Scale-invariant Fermi gas in a time-dependent harmonic potential

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On a scale-invariant Fermi gas in a time-dependent harmonic potential

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We investigate a scale-invariant two-component Fermi gas in a time-dependent isotropic harmonic potential. The exact time evolution of the density distribution in position space in any spatial dimension is obtained. Two experimentally relevant examples, an abrupt change and a periodic modulation of the trapping frequency are solved. Small deviations from scale invariance are addressed within first order perturbation theory. We discuss the consequences for experiments with ultracold quantum gases such as the excitation of a tower of undamped breathing modes and a new alternative for measuring the Tan contact.

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Introduction: Symmetries play a key role in modern physics. They can provide useful insights into understanding of systems whose microscopic dynamics is not known or poorly understood. If the microscopic description is available, symmetries serve as a guiding principle for the construction of solutions. In this work we study consequences of scale invariance which implies that physical observables do not depend on absolute lengths. We consider a two-component Fermi gas with contact interactions in any spatial dimension d governed by the Hamiltonian [1]

$$H = \int d\mathbf{x}\Big[-\sum_{i=\uparrow,\downarrow}\psi_i^\dagger\frac{\nabla^2}{2m}\psi_i + c\psi_\downarrow^\dagger\psi_\uparrow^\dagger\psi_\uparrow\psi_\downarrow\Big] \qquad (1)$$

tuned to a scale-invariant regime and trapped in a time-dependent isotropic harmonic potential. The symmetries of this problem allow to find an exact time evolution provided the initial state is known. No knowledge of the equation of state or spectral functions is needed. This is especially beneficial for the much studied Fermi gas with infinite scattering length in $d = 3$ which is theoretically one of the most interesting scale-invariant system. Symmetries alone predict a number of robust dynamical phenomena, which we illustrate here using two examples.

Exact time evolution: Consider a Fermi gas loaded in a time-dependent isotropic harmonic potential described by the Hamiltonian $H_{\rm osc} = H + \int d\mathbf{x}\frac{m\omega^2(t)\mathbf{x}^2}{2}\sum_{i=\uparrow,\downarrow}\psi_i^\dagger\psi_i$. In the following, we assume that for $t<0$ the trap is static, i.e. $\omega(t<0)=\omega_{\rm in}$, and that at $t=0$ the given N-body system is in the eigenstate $\psi(\mathbf{X})$ of the Hamiltonian $H_{\rm osc}$ with the energy E. Here $\mathbf{X}$ collectively denotes the set of positions $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ of N Fermi particles [2]. Subsequently, for $t>0$ the trap frequency is varied with an arbitrary time dependence $\omega(t)$. The time evolution of a scale-invariant Fermi gas in d spatial dimensions can be obtained from the initial wave-function by a combined gauge and scale transformation

$$\psi(\mathbf{X},t) = \frac{e^{-i\theta(t)}}{\lambda^{dN/2}(t)}\exp\left[\frac{im\dot\lambda(t)}{2\lambda}X^2\right]\psi(\mathbf{X}/\lambda(t)), \qquad (2)$$

where $\dot\lambda(t)\equiv\frac{d\lambda(t)}{dt}$. Both $\theta(t)$ and $\lambda(t)$ are determined by the shape of $\omega(t)$. For one particle Eq. (2) goes back to works [3] that is easily generalized to any number N of noninteracting particles [4]. Recently Castin made an insightful observation that the solution (2) is also valid for the three-dimensional strongly-coupled unitary Fermi gas [5]. This was achieved by showing that Eq. (2) obeys the Bethe-Peierls contact condition at unitarity. We find that the solution (2) is valid for a scale-invariant Fermi gas in any spatial dimension [6].

One can check that for $\psi(\mathbf{X},t)$ to be a solution, the gauge angle $\theta(t)$ must solve $\dot\theta(t)=\frac{E}{\lambda^2(t)}$ with $\theta(0)=0$, while the scaling function $\lambda(t)$ obeys the differential equation

$$\ddot\lambda(t) = \frac{\omega_{\rm in}^2}{\lambda^3(t)} - \omega^2(t)\lambda(t), \qquad \omega_{\rm in}\equiv\omega(t=0_-) \qquad (3)$$

with the initial conditions

$$\lambda(0)=1, \qquad \dot\lambda(0)=0. \qquad (4)$$

We recognize a one-dimensional Newton equation for a particle in an inverse square and a time-dependent harmonic potential. Physically, the scaling function $\lambda(t)$ is of a great interest, since it governs the time evolution of various observables. Among them the most experimentally relevant is the density distribution in position space that evolves as $n(\mathbf{x},t)=\frac{1}{\lambda^d(t)}n_0(\frac{\mathbf{x}}{\lambda(t)})$, where $n_0(\mathbf{x})$ is an initial density profile at $t=0$. For the cloud of initial radius $r_{\rm cl,0}$ this implies $r_{\rm cl}(t)=\lambda(t)r_{\rm cl,0}$. Since $\lambda(t)$ does not depend on energy E, the latter two formulae are valid for any initial statistical mixture of stationary states such as, for example, a thermal state.

Abrupt perturbation: First, we consider an experimental setting, where the frequency is changed abruptly at $t=0$ from the initial positive value $\omega_{\rm in}$ to the final positive value $\omega_{\rm f}$, i.e. $\omega(t)=\omega_{\rm in}+(\omega_{\rm f}-\omega_{\rm in})\theta(t)$. For $t>0$ the potential corresponding to the Newton equation (3) has a minimum at $\lambda_{\rm min}=\sqrt{\frac{\omega_{\rm in}}{\omega_{\rm f}}}$ around which $\lambda(t)$ oscillates periodically starting from its initial state (4). In

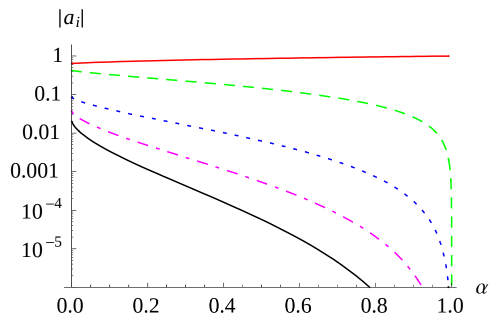

FIG. 1: Absolute values of the amplitudes a_0 (solid red), a_1 (dashed green), a_2 (dotted blue), a_3 (dashed-dotted magenta), a_4 (solid black) as a function of $\alpha = (\omega_{\rm in}/\omega_{\rm f})^2$.

this case the exact solution of Eqs. (3) and (4) for $t > 0$ can be found to be [7]

$$\lambda(t) = \sqrt{\frac{(1+\alpha) + (1-\alpha)\cos(2\omega_{\rm f} t)}{2}} \tag{5}$$

where $\alpha \equiv \left(\frac{\omega_{\rm in}}{\omega_{\rm f}}\right)^2$. The solution (5) can be expressed as the Fourier series $\lambda(t) = \sum_{n=0}^{\infty} a_n \cos(2n\omega_{\rm f} t)$ which physically corresponds to a decomposition into undamped isotropic breathing modes with frequencies $\omega_n = 2n\omega_{\rm f}$ and amplitudes a_n. The lowest amplitudes a_n can be computed analytically as a function of α. Their absolute values are plotted in Fig. 1 for $\alpha < 1$. Quite intuitively, the higher modes have smaller amplitudes compared to the lower ones. One can excite the higher modes most efficiently by a strong perturbation with $\omega_{\rm f} \gg \omega_{\rm in}$. In this limit we find $|a_0| = \frac{2}{\pi}$ and $|a_n| = \frac{4}{\pi(4n^2-1)}$ for $n \in \mathbf{N}$.

A two-dimensional Fermi gas in a time-dependent isotropic trap was recently investigated at different values of the scattering length $a_{\rm 2d}$ in [9]. In this experiment the collective breathing excitations were created by adiabatic reduction of the strength of the trapping frequency $\omega_\perp$ followed by an abrupt restoration to its original value. Provided the first adiabatic step does not excite collective modes, this experimental setting can be well described by Eq. (5). In [9] two different perturbations were studied: a weak perturbation with $\alpha = 0.64$ and a strong one with $\alpha = 0.36$. In both cases only the lowest breathing mode $\omega_1 = 2\omega_\perp$ was measured and no signature of the higher ones was detected. In the regime of asymptotic scale invariance [10], i.e. in the limit $a_{\rm 2d} \to \infty$ or $a_{\rm 2d} \to 0$, this fact can be understood from our calculation (see Fig. 1) which predicts $|\frac{a_2}{a_1}| \approx 3\%$ for the weak perturbation and $|\frac{a_2}{a_1}| \approx 6\%$ for the strong one. These are significantly below the experimental resolution limit $|\frac{a_2}{a_1}| \approx 20\%$ of the experiment [12]. In future the higher breathing modes can be directly measured either by increasing the resolution limit of experiments or by enhancing the perturbation of the trap to values $\alpha \approx 0$.

Periodic perturbation: Second, we investigate another experimentally relevant setting, where the trapping frequency oscillates periodically around its initial value $\omega_{\rm in}$ as $\omega^2(t) = \omega_{\rm in}^2 + \Delta\omega^2 f(t)$ with $f(t+T) = f(t)$ [13]. As the frequency varies in time, the initially stable equilibrium position $\lambda(0) = 1$ of Eq. (3) can become unstable as more and more energy is pumped in. We will first identify the condition for instability for a small perturbation with $0 < (\Delta\omega/\Omega)^2 \ll 1$, where $\Omega = 2\pi/T$. To this end we notice that the solution of the nonlinear equation (3) with the initial conditions (4) can be related to the solution of the linear Newton equation for a time-dependent harmonic oscillator (with the same initial conditions)

$$\ddot{\gamma}(t) = -\omega^2(t)\gamma(t) \tag{6}$$

via the formula

$$\lambda^2(t) = \gamma^2(t)\left[1 + \omega_{\rm in}^2 \xi^2(t)\right] \tag{7}$$

with $\xi(t) = \int_0^t \frac{d\tau}{\gamma^2(\tau)}$ [6].

The stability analysis of the time-dependent harmonic oscillator (6) is a textbook problem [6]. As a result, the instability known as a parametric resonance occurs if one period T of the frequency modulation contains approximately a whole number of half-periods of the characteristic oscillations. For the resonant modulation frequencies we obtain $\Omega_n = \frac{2\omega_{\rm in}}{n}$, where $n \in \mathbf{N}$. Since the inverse cube force in Eq. (3) is time-independent, it can not produce any additional resonances for $\lambda(t)$ in Eq. (7). Hence the above relation is also a necessary and sufficient condition for the parametric resonance in the original nonlinear problem (3). Therefore we arrive at a conclusion that there is an infinite set of modulation frequencies Ω_n that will cause the atomic cloud of a scale-invariant Fermi gas to oscillate with the ever increasing amplitude up until energies where the zero-range description (1) breaks down.

What happens if the periodic perturbation is not small? In order to make quantitative predictions in this regime, we must specify the modulation function $f(t)$. For simplicity we consider

$$f(t) = \begin{cases} +1, & t \in (0, T/2), \\ -1, & t \in (T/2, T). \end{cases} \tag{8}$$

As before, due to the mapping (7), it is sufficient to analyze the stability of the time-dependent harmonic oscillator described by Eq. (6). The resonance condition now reads [6]

$$\begin{aligned} 2 = \Bigg| & 2\cos\left(\frac{\omega_+ T}{2}\right)\cos\left(\frac{\omega_- T}{2}\right) - \\ & \left(\frac{\omega_-}{\omega_+} + \frac{\omega_+}{\omega_-}\right)\sin\left(\frac{\omega_+ T}{2}\right)\sin\left(\frac{\omega_- T}{2}\right)\Bigg|, \end{aligned} \tag{9}$$

where $\omega_\pm = \sqrt{\omega_{\rm in}^2 \pm \Delta\omega^2}$. The solution of this transcendental equation can be found numerically and is plotted in solid red in Fig. 2. For a weak perturbation with $(\Delta\omega/\Omega)^2 \ll 1$ we recover the previously found discrete set of resonant frequencies. As the strength of the perturbation increases the instability regions become broader.

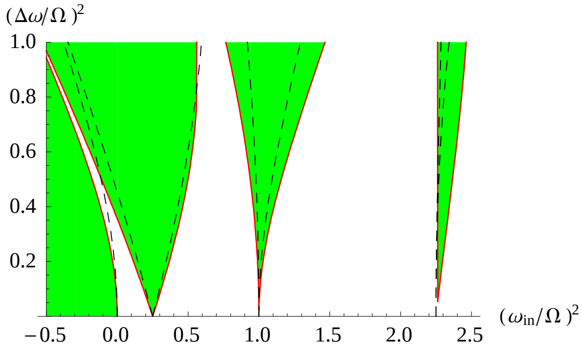

FIG. 2: Stability diagram parametrized by the dimensionless perturbation strength versus the dimensionless initial frequency. For $f(t)$ given by Eq. (8) the solutions are unstable in the green-shaded region. Black dashed curves illustrate the instability boundary for $f(t) = \cos(\Omega t)$.

A notable feature of Fig. 2 is that even the antitrapped Fermi gas with $(\omega_{\rm in}/\Omega)^2 < 0$ can be stabilized by the properly tuned periodic perturbation. This is a direct analogue of the inverted (Kapitza) pendulum stabilized by a vertically oscillating point of suspension.

Symmetries and beyond: In fact, the infinite tower of breathing modes in an isotropic trap is a general consequence of scale or more precisely of nonrelativistic conformal invariance [14–16]. Indeed, using solely the generators $\mathbf{P}$, $\mathbf{K}$, H, C and D of the Schrödinger group (see [16] for the Schrödinger algebra and the definitions of these generators for the Fermi gas) we can construct the operators $\mathbf{Q}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{P}}{\sqrt{\omega}} + i\sqrt{\omega}\mathbf{K}\right)$, $L^\dagger = \frac{1}{2}\left(\frac{H}{\omega} - \omega C + iD\right)$. One can show that in the harmonic trap with the frequency ω, the operator $\mathbf{Q}^\dagger$ excites center-of-mass energy eigenstates by acting repeatedly on a N-body primary state. Since $[H_{\rm osc}, \mathbf{Q}^\dagger] = \omega\mathbf{Q}^\dagger$, the excited states have energies $E_0 + n\omega$, where $n \in \mathbf{N}$ and E_0 denotes the energy of the primary state. In a similar fashion $L^\dagger$ excites breathing eigenstates with energies $E_0 + 2n\omega$. This is true since L, $L^\dagger$ and $H_{\rm osc}$ satisfy $[L, L^\dagger] = \frac{H_{\rm osc}}{\omega}$, $[H_{\rm osc}, L^\dagger] = 2\omega L^\dagger$. The primary state must be annihilated by $\mathbf{Q}$ and L. While for $N = 1$ the unique primary state is the ground state of the total Hamiltonian, for $N \geq 2$ there is an infinite number of the primary states. The energy spectrum is thus organized in infinite ladders with a ladder built on top of every primary state. While the individual center-of-mass (breathing) states do not actually deform in time because they are the eigenstates of the total Hamiltonian, the time evolution of a linear combination of the states from a given ladder produces dipole (breathing) density oscillation decomposable into modes with frequencies $n\omega$ $(2n\omega)$. A simple way how to coherently excite such a linear combination is to perform the abrupt quench (5). It is clear from the solution (5) that the states from different ladders do not mix under such a rapid change of the trapping frequency [17].

Due to separability of the center-of-mass and internal motion in a harmonic trap, one can construct the operator $B^\dagger = L^\dagger - \frac{\mathbf{Q}^\dagger\cdot\mathbf{Q}^\dagger}{2mN}$, which excites internal breathing eigenstates. Indeed, B, $B^\dagger$ and $H_{\rm osc}$ satisfy $[B, B^\dagger] = \frac{H_{\rm osc}}{\omega} - \frac{\{Q_i, Q_i^\dagger\}}{2mN}$, $[H_{\rm osc}, B^\dagger] = 2\omega B^\dagger$ and B and $B^\dagger$ act only on the internal degrees of freedom of the atomic cloud [18].

As argued above, the infinite equidistant tower of internal breathing modes is a generic feature of a scale-invariant many-body system loaded into an isotropic harmonic trap. But what happens to these modes if the symmetries are realized only approximately? Within first order perturbation theory the correction to the energy of the $n^{\rm th}$ internal breathing state caused by a small symmetry-breaking Hamiltonian perturbation δH is

$$\delta E_n = \frac{\langle 0|B^n \delta H B^{\dagger n}|0\rangle}{\langle 0|B^n B^{\dagger n}|0\rangle}, \tag{10}$$

where $|0\rangle$ stands for a N-body primary state in the harmonic trap. Here we assume that the internal breathing states are non-degenerate with other energy eigenstates in the trap. This should be fulfilled in the strongly interacting unitary Fermi gas in three spatial dimensions which we restrict our attention to in the following.

Consider first the breaking of scale invariance by a finite (but large) scattering length a_{3d}. For the Fermi gas near the unitarity regime the Hamiltonian perturbation can be expressed using the local composite dimer field ϕ via

$$\delta H = -\frac{m a_{3d}^{-1}}{4\pi}\int d\mathbf{x}\phi^\dagger\phi. \tag{11}$$

Since this perturbation does not affect the motion of the center of mass, δE_n equals to the energy shift $\delta\mathscr{E}_n$ associated with the internal motion only. By substituting this perturbation into Eq. (10) and using general properties of nonrelativistic scale invariance, we derive [6] for the shift of the level spacing $\delta\Delta_n = \delta(\mathscr{E}_n - \mathscr{E}_{n-1})$

$$\delta\Delta_n = \frac{S_{n-2}}{S_{n-1}}\delta\Delta_{n-1} + \frac{\zeta\omega}{S_{n-1}}\delta\mathscr{E}_{n-1}, \tag{12}$$

where $\zeta = -1/4$, $S_k = (k+1)(\mathscr{E}_0 + k\omega)$ and $\delta\Delta_0 = 0$. Provided the internal part of the energy $\mathscr{E}_0 = E_0 - 3\omega/2$ at unitarity and its shift $\delta\mathscr{E}_0$ are known for a given primary state of the N-particle system, the recursion relation predicts the frequency shifts of the whole tower of breathing modes. The deviations from the scale-invariant value $\Delta_n = 2\omega$ are the largest for the lowest breathing modes. At high energies as $n \to \infty$ the shift $\delta\Delta_n \sim n^{-3/2} \to 0$ [6].

A precise experimental measurement of the lowest level spacing shifts $\delta\Delta_n$ in a many-particle Fermi gas near unitarity can provide a new way to measure the Bertsch parameter $\xi_{\rm B}$ and the Tan contact $C_{\rm trap}$. Indeed, at $T = 0$ the local density approximation predicts for the ground state energy $\mathscr{E}_0 \approx E_0 = 3^{4/3}\sqrt{\xi_{\rm B}}N^{4/3}\omega/8$ [19]. On the other hand, the contact can be directly extracted from the energy shift via $\delta\mathscr{E}_0 = -m a_{3\rm d}^{-1} C_{\rm trap}/4\pi$ [20]. By substituting these two expressions into Eq. (12) we obtain for the lowest shift $\delta\Delta_1 = \frac{mC_{\rm trap}a_{\rm 3d}^{-1}}{2\cdot 3^{4/3}\pi\sqrt{\xi_{\rm B}}N^{4/3}}$, which allows to determine the ratio $C_{\rm trap}/\sqrt{\xi_{\rm B}}$. An additional measurement of $\delta\Delta_2$ would allow to extract separate values of $\xi_{\rm B}$ and $C_{\rm trap}$ from Eq. (12).

In experiments scale invariance is broken by a finite effective range $r_{\rm eff}$. In this case the Hamiltonian perturbation can be expressed as

$$\delta H = \frac{mr_{\rm eff}}{16\pi}\int d\mathbf{x}\left(\phi^\dagger(-i\partial_t - H_{\rm CM})\phi + \text{c. c.}\right), \qquad (13)$$

where $H_{\rm CM} = \omega\{Q_i, Q_i^\dagger\}/2mN$. This perturbation is invariant under translations and Galilean boosts, in addition it affects only the internal motion. For the perturbation (13) we obtain the recursion relation (12) with $\zeta = 3/4$ [6]. For high levels the shift scales as $\delta\Delta_n \sim n^{-1/2} \to 0$ when $n \to \infty$ [6]. For the scattering length and effective range perturbations, we found that the relation (12) is in agreement with the perturbation expansion around unitarity, done recently in [21], of the analytical solution for two particles in a harmonic trap [22].

Let us also consider a long-range two-body (an)isotropic perturbation of the form

$$\delta H \sim \int d\mathbf{x}d\mathbf{y}n(\mathbf{x})\frac{g(\theta)}{|\mathbf{x}-\mathbf{y}|^\rho}n(\mathbf{y}) \qquad (14)$$

with $n = \sum_{i=\uparrow,\downarrow}\psi_i^\dagger\psi_i$, $\rho \in \mathbf{R}$ and $g(\theta)$ is some function of the angle between the unit vector pointing at some fixed direction (e.g. induced by an external field) and the vector $\mathbf{x}-\mathbf{y}$. In this case we again get the recursion relation (12) with $\zeta = \alpha(\alpha-2)/4$ [6]. Note that for the inverse-square interaction potential ($\rho = 2$) the perturbation is scale-invariant and thus does not modify the breathing frequencies. In the context of cold atom experiments, Eq. (12) allows to estimate the effect of weak magnetic dipole-dipole interactions ($\rho = 3$) on the tower of breathing frequencies.

Conclusion: In this work we studied a scale-invariant Fermi gas in a time-dependent isotropic harmonic potential. Within the zero-range model (1) the exact time evolution can be found by solving an effective one-dimensional Newton equation. As examples we considered two experimentally relevant settings. First, an abrupt change of the trapping frequency $\omega(t)$ in the form of a step function was studied. We found the exact solution of this problem, decomposed it into a series of breathing modes and discussed why only the lowest mode was observed in the recent experiment [9]. The influence of a small deviation from scale invariance on the frequencies of breathing modes was studied using first order perturbation theory. Second, periodic oscillations around the initial value of the trapping frequency were investigated. We identified modulation frequencies at which the system becomes unstable and exhibits parametric resonances. We also observed that an antitrapped Fermi gas can be stabilized by periodic frequency oscillations. The findings of this paper are valid at arbitrary temperature provided the zero-range model (1) accurately describes the Fermi gas. Higher breathing modes, parametric resonances and the Kapitza pendulum investigated in this work can be directly realized in future experiments with ultracold quantum gases.

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