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Singularity of dynamical maps

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For a dynamical map $\Lambda(t,0)$, which sends a state $\rho(0)$ of quantum open system to a state $\rho(t) = \Lambda(t,0)\rho(0)$, the decomposition law $\Lambda(t,0) = \Lambda(t,t_c)\Lambda(t_c,0)$ may break down at a specific time t_c. In this paper, we present a method to find the singular points t_c and propose a measure for the singularity of the dynamical map. Two examples are portrayed to illustrate the method, the measure of singularity for these singular points is calculated and discussed. An extension to high-dimensional system is presented.

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I. INTRODUCTION

The actual dynamics of any real open quantum system is expected to deviate to some extent from the Markovian evolution. This deviation can be measured by non-Markovianity and it has attracted much attention in recent years, leading to a deeper understanding of quite a few issues in the theory of open quantum system [1–8].

Non-Markovian systems can be found in many branches of physics, including quantum optics [9, 10], solid state physics [11], quantum chemistry[12], and quantum information processing [13]. Since non-Markovian dynamics modifies monotonic decay of quantum coherence, it may protect quantum entanglement in composite systems for longer time than standard Markovian evolution [14]. In particular it may protect the system against the sudden death of entanglement [15]. Therefore, it is interesting to quantify the non-Markovianity within the description of quantum open system.

There are two approaches to quantify the measure of the degree of non-Markovianity. One approach is based on the idea of the composition law which is essentially equivalent to the idea of divisibility [16]. This approach was used recently in Ref.[17, 18] to construct the measure of non-Markovianity, quantifying actually the deviation of the dynamical map from divisibility. Another approach is as in Ref.[19], where the authors define non-Markovianity dynamics as the information flow from the environment back into the system, the measure manifests itself as an increase in the distinguishability of pairs of evolving quantum states, and the information might be the Fisher information [20].

The measure for non-Markovianity proposed in Ref.[17] is based on the completely positive divisibility of a dynamical map: a trace preserving completely positive map $\Lambda(t,0)$ is completely positive divisible (CP-divisibility) if it can be written as,

$$\Lambda(t_2,0) = \Lambda(t_2,t_1)\Lambda(t_1,0), \tag{1}$$

and $\Lambda(t_2,t_1)$ is completely positive for any t_2 and t_1 ($t_2 > t_1 > 0$). By contrast, we say that the map $\Lambda(t_2,0)$ is positively divisible (P-divisibility) if $\Lambda(t_2,t_1)$ sends states into states but it is only positive, and that $\Lambda(t_2,0)$ is indivisible if neither P-divisibility nor CP-divisibility of $\Lambda(t,0)$ holds.

In this paper, we shall consider the other situation where,

$$\Lambda(t_2,0) \neq \Lambda(t_2,t_c)\Lambda(t_c,0), \tag{2}$$

at a special time t_c, $t_2 < t_c < 0$. We will refer to this instance of time t_c as the singular point of the dynamical map $\Lambda(t_2,0), t_2 \in (0,\infty)$. Taking a qubit (two-level system) as an example, a method to find the singular point is presented, a measure to quantify this singularity is proposed and discussed.

This paper is organized as follows. In Sec.II, we present a general formalism for a qubit dynamics, exhibiting the method to find the singular point t_c. A measure to quantify the singularity is constructed. Two examples, one describes a qubit coupled to a harmonic oscillator bath and the other includes a qubit coupled to a finite spin bath, are given to illustrate the critical point in Sec.III, the measure of singularity is also calculated and discussed in this section. A generalization of the representation to d-dimensional open systems is presented in Sec. IV. Finally, we conclude our results in Sec. V.

II. GENERAL FORMALISM FOR A QUBIT DYNAMICAL MAP

To simplify the representation and make the discussion clear, we first examine the case of qubit. Consider a dynamical map $\Lambda(t,0)$ for a qubit (or two-level system), which sends an arbitrary initial state $\rho(0) = (1 + \vec{n}(0)\cdot\vec{\sigma})/2$ with Bloch vector $\vec{n}(0) = (n_x(0), n_y(0), n_z(0))$ into a state $\rho(t)$,

$$\rho(t) = \Lambda(t,0)\rho(0) = \frac{1}{2}(1 + \vec{n}(t)\cdot\vec{\sigma}). \tag{3}$$

Without loss of generality, the Bloch vector $\vec{n}(t) = (n_x(t), n_y(t), n_z(t))$ can be written as,

$$\vec{n}(t) = \vec{n}(0) \cdot D(t) + \vec{f}(t), \quad (4)$$

where $D(t)$ is a 3×3 matrix and $\vec{f}(t)$ is a time-dependent vector.

Now we elicit the condition for $\Lambda(t_2, 0) \neq \Lambda(t_2, t_c)\Lambda(t_c, 0)$. To this aim we introduce an ancilla A and define,

$$M_{SA} \equiv \Lambda(t_2, t_c) \otimes \mathrm{I}_A(|\Phi_{SA}\rangle\langle\Phi_{SA}|), \quad (5)$$

where $|\Phi_{SA}\rangle = |0\rangle_S \otimes |0\rangle_A + |1\rangle_S \otimes |1\rangle_A$ is an unnormalized maximally entangled state of the qubit and ancilla, and I_A denotes the identity operator of the ancilla. Since the ancilla is also a qubit, M_{SA} can be written as,

$$\begin{aligned} M_{SA} &= \frac{1}{2}(x + \vec{r}\cdot\vec{\sigma}_A^T + \vec{s}\cdot\vec{\sigma}_S + \vec{\sigma}_S \cdot V \cdot \vec{\sigma}_A^T) \\ &= \frac{1}{2}(\mathrm{I}, \vec{\sigma}_S) F \begin{pmatrix} 1 \\ \vec{\sigma}_A^T \end{pmatrix}. \end{aligned} \quad (6)$$

Here x is a constant, $\vec{r}$ and $\vec{s}$ are vectors, I is an identity matrix, $F = \begin{pmatrix} x & \vec{r} \\ \vec{s} & V \end{pmatrix}$ and V is a 3×3 matrix, which is determined by the map $\Lambda(t_2, t_c)$ and will be derived in the following. If $\Lambda(t_2, 0) = \Lambda(t_2, t_c)\Lambda(t_c, 0)$ holds, the map $\Lambda(t_2, t_c)$ would send the state $\rho(t_c) = \Lambda(t_c, 0)\rho(0)$ to state $\rho(t_2)$. In terms of M_{SA}, this can be expressed as,

$$\begin{aligned} \rho(t_2) &= \Lambda(t_2, t_c)\rho(t_c) = \mathrm{Tr}_A[M_{SA}\ I_S \otimes \rho_A^T(t_c)] \\ &= \frac{1}{2}(\mathrm{I}, \vec{\sigma}_S) F \begin{pmatrix} 1 \\ \vec{n}(t_c) \end{pmatrix}. \end{aligned} \quad (7)$$

Writing $\rho(t_2) = \frac{1}{2}(\mathrm{I}, \vec{\sigma}_S)\begin{pmatrix} 1 \\ \vec{n}(t_2) \end{pmatrix}$, we obtain from Eq.(7),

$$\begin{pmatrix} 1 \\ \vec{n}(t_2) \end{pmatrix} = F \begin{pmatrix} 1 \\ \vec{n}(t_c) \end{pmatrix} = \begin{pmatrix} x + \vec{r}\cdot\vec{n}(t_c) \\ \vec{s} + \vec{n}(t_c)\cdot V \end{pmatrix}. \quad (8)$$

It is easy to find that,

$$\begin{aligned} x = 1, \quad \vec{r} = 0, \\ \vec{n}(t_2) = \vec{n}(t_c)\cdot V + \vec{s}. \end{aligned} \quad (9)$$

Considering that $\rho(0)$ is an arbitrary initial state, namely $\vec{n}(0)$ is arbitrary, Eqs.(4,9) together yield,

$$\begin{aligned} D(t_2) = D(t_c)\cdot V, \\ \vec{f}(t_2) = \vec{f}(t_c)\cdot V + \vec{s}. \end{aligned} \quad (10)$$

The condition for $\Lambda(t_2, 0) \neq \Lambda(t_2, t_c)\Lambda(t_c, 0)$ now is equivalent to that there does not exist a matrix V to satisfy Eq. (10). If the determinant of $D(t_c)$ is non-zero, we have

$$V = D^{-1}(t_c)\cdot D(t_2), \quad (11)$$

and

$$\vec{s} = \vec{s}(t_2, t_c) = \vec{f}(t_2) - \vec{f}(t_c)\cdot V. \quad (12)$$

From the above derivations, we find that once x, $\vec{r}$, $\vec{s}$, V are (uniquely or non-uniquely) established for any t_c ($t_2 > t_c > 0$), the decomposition $\Lambda(t_2, 0) = \Lambda(t_2, t_c)\Lambda(t_c, 0)$ holds true, namely there are no singular points in the time interval $[0, t_2]$. We notice that the null-determinant of $D(t_c)$ plays a key role in finding V, it can thus be taken as a condition to find the singular point t_c if the matrix $D(t_2)$ is of full rank. Mathematically, the necessary and sufficient condition for Eq. (10) to have no solution is that the rank of $D(t_c)$ must be smaller than the rank of $[D(t_c)|D(t_2)]$, where $[D(t_c)|D(t_2)]$ is an augmented matrix obtained by attaching the columns of $D(t_2)$ to the columns of $D(t_c)$.

To quantify the singularity of the singular point, we introduce the trace distance, $D(\rho_1, \rho_2) = \frac{1}{2}\mathrm{Tr}|\rho_1 - \rho_2|$, which is an appropriate measure for the distinguishability between two quantum states ρ_1 and ρ_2. Here $|A| = \sqrt{A^\dagger A}$. We define the singularity measure of a dynamical map $\Lambda(t_2, 0)$ at time t_c by

$$S_\Lambda(t_c) = \max_{\rho(0), T} D(\rho(T), \rho_{t_c}(T)), \quad (13)$$

where $\rho(T) \equiv \Lambda(T, 0)\rho(0)$ and $\rho_{t_c}(T) \equiv \Lambda(T, t_c)\cdot\Lambda(t_c, 0)\rho(0)$. **The latter can be viewed as the resulting state of the dynamical map $\Lambda(T, t_c)$ taking $\rho(t_c) = \Lambda(t_c, 0)\rho(0)$ as an initial state. The maximum is taken over all initial states and the final time T.**

III. EXAMPLES

In this section, we will present two examples to illustrate the singular point and the measure of singularity. The first example is a dephasing model that consists of a spin-$\frac{1}{2}$ particle coupling to a spin-bath. The coupling Hamiltonian commutes with the free Hamiltonian of the central spin, thus the central spin conserves its energy. In the second example, we consider a dissipative system, the energy of the system is no longer conserved.

A. A two-level system coupling to a finite spin bath

Consider a central spin-$\frac{1}{2}$ coupling to a bath of N spin-$\frac{1}{2}$ particles. The interaction Hamiltonian is,

$$H = \sum_{k=1}^{N} A_k \sigma_z \sigma_z^k, \quad (14)$$

where $A_k = A/\sqrt{N}$ represents the coupling constants. Assume the initial state of the whole system is $\rho_s(0) \otimes (\frac{1}{2^N} I)$, i.e., all spins in the reservoir are in a maximal mixed state. The density matrix of the central spin at time t takes,

$$\rho(t) = \begin{pmatrix} \rho_{11} & \rho_{12}\cos^N(\frac{2At}{\sqrt{N}}) \\ \rho_{21}\cos^N(\frac{2At}{\sqrt{N}}) & \rho_{22} \end{pmatrix}. \quad (15)$$

In terms of dynamical map, the dynamics can be represented as, $\Lambda(t,0)\rho = \frac{1}{2}(1-\cos^N(\frac{2At}{\sqrt{N}}))\sigma_z\rho\sigma_z + \frac{1}{2}(1+\cos^N(\frac{2At}{\sqrt{N}}))\rho$. This is equivalent to the following master equation,

$$\dot{\rho} = \gamma(t)\mathcal{L}(\rho), \quad (16)$$

where $\mathcal{L}(\rho) = \sigma_z\rho\sigma_z - \rho$, and the time-dependent decay rate is $\gamma(t) = A\sqrt{N}\tan(\frac{2At}{\sqrt{N}})$. This model is discussed in several papers as a typical example to quantify non-Markovianity.

Writing $\rho(t)$ in Eq. (15) in the form of Eq. (4), we find

$$\begin{aligned} D(t) &= \begin{pmatrix} C(t) & 0 & 0 \\ 0 & C(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \vec{f}(t) &= 0, \end{aligned} \quad (17)$$

where $C(t) = \cos^N(\frac{2At}{\sqrt{N}})$. The singular point t_c can be found by solving $C(t_c) = 0$, it yields,

$$t_c = \frac{\sqrt{N}}{4A}(2n+1)\pi, \quad n = 0, 1, 2, ... \quad (18)$$

By the definition of the measure of singularity, we obtain,

$$S_\Lambda(t_c) = \max_{|\rho_{12}|, T} |C(T)||\rho_{12}(0)|, \quad (19)$$

where T is a time, $T > t_c$, and $\rho_{12}(0)$ denotes the element of the initial density matrix $\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}$. After a simple algebra, we have $S_\Lambda(t_c) = \frac{1}{2}$ for any singular point given in Eq.(18). It is interesting that the singularity measure of these singular points are equal. Indeed, examining the dynamical map $\Lambda(t,0)$, we find that the features of $\Lambda(t,0)$ around any t_c are the same.

B. The damping J-C model

This example consists of a two-level system coupling to a reservoir at zero temperature. The reservoir consists of infinite number of harmonic oscillators that is also referred in the literature as the spin-boson model. The Hamiltonian for such a system reads,

$$H = H_0 + H_I, \quad (20)$$

where $H_0 = \hbar\omega_0\sigma_+\sigma_- + \sum_k \hbar\omega_k b_k^\dagger b_k$, $H_I = \sigma_+ B + \sigma_- B^\dagger$, and $B = \sum_k g_k b_k$. The Rabi frequency of the two-level system and the frequency for the $k-th$ harmonic oscillator are denoted by ω_0 and ω_k, respectively. $b_k^\dagger$ and b_k are the creation and annihilation operators of $k-th$ oscillator, which couples to the system with coupling constant g_k.

This model is exactly solvable [9]. Assuming the system and the reservoir initially uncorrelated, we can obtain a time-dependent master equation in the interaction picture,

$$\begin{aligned} \dot{\rho} &= -i\frac{e(t)}{2}[\sigma_+\sigma_-, \rho] \\ &+ \gamma(t)(\sigma^-\rho\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\rho - \frac{1}{2}\rho\sigma^+\sigma^-), \end{aligned} \quad (21)$$

where $e(t) = -2\mathrm{Im}[\frac{\dot{c}(t)}{c(t)}]$ and $\gamma(t) = -2\mathrm{Re}[\frac{\dot{c}(t)}{c(t)}]$. $e(t)$ plays the role of Lamb shift and $\gamma(t)$ is the decay rate. Both $e(t)$ and $\gamma(t)$ are time-dependent. $c(t)$ is determined by $\dot{c}(t) = -\int_0^t f(t-\tau)c(\tau)d(\tau)$, where $f(t-\tau) = \int d\omega J(\omega)exp(i(\omega_0-\omega)(t-\tau))$ is the environmental correlation function. In the derivation of the master equation, the reservoir is assumed in its vacuum at $t = 0$.

Consider the following spectral density, $J(\omega) = \frac{1}{\pi}\frac{\gamma_0\lambda^2}{(\omega_0-\omega)^2+\lambda^2}$, where γ_0 represents the coupling constant between the system and reservoir, λ defines the spectral width of the coupling at the resonance point ω_0. For the spectral density $J(\omega)$, we have $e(t) = 0$, $c(t) = c_0 e^{-\lambda t/2}[\cosh(\frac{dt}{2}) + \frac{\lambda}{d}\sinh(\frac{dt}{2})]$, and

$$\gamma(t) = \frac{2\gamma_0\lambda\sinh(dt/2)}{d\cosh(dt/2) + \lambda\sinh(dt/2)} \quad (22)$$

with $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$ in Eq.(21). Assume the system initially in $\rho(0) = \begin{pmatrix} \rho_{ee}(0) & \rho_{eg}(0) \\ \rho_{ge}(0) & \rho_{gg}(0) \end{pmatrix}$, by the effective Hamiltonian approach[21], we have the density matrix at time t, $\rho(t) = \begin{pmatrix} \rho_{ee}(t) & \rho_{eg}(t) \\ \rho_{ge}(t) & \rho_{gg}(t) \end{pmatrix}$, where

$$\begin{aligned} \rho_{ee}(t) &= \rho_{ee}(0)e^{-\int_0^t \gamma(t')dt'}, \\ \rho_{gg}(t) &= 1 - \rho_{ee}(t), \\ \rho_{eg}(t) &= \rho_{ge}^*(t) = e^{-\frac{1}{2}\int_0^t \gamma(t')dt'}\rho_{eg}(0). \end{aligned} \quad (23)$$

It is easy to show that the matrix $D(t)$ and $\vec{f}(t)$ in this example are,

$$D(t) = \begin{pmatrix} D_{11}(t) & 0 & 0 \\ 0 & D_{22}(t) & 0 \\ 0 & 0 & D_{33}(t) \end{pmatrix} \quad (24)$$

and

$$\vec{f}(t) = (f_x, f_y, f_z) = (0, 0, (e^{-\int_0^t \gamma(t')dt'} - 1)), \quad (25)$$

where

$$\begin{aligned} D_{11} &= D_{22} = e^{-\frac{1}{2}\int_0^t \gamma(t')dt'}, \\ D_{33} &= D_{11}^2. \end{aligned} \quad (26)$$

We find from Eq. (24) that $D_{jj}(t_c) = 0$, $j = 1, 2$, or 3, gives the singular points. $D_{jj}(t_c) = 0$ can happen only when $\gamma_0/\lambda > 1/2$. Noticing that $D_{11}(t) = e^{-\lambda t/2}[\cos(\frac{d_0 t}{2}) + \frac{\lambda}{d_0}\sin(\frac{d_0 t}{2})]$ for $\gamma_0/\lambda > 1/2$, where $d_0 = \sqrt{|\lambda^2 - 2\gamma_0\lambda|}$, we obtain the nth singular point $t_c^{(n)} = \frac{2}{d_0}(\cot^{-1}(-\frac{1}{\sqrt{|1-2\gamma_0/\lambda|}}) + n\pi), n = 0, 1, 2, ...$ At these singular points, the singularity measure can be given by maximizing the distance $D(\rho(T), \rho_{t_c}(T)) = \frac{1}{2}\sqrt{D_{33}(T)(n_1^2(0) + n_2^2(0)) + D_{33}^2(T)(1 + n_3(0))^2}$ over T and the Bloch vector $\vec{n}(0) = (n_1(0), n_2(0), n_3(0))$ with constraint $0 \leq n_1^2(0) + n_2^2(0) + n_3^2(0) \leq 1$. Simple algebra shows that the maximum of the nth singular point arrives at $T = T^{(n)} = \frac{2(n+1)\pi}{d_0}$, $n = 0, 1, 2...$, and $n_3(0) = \frac{D_{33}(T^{(n)})}{1-D_{33}(T^{(n)})}, n_1^2(0) + n_2^2(0) = 1 - n_3^2(0)$. The measure of singularity for the nth singular point $t_c^{(n)}$ is then

$$
\begin{aligned}
S_\Lambda(t_c^{(n)}) &= \frac{1}{2}\sqrt{\frac{e^{-\frac{2(n+1)\pi\lambda}{d_0}}}{1 - e^{-\frac{2(n+1)\pi\lambda}{d_0}}}}, \text{ for } 0 \leq D_{33}(T^{(n)}) < 0.5. \\
S_\Lambda(t_c^{(n)}) &= e^{-\frac{2(n+1)\pi\lambda}{d_0}}, \text{ for } 0.5 \leq D_{33}(T^{(n)}) \leq 1. \qquad (27)
\end{aligned}
$$

We find that the values of singularity measures are different in the two examples. In the first example, the singularity for all singular points are the same, $S_\Lambda(t_c) = \frac{1}{2}$, while in the second one, the singularity depends on the singular points. This results from the difference in the states at the singular point t_c in the two examples. Especially, as n increases, the singularity decreases and finally tends to zero as $n \to \infty$. In other words, the singularity of larger t_c is smaller than that for a smaller t_c. This can be understood as that at large t_c, the state of the open system is more close to the steady state, leading to a small difference in the states. Furthermore, the difference in singularity is a reflection of the system-environment coupling. The first example is a dephasing model, it conserves the system energy but spoils the off-diagonal elements of the density matrix. By contrast, the system would decay to its ground state at the singular points in the second example.

IV. EXTENSION TO d-DIMENSIONAL SYSTEMS

We consider now an arbitrary dynamical map $\Lambda(t, 0)$ with $t > 0$ for a quantum d-dimensional system (qudit). Let $\{\lambda_\mu\}_{\mu=1}^n$ with $n = d^2 - 1$ be a set of traceless qudit observable satisfying $\mathrm{Tr}(\lambda_\mu\lambda_\nu) = d\delta_{\mu\nu}$. Together with the identity operator they form an orthonormal basis for all the qudit operators. Thus we have expansions $\varrho = (I + \vec{n}\cdot\vec{\lambda})/d$ for the initial state and $\Lambda(t, 0)\varrho = (I + \vec{n}(t)\cdot\vec{\lambda})/d$ with $\vec{n}(t) = D(t)\cdot\vec{n} + \vec{e}(t)$ for the final state $\varrho(t) = \Lambda(t, 0)\varrho$. Here $\vec{n} = \mathrm{Tr}(\varrho\vec{\lambda})$, $\vec{n}(t) = \mathrm{Tr}(\varrho(t)\vec{\lambda})$, and $\vec{e}(t) = \mathrm{Tr}[(\Lambda(t, 0)I)\vec{\lambda}]$ are n-dimensional real vectors and $D(t)$ is an $n \times n$ real matrix with matrix elements given by $[[D(t)]]_{\mu\nu} = \mathrm{Tr}((\Lambda(t, 0)\lambda_\nu)\lambda_\mu)$. The linear trace-preserving map $\Lambda(t, 0)$ is determined uniquely by $D(t)$ and $\vec{e}(t)$ and vice versa. For later use we denote by $V(t) = \{\vec{a} \in R^n | D(t)\cdot\vec{a} = 0\}$ the null space of $D(t)$.

Let $t > t_c > 0$ and consider the possible decomposition of a dynamical map $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$ for some linear trace-preserving map $\Lambda(t, t_c)$. Any linear qudit map, e.g., $\Lambda(t, t_c)$, is in a one-to-one correspondence with a 2-qudit operator, e.g., $R = \Lambda(t, t_c) \otimes \mathcal{I}(\Phi)$ with Φ being the projector of the subnormalized 2-qudit state $|\Phi\rangle = \sum_n |n, n\rangle$. We denote by S the $n \times n$ matrix with elements $[[S]]_{\mu\nu} = \mathrm{Tr}(R\lambda_\mu \otimes \lambda_\nu^T)/d$ for $\mu, \nu = 1, 2, \ldots, n = d^2 - 1$ and $\vec{r} = \mathrm{Tr}(R\vec{\lambda} \otimes I)/d$. For a trace-preserving map it holds $\mathrm{Tr}(RI \otimes \vec{\lambda}) = 0$ and therefore S and $\vec{r}$ determine uniquely R and consequently the linear trace-preserving map $\Lambda(t, t_c)$. By definition the linear map $\Lambda(t, t_c)$ is a possible decomposition if and only if for an arbitrary initial state ϱ it holds

$$\varrho(t) = \Lambda(t, 0)\varrho = \Lambda(t, t_c)(\Lambda(t_c, 0)\varrho) = \Lambda(t, t_c)\varrho(t_c). \qquad (28)$$

Lemma. If a qudit operator O satisfies $\langle\psi|O|\psi\rangle = 0$ for an arbitrary pure qudit state $|\psi\rangle$ then $O = 0$.

Proof. Let $V_{12} = \sum_{ij} |i, j\rangle\langle j, i|$ be the swapping operator of 2 qudits and I_{12} be the identity operator. From the following identity

$$W_{12} := \int d\psi|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| = \frac{I_{12} + V_{12}}{d(d+1)} \qquad (29)$$

it follows that $O = d(d+1)\mathrm{Tr}_1\big((O_1 \otimes I_2)W_{12}\big) = 0$.

Theorem. Given a qudit channel $\Lambda(t, 0)$ with $t > t_c > 0$ there exists a linear trace-preserving map $\Lambda(t, t_c)$ such that $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$ if and only if $V(t_c) \subseteq V(t)$. Moreover the decomposition $\Lambda(t, t_c)$ is unique if and only if $\det D(t_c) \neq 0$.

Proof. Necessity (only if part), i.e., $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$ infers $V(t_c) \subseteq V(t)$. From Eq.(28) it follows that for arbitrary ϱ it holds $\mathrm{Tr}(\vec{\lambda}\varrho(t)) = \mathrm{Tr}\big(\vec{\lambda}\Lambda(t, t_c)\varrho(t_c)\big) = \mathrm{Tr}(R\vec{\lambda} \otimes \varrho^T(t_c)) = S\cdot\vec{n}(t_c) + \vec{r}$. Taking into account $\vec{n}(t) = \mathrm{Tr}(\vec{\lambda}\varrho(t)) = D(t)\cdot\vec{n} + \vec{e}(t)$ for t and t_c we see that $(D(t) - S\cdot D(t_c))\cdot\vec{n} = S\cdot\vec{e}(t_c) + \vec{r} - \vec{e}(t)$ must hold for arbitrary $\vec{n} = \mathrm{Tr}(\varrho\vec{\lambda})$ with ϱ being a density matrix. If we let $\vec{n} = 0$ with corresponding state being $\varrho = I/d$ then we obtain $\vec{r} = \vec{e}(t) - S\cdot\vec{e}(t_c)$. As a result $\Delta\cdot\vec{n} = 0$, where $\Delta := D(t) - S\cdot D(t_c)$, for arbitrary $\vec{n} = \mathrm{Tr}(\varrho\vec{\lambda})$ with ϱ being a density matrix. From $\Delta\cdot\mathrm{Tr}(\varrho\vec{\lambda}) = 0$ it follows that $\mathrm{Tr}(\varrho\Delta_\mu) = 0$ for arbitrary μ and arbitrary qudit state ϱ where $\Delta_\mu = \sum_\nu \Delta_{\mu\nu}\lambda_\nu$ is traceless, i.e., $\mathrm{Tr}\Delta_\mu = 0$. As a result of lemma, $\Delta_\mu = 0$ for all μ, i.e., $D(t) = S\cdot D(t_c)$. Thus we have $V(t_c) \subseteq V(t)$ since $D(t_c)\cdot\vec{n} = 0$ infers $D(t)\cdot\vec{n} = S\cdot D(t_c)\cdot\vec{n} = 0$.

Sufficiency (if part), i.e., $V(t_c) \subseteq V(t)$ infers $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$. Let $\{\vec{e}_i\}_{i=1}^K$ span $V(t_c)$ where $K = \dim V(t_c)$ and $\{\vec{e}_i\}_{i=K+1}^n$ span the orthogonal complement of $V(t_c)$. As a result $D(t_c)\cdot\vec{e}_i = 0$ and thus $D(t)\cdot\vec{e}_i = 0$ for $i = 1, 2, \ldots, K$ since we have assumed $V(t_c) \subseteq V(t)$. The equation $D(t) = S\,\cdot$

$D(t_c)$ is equivalent to $D(t) \cdot \vec{e}_i = S \cdot D(t_c) \cdot \vec{e}_i$ for $i = 1, \ldots n$. Since $D(t_c) \cdot \vec{e}_i = D(t) \cdot \vec{e}_i = 0$ for $i = 1, \ldots, K$ the equation becomes $D'(t) = S \cdot D'(t_c)$ with $D'(t_c) = [D(t_c)\vec{e}_{K+1}, D(t_c)\vec{e}_{K+2}, \ldots D(t_c)\vec{e}_n]$ and $D'(t) = [D(t)\vec{e}_{K+1}, D(t)\vec{e}_{K+2}, \ldots, D(t)\vec{e}_n]$ being of dimension $n \times (n-K)$. Since the rank of $D(t_c)$ is $n-K$ there are exactly $n-K$ (among n) linearly independent row vectors of $D'(t_c)$. Therefore it is always possible to expand each row vector of $D'(t)$, an $(n-K)$-dimensional vector, by those n row vectors of $D'(t_c)$, i.e., for any given i there exist real numbers S_{ik} such that

$$([D'(t)]_{i1}, [D'(t)]_{i2}, \ldots, [D'(t)]_{i,n-K}) = \sum_{k=1}^{n} S_{ik}([D'(t_c)]_{k1}, [D'(t_c)]_{k2}, \ldots, [D'(t_c)]_{k,n-K}). \quad (30)$$

The $n \times n$ matrix S formed by the coefficients S_{ik} with $i, k = 1, 2, \ldots, n$ satisfies $D(t) = S \cdot D(t_c)$ and together with $\vec{r} = \vec{e}(t) - S \cdot \vec{e}(t_c)$ determines R and thus $\Lambda(t, t_c)$ such that $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$. Moreover if and only if $K = 0$, i.e., $V(t_c)$ is empty, i.e, $\det D(t_c) \neq 0$, S as well as $\vec{r} = \vec{e}(t) - S \cdot \vec{e}(t_c)$ is unique and therefore $\Lambda(t, t_c)$ is unique.

We note that the condition $V(t_c) \subseteq V(t)$ is equivalent to $\mathrm{Rank}[D(t_c)|D(t)] = \mathrm{Rank}(D(t_c))$, where $[D(t_c)|D(t)]$ is an augmented matrix. The decomposition $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$ does not exist if and only if $\mathrm{Rank}[D(t_c)|D(t)] > \mathrm{Rank}(D(t_c))$.

The measure proposed here quantifies the non-divisibility of the dynamical map, this measure of non-divisibility is obviously different from the measure of non-completely positive divisibility (or non-Markovianity), where $\Lambda(t, 0) = \Lambda(t, t_c)\Lambda(t_c, 0)$ holds, but $\Lambda(t, t_c)$ is not completely positive. Once $\Lambda(t, 0) \neq \Lambda(t, t_c)\Lambda(t_c, 0)$, the non-Markovianity measure (at time t_c) is no longer available, since state $\Lambda(t, 0)\rho(0)$ differs from $\Lambda(t, t_c)(\Lambda(t_c, 0)\rho(0))$. In this case, a measure of non-Markovianity due to this non-divisibility is required. The non-Markovianity in this situation depends both on the measure of the singularity and the number of singular points. Therefore, we propose,

$$N_M = \sum_j S_\Lambda(t_c^{(j)}), \quad (31)$$

to quantify the non-Markovianity caused by the singular points $t_c^{(j)}$, $(j = 1, 2, 3, ...)$. Physically, once a dynamical map has a singular point t_c, the state at time $t > t_c$ would depend on the state at an earlier time $t' < t_c$, although the state at t_c is the same. This feature can be found by examining Eq.(15), which is a reminiscence of the classical non-Markovian process. **We should stress that the measure N_M in Eq.(31) itself might not quantify the non-Markovianity, because for a dynamical map without singularity, N_M is zero, but it might be non-Markovian.**

The present prediction for qubits can be observed in the experimental setup in [22], where the polarization degree of freedom of photons plays the role of open system, the environment was simulated by the frequency degree of freedom with two central frequencies at ω_1 and ω_2. The evolution of the off-diagonal elements of the photon density matrix takes, $|H\rangle\langle V| \to \kappa^*(t)|H\rangle\langle V|$, $|V\rangle\langle H| \to \kappa(t)|V\rangle\langle H|$. Here $\kappa(t)$ is adjustable and can be manipulated to zero at times $t_c = -\frac{(2n+1)\pi}{\Delta\omega \cdot \Delta n}, n = 0, 1, 2, ...,$ where $\Delta\omega = \omega_2 - \omega_1$, is the difference in the central frequencies of the environment. Δn denotes the difference in the refraction indices of horizontally and vertically polarized photons. The observed final states are different that depend on whether an observation is made at the singular points t_c. By measuring the difference in the final states, the singularity can be quantified in the experiment.

V. CONCLUSION

In summary, we have explored the singular point t_c where the dynamical map $\Lambda(t, 0) \neq \Lambda(t, t_c)\Lambda(t_c, 0)$, i.e., the dynamical map is indivisible at the instance of time t_c. We quantify the singularity of the singular point t_c and present examples to show the singularity. Until now these points were not aware in the divisibility-based measure of non-Markovianity, hence it would contribute to the understanding of quantum non-Markovian process.

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