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We investigate the quality of quantum state transfer through a uniformly coupled antiferromagnetic spin chain in a multi-excitation subspace. The fidelity of state transfer using multi-excitation channels is found to compare well with communication protocols based on the ground state of a spin chain with ferromagnetic interactions. Our numerical results support the conjecture that the fidelity of state transfer through a multi-excitation subspace only depends on the number of initial excitations present in the chain and is independent of the excitation ordering. Based on these results, we describe a communication scheme which requires little effort for preparation.

I. INTRODUCTION

Spin qubits have been considered in many quantum communication analyses due to their wide applicability in various solid-state devices. A prominent form of interaction between the spins is the exchange coupling which can be described by the Heisenberg model. The Heisenberg spin chain has been extensively studied as a communication channel for many quantum information processing tasks [1]. In most theoretical treatments of the subject one assumes a ferromagnetic (FM) coupling and the channel spins are assumed to be initialized to the completely polarized ground state. However, most physical realizations of quantum spin channels have an antiferromagnetic (AFM) ordering [2]. The ground state wave function for the antiferromagnetic XY Hamiltonian contains numerous amplitudes within a multi-excitation subspace. Although the ground state configuration is more complicated than the FM ground state, it may prove to be a more suitable pathway for quantum communication. It appears that the first proposal for using antiferromagnetic spin chains for quantum state transfer was provided in [3]. There it was shown that AFM Heisenberg chains can represent good quantum channels for robust finite temperature teleportation and state transfer.

Recently, an experimental proposal for the quantum simulation of an AFM spin chain in an optical lattice has been provided [4]. It has also been shown that the ground state of some one-dimensional spin models with finite correlation length can distribute entanglement between long distance sites [5]. As the chain length increases, true long-distance entanglement, characterized by energy gaps above the ground state, vanishes exponentially. However, long distance entanglement can be supported by the ground state of spin models with infinite correlation length defined on one-dimensional open chains with small end bonds [5]. Open quantum spin chains endowed with XY-like Hamiltonians containing nearest-neighbor interactions have also investigated [6]. For dimerized XY chains, true long distance entanglement has been found to exist only at zero temperature although "quasi long-distance" entanglement can be realized in open XY chains with small end bonds [6]. There it was found that the entanglement properties slowly fall off with the size of the chain and that efficient qubit teleportation can be realized with high fidelity in long chains even at moderately low temperatures.

Quantum computing is possible using a wide variety of systems assembled from antiferromagnetically coupled spins [7]. Entanglement properties in two-dimensional AFM models have been studied [8] and it has been shown that multiqubit entanglement can be generated efficiently via a quantum data bus consisting of spin chains with strong static AFM couplings [9, 10]. The effects of fluctuating exchange couplings and magnetic fields on the fidelity of data bus transfer have also been investigated [11]. Bayat *et al.* studied the entanglement transfer through an AFM spin chain and found that when compared to the FM case, the entanglement can be transmitted faster, with less decay, and with a much higher purity [12]. Furthermore, Wang *et al.* demonstrated that near-perfect entanglement can be generated between the first and last spins of an AFM isotropic Heisenberg chain by applying a magnetic field to a single site in a specific direction [13]. Moreover, perfect state transfer across a strongly coupled AFM spin chain or ring has been shown to be possible using weakly coupled external qubits [14]. Detrimental dispersion effects on the transmission are found to be strongly reduced by modifying only one or two bonds in an XX spin chain and a transmission fidelity more than 99% for arbitrary long chains is gained [15, 16].

It is known that quantum information propagates dispersively through most spin chains due to the nontrivial structure of the many-body Hamiltonian that describes the channel [17]. Designing a non-dispersive channel requires the intricate engineering of the local couplings [18]. Dispersion is always detrimental to the information transmission, usually there is always some portion of information left in the chain after measurement and hence lost to the receiver [19]. Although perfect state transfer cannot typically be achieved using uniformly coupled chains alone, the investigation of state transfer through

chains of this sort is warranted due to the relative ease of preparation compared to more elaborate schemes.

In this work, we compare the quality of state transmission for several initial configurations of a spin chain. Specifically, we examine spin chains initialized to the Néel state (AFM arrangements) and find that the average fidelity of state transfer using these channels is similar to that which can be obtained using the completely polarized FM ground state. The results follow the original proposal of Bose [20]. Higher average fidelities occur using this AFM arrangement for chains having an appropriate length, and in some cases these higher values are accompanied by shorter arrival times as well. It is also found that the quality of transmission through chains prepared in a multi-excitation state which contain a fixed number of initial excitations remains nearly identical regardless of the order in which these excitations occur. These results bode well for potential realizations of these communication channels as they suggest a simplification of the initialization process. We analyze the magnetic field dependence of the fidelity in all cases and find that this measure is influenced strongly by fluctuations in the external field.
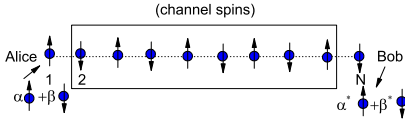

FIG. 1: (Color online) Schematic of our spin chain communication channel. After initializing the chain to the Néel state Alice encodes an arbitrary qubit state at one end and allows it to propagate freely to the other end.

We first consider the Néel state configuration shown schematically in Fig. 1. We will provide our mathematical model next and derive expressions for the fidelity measure. In Sec. III we discuss our results for this and other AFM arrangements and compare these results to the FM case. Sec. IV concludes with a summary of our findings.

II. THE MODEL

We consider a linear spin chain with uniform nearest-neighbor XY couplings. The Hamiltonian is given by

$$H = \frac{J}{2}[\sum_{l=1}^{N-1}(\sigma_l^x\sigma_{l+1}^x + \sigma_l^y\sigma_{l+1}^y) - \frac{2h}{J}\sum_{l=1}^{N}\sigma_l^z] \quad (1)$$

In this expression J denotes the exchange constant between adjacent spins. For FM chain, $J < 0$ and for AFM chain, $J > 0$. h represents the external magnetic field strength of a field applied along the z direction, and $\sigma_l^{x,y,z}$ signifies the Pauli operators acting on spin l. We take

$\hbar = 1$ throughout. Note that for this Hamiltonian the z-component of the total spin $\sigma^z = \sum \sigma_l^z$ is a conserved quantity, i.e. $[H, \sigma^z] = 0$, which indicates that the system contains a fixed number of excitations. The number of excitations in the chain corresponds to the number of $|1\rangle$'s appearing in the state vector, where $|0\rangle$ represents the spin-down state of a spin qubit and $|1\rangle$ represents a spin-up state. For instance, the state $|111...1\rangle$ labels a chain containing N excitations.

This Hamiltonian can be diagonalized by means of the Jordan-Wigner transformation which maps spins to one dimensional spinless fermions with creation operators defined by $c_k^\dagger = (\prod_{l=1}^{k-1} -\sigma_l^z)\sigma_k^+$. Here $\sigma_k^\pm = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$ denotes the spin raising and lowering operations at site k. The action of $c_k^\dagger$ is to flip the spin at site k from down to up. For indices l and m the operators c_l and $c_m^\dagger$ satisfy the anticommutation relation $\{c_l, c_m^\dagger\} = \delta_{lm}$. The z component of the total spin is a conserved quantity and thus the total number of excitations $M = \sum_k c_k^\dagger c_k$ in the chain remains constant.

The time dependence of the operator $c_k^\dagger$ has been calculated [22] and is given by

$$c_k^\dagger(t) = \sum_{l=1}^{N} f_{k,l}(t)c_l^\dagger. \quad (2)$$

In this expression the transition amplitudes $f_{k,l}$ evolve according to the relation

$$f_{k,l}(t) = \frac{2}{N+1}\sum_{m=1}^{N}\sin(q_m k)\sin(q_m l)e^{-iE_m t}, \quad (3)$$

where $q_m = \pi m/(N+1)$, $E_m = 2h + 2J\cos q_m$. In what follows, let us define the completely polarized state $|000...0\rangle$ to be $|\mathbf{0}\rangle \equiv |000...0\rangle$ and let S denote a set of M different numbers from $1, 2, ..., N$. The set $S = \{k_1, k_2, ..., k_M\}$ serves to label the sites where the M excitations initially exist. In this notation, we can express our initial chain configuration as

$$|\Psi(0)\rangle = \left(\prod_{k\in S}c_k^\dagger\right)|\mathbf{0}\rangle. \quad (4)$$

In this work we will only consider spin channels which are initialized to single ket states of this form. This initial state then evolves to [23]

$$|\Psi(t)\rangle = \sum_{l_1<l_2<...<l_M}\det(A)\left(\prod_{m=1}^{M}c_{l_m}^\dagger\right)|\mathbf{0}\rangle, \quad (5)$$

where

$$A = \begin{vmatrix} f_{k_1,l_1} & f_{k_1,l_2} & \cdots & f_{k_1,l_M} \\ f_{k_2,l_1} & f_{k_2,l_2} & \cdots & f_{k_2,l_M} \\ \cdots & \cdots & \cdots & \cdots \\ f_{k_M,l_1} & f_{k_M,l_2} & \cdots & f_{k_M,l_M} \end{vmatrix}. \quad (6)$$

Note that we have suppressed the explicit time dependence of the amplitudes $f_{i,j}(t)$ in the determinant above. The indices $l_1, l_2, ... l_M$ have a similar meaning to the k_i and mark the sites where the excitations have spread to. We are interested in the fidelity of the state which Bob receives at site N so we will also need an expression for the reduced density matrix at this site. This operator has the form [24]

$$\rho_N(t) = \begin{pmatrix} \langle \sigma_N^+ \sigma_N^- \rangle & \langle \sigma_N^- \rangle \\ \langle \sigma_N^+ \rangle & \langle \sigma_N^- \sigma_N^+ \rangle \end{pmatrix} \qquad (7)$$

where the symbol $\langle \cdot \rangle$ represents the average value of $\langle \Psi_l^M(t) | \cdot | \Psi_l^M(t) \rangle$.

Let us assume that the interaction between spins 1 and 2 can be turned on or off. First the interaction is turned off and Alice prepares an arbitrary qubit state $\alpha |0\rangle + \beta |1\rangle$ at the first site of the chain. Now suppose the channel which includes spins 2 to N is prepared in the Néel state. The initial state of the system is given by

$$|\Psi(0)\rangle = \begin{cases} (\alpha |0\rangle + \beta |1\rangle) \otimes |1010...10\rangle, & \text{for odd } N \\ (\alpha |0\rangle + \beta |1\rangle) \otimes |1010...01\rangle, & \text{for even } N \end{cases} \qquad (8)$$

Now turn on the interaction and let the system evolve freely. At an appropriate time Bob will receive a state which resembles the one Alice prepared. An ideal transmission would result when Bob receives a state which is identical to the one Alice prepared, in that case the fidelity would be perfect. Then

$$\begin{aligned} |\Psi(t)\rangle &= \alpha \sum_{l_1 < l_2 < ... < l_{M_1}} A_1 \left(\prod_{m=1}^{M_1} c_{l_m}^\dagger \right) |\mathbf{0}\rangle \\ &+ \beta \sum_{l_1 < l_2 < ... < l_{M_2}} A_2 \left(\prod_{m=1}^{M_2} c_{l_m}^\dagger \right) |\mathbf{0}\rangle . \end{aligned} \qquad (9)$$

In the expression above we have

$$M_1 = \frac{N-1}{2}, \; M_2 = \frac{N+1}{2}, \; \text{for odd } N \qquad (10)$$

and

$$M_1 = \frac{N}{2}, \; M_2 = \frac{N}{2} + 1, \qquad \text{for even } N \qquad (11)$$

The determinants A_1 and A_2 have the same form as A except that M is replaced with M_1 and M_2 respectively. Now the matrix elements in Eq. (7) can be calculated as

$$\begin{aligned} \langle \sigma_N^+ \sigma_N^- \rangle &= |\alpha|^2 \Gamma_1 + |\beta|^2 \Gamma_2, \\ \langle \sigma_N^- \sigma_N^+ \rangle &= |\alpha|^2 \Gamma_3 + |\beta|^2 \Gamma_4, \end{aligned}$$

and

$$\langle \sigma_N^+ \rangle = \alpha \beta^* \Gamma_5, \quad \langle \sigma_N^- \rangle = \langle \sigma_N^+ \rangle^* \qquad (12)$$

with

$$\begin{aligned} \Gamma_1 &= \sum_{l_1 < l_2 < ... < (l_{M_1} = N)} (\det A_1)^* \det A_1, \\ \Gamma_2 &= \sum_{l_1 < l_2 < ... < (l_{M_2} = N)} (\det A_2)^* \det A_2, \\ \Gamma_3 &= \sum_{l_1 < l_2 < ... < (l_{M_1} \neq N)} (\det A_1)^* \det A_1, \\ \Gamma_4 &= \sum_{l_1 < l_2 < ... < (l_{M_2} \neq N)} (\det A_2)^* \det A_2, \\ \Gamma_5 &= \sum_{l_1 < l_2 < ... < l_{M_1} < (l_{M_2} = N)} (\det A_2)^* \det A_1. \quad (13) \end{aligned}$$

Let $\alpha = \cos \frac{\theta}{2}$ and $\beta = \sin \frac{\theta}{2} e^{i\varphi}$, the average fidelity of transmission at Bob's end can then be calculated by integration over the unit sphere

$$\begin{aligned} F &= \frac{1}{4\pi} \int \langle \varphi_{in} | \rho_N(t) | \varphi_{in} \rangle \, d\Omega \\ &= \frac{1}{3} [\Gamma_2 + \Gamma_3 + \text{Re}(\Gamma_5)] + \frac{1}{6} [\Gamma_1 + \Gamma_4]. \quad (14) \end{aligned}$$

When two excitations exist (M=2) the fidelity can be found to agree with Eq. (7) of Ref [25] where we discuss duplex quantum communication in a FM spin chain.

Here we define the perfect state transfer as "Bob receives a state which is identical to the one Alice prepared". However, other works[1, 20] about the definition of perfect state transfer allow the receiver's state has an arbitrary phase factor compared with the sender's state. In the latter definition, the third term in Eq.(14) is changed to $(|\Gamma_5| \cos \gamma)/3$, where $\gamma = \arg\{\Gamma_5\}$. To maximize the average fidelity, the magnetic field must be properly chosen such that $\cos \gamma = 1$.

When $M = 1$, i.e. only one excitation exists in the chain, the excited state contributes as a term in Alice's encoded qubit state ($k_1 = 1$). In this case we have $\Gamma_1 = 0$, $\Gamma_2 = |f_{1,N}|^2$, $\Gamma_3 = 1$, $\Gamma_4 = 1 - |f_{1,N}|^2$, and $\Gamma_5 = f_{1,N}^*$. The corresponding average fidelity can be simplified to

$$F = \frac{1}{2} + \frac{1}{3} |f_{1,N}| \cos \gamma + \frac{1}{6} |f_{1,N}|^2. \qquad (15)$$

Where $\gamma = \arg\{f_{1,N}\}$. This result is in accordance with Bose's expression (Eq. (6) in Ref [20]) where the channel spins are prepared in the ground state of a FM chain.

III. RESULTS AND DISCUSSIONS

Starting from the initial state (Eq.(8)) the system undergoes a time evolution described by Eq's. (9)-(12). This evolution can be viewed as the propagation of the half site excitations. After initialization these excitations
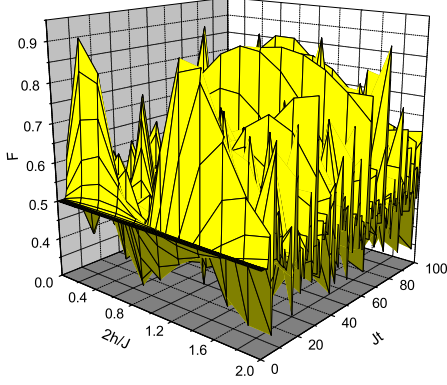
FIG. 2: (Color online) Plot of the average fidelity of state transfer as a function of scaling time Jt and scaling external field $2h/J$. The figure corresponds to a N=10 site chain which has been initialized to the Néel state.
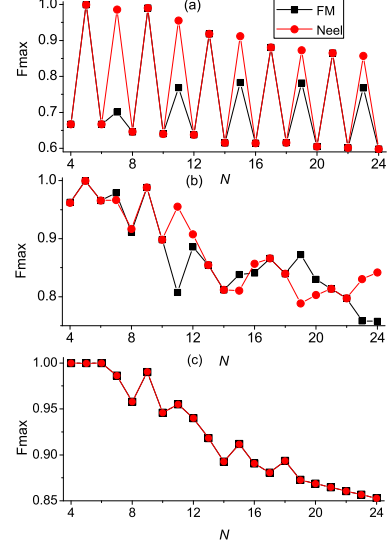

FIG. 3: (Color online) Comparison of the maximum average fidelity that can be achieved as a function of chain length N for different initial state configurations; the FM ground state and the Néel state. The results are obtained within the time interval $[0,500/|J|\,]$ and for field strengths (a) h=0.0, (b) h=1.0, (c) optimal field strengths.

begin to spread outward and at a later time there is typically a nonzero probability of finding any one of the spins in an excited state. From the point view of wave mechanics, the transition amplitude $f_{k,l}(t)$ can be viewed as the propagator and the state transfer can be characterized in terms of the dispersion of all propagators. At some time Tmax maximum constructive interference occurs at the receiving end and the state of the spin at Bob's site now has its strongest resemblance to the state which Alice prepared. First we investigate the effect of a magnetic field on the evolution of the average fidelity. As an example, Fig.2 illustrates the average fidelity of transmission through an N=10 site chain as a function of Jt and $2h/J$. We find that the average fidelity changes abruptly with Jt and $2h/J$ as it oscillates around the value 0.5. The magnetic field has a pronounced effect on the average fidelity, with increasing $2h/J$ the oscillation of the average fidelity becomes more rapid. The maximum average fidelity (MAF) Fmax=0.909 is achieved at JTmax $= 6.0$ with $2h/J$=0.2.

In Fig.3 we compare the communication fidelity when an arbitrary qubit state is transferred through two initial channel configurations of various length. One of these channels is chosen to be the ground state $|\mathbf{0}\rangle$ of a FM chain while the other channel corresponds to the Néel state. We note that it has recently been shown that for chains containing $N = p - 1$ or $N = 2p - 1$ sites, where p is prime, or if $N = 2^m - 1$, a maximum fidelity greater than $1 - \epsilon$ ($\epsilon > 0$) can be achieved if the waiting time is not an issue. We set the magnetic field to h=0.0 (h=1.0) for plots Fig. 3(a) (Fig. 3(b)). First notice that when N=5, both channels yield Fmax ≈ 1 indicating that near perfect state transfer can be realized. Secondly, in the absence of a magnetic field (Fig. 2(a)) the MAF associated with the Néel channel is typically greater than or equal to the values which occur for the FM ground state

configuration. The figure suggests that for h=0.0 the Néel channel supports a better state transfer than $|\mathbf{0}\rangle$ for chains containing $7 + 4n$ ($n = 0, 1, 2...$) sites. In Fig. 3(b) we compare the results when an external magnetic field is present. In this case the MAF associated with the Néel state can be lower than the $|\mathbf{0}\rangle$ channel for certain N. We also find that when $N = 5 + 4n$ and $N = 6 + 4n$ ($n = 0, 1, 2...$), the MAF is equal for the Néel channel and the FM ground state channel. In Fig.3(c) we plot the MAF for optimal choice of magnetic field. Through numerical calculation we find that the MAF and the time Tmax at which the average fidelity gains its maximum value (Fig.4(c)) are always equal when using the the FM ground state and the Néel state as the initial state. Then half excitations of the chain length and a single excitation shows same transmission quality when considering optimal choice of magnetic field. Fig.3(c) also shows that the MAF are greatly enhanced compared with the fixed magnetic field which is plotted in Fig.3(a),(b). $N = 4, 5, 6$ gives nearly perfect (Fmax=0.999) state transfer.

In Fig.4 we plot the time Tmax as a function of site number N. In the absence of a magnetic field (Fig.4(a)) we find that when Fmax is equal for chains differing in length by one unit the associated arrival times Tmax are also equal. In the presence of a magnetic field (Fig. 4(b)) this behavior also exists. For certain chain lengths a shorter arrival time accompanies the higher MAF which can be obtained using the Néel channel, e.g., N =11, 19

(h=0.0) and N=12, 16 (h=1.0).
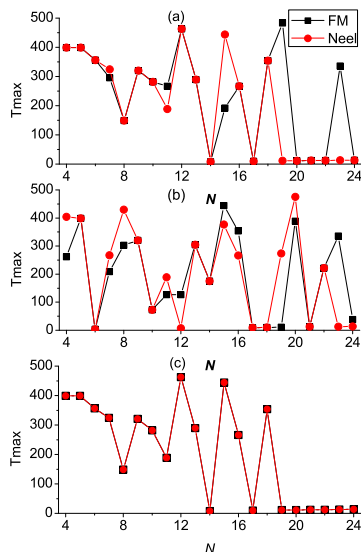

FIG. 4: (Color online) The time Tmax at which the average fidelity gains its maximum. (a) h=0.0, (b) h=1.0 (c) optimal field strengths.

Since the Néel state is not an eigenstate of the AFM chain there is a large probability that it will collapse to another state when we attempt to obtain it through measurement. These other possible states will contain the same number of excitations as the Néel state but the location of the excitations will generally be different. For instance, when N=6 and M=3 the locations of the excitations for the Néel state are at sites 2, 4, and 6. Suppose these excitations occupied other sites, say sites 2, 3, and 4. We now check to see how the average fidelity is affected by such a re-ordering. We sample random configurations for chains containing N=6 and N=15 sites having M=3 and M=7 excitations respectively. For the N=6 site chains we select the excitation locations to be (2, 3, 4), (3, 4, 5), (2, 4, 6), and for the N=15 site chains we choose (3, 4, 6, 10, 11, 12, 14), (2, 3, 7, 8, 10, 11, 13). Using the arrival times Tmax associated with the Néel ordered states we calculate the difference in the average fidelity between these different configurations. For h = 1.0 we find a difference of only 5.55×10^{-16} between the orders (2, 3, 4) and (3, 4, 5). A comparison between

orders (2, 4, 6) and (2, 3, 4) yields an even smaller difference for the same value of h. For h=0.0 and N=15 the difference in the fidelity for the two configurations above is 3.55×10^{-15}. We have also checked other initial state configurations using various values for N and find that the average fidelity is nearly equal to the corresponding N-site Néel state at the same time which maximizes the Néel channel average fidelity. We conjecture that when the number of excitations is roughly similar to half of the system size, the evolution of the average fidelity only depends on the number of excitations in the chain and is independent of their ordering. If this prediction holds, the initialization process of the AFM chain would be simplified. If the chains state collapses via measurement to any state containing a fixed and known number of excitations we could predict the behavior of the subsequent evolution of the fidelity.

IV. CONCLUSIONS

In this work we have shown that multi-excitation channels can provide suitable pathways for quantum communication. Some of the AFM chains we have considered have been found to outperform state transfer protocols based on ferromagnetic media which are initialized to the ground state. Specifically, we have found certain Néel state configurations which allow a quantum state to be transmitted in a shorter amount of time and arrive with a higher average fidelity than in the FM case. Moreover, numerical calculations support our conjecture that the quality of state transfer through a multi-excitation subspace only depends on the number of excitations present in the initial state of the system. Since the fidelity of state transfer appears to be independent of the ordering of the initial excitations, we believe that the AFM ground state can serve as a communication channel.

These results should be interesting to test experimentally, perhaps using NMR methods [26], fabricated AFM nano-chains [2], or optical lattices [4, 27].

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