) \frac{\alpha^3(Z\alpha)}{\pi^2} E_F, \quad (26)$$

where

$$\Delta\epsilon_{L(C)}^{vp} = \sum_i \Delta\epsilon_{L(C)}^{vp(i)}. \quad (27)$$

The fourth columns in Tables III and IV contain the integrals  $\Delta\epsilon_L^{vp(i)}$  and  $\Delta\epsilon_C^{vp(i)}$ , and their

sums in the last line. Collecting these contributions we obtain the total contribution to HFS generated by the diagrams in Fig. 2

$$\Delta E_u = -0.8242(1) \frac{\alpha^3(Z\alpha)}{\pi^2} E_F. \quad (28)$$

### III. CONCLUSIONS

Collecting results in Eq. (22) and Eq. (28) we obtain the total contribution to HFS generated by the polarization insertions in Figs. 1 and 2

$$\Delta E_{vp} = -2.056(1) \frac{\alpha^3(Z\alpha)}{\pi^2} E_F, \quad (29)$$

or numerically

$$\Delta E_{vp} = -2.63 \text{ Hz}. \quad (30)$$

Three-loop contribution to HFS containing closed electron loops and factorized one-loop radiative insertions in the electron line were calculated earlier [9–11]. Combining those corrections with the result in Eq. (29) we obtain the sum of all gauge invariant three-loop radiative corrections to HFS calculated thus far

$$\Delta E_t = -3.338(1) \frac{\alpha^3(Z\alpha)}{\pi^2} E_F, \quad (31)$$

or numerically

$$\Delta E_t = -4.28 \text{ Hz}. \quad (32)$$

Work on calculation of the remaining three-loop contributions to HFS is now in progress.

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