Light transport in random media with PT symmetry

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Light Transport in Random Media with $\mathcal{PT}$-symmetry

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The scattering properties of randomly layered optical media with $\mathcal{PT}$-symmetric index of refraction are studied using the transfer-matrix method. We find that the transmitance decays exponentially as a function of the system size, with an enhanced rate $\xi_\gamma(W)^{-1} = \xi_0(W)^{-1} + \xi_\gamma(0)^{-1}$, where $\xi_0(W)$ is the localization length of the equivalent passive random medium and $\xi_\gamma(0)$ is the attenuation/amplification length of the corresponding perfect system with a $\mathcal{PT}$-symmetric refraction index profile. While transmitance processes are reciprocal to left and right incident waves, the reflectance is enhanced from one side and is inversely suppressed from the other, thus allowing such $\mathcal{PT}$-symmetric random media to act as unidirectional coherent absorbers.

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Introduction–Wave propagation in naturally occurring or engineered complex media, is an interdisciplinary field of research that addresses systems as diverse as classical, quantum and atomic-matter waves. Despite this diversity, the wave nature of these systems provides a common framework for understanding their transport properties. One such characteristic is wave interference phenomena. Their existence results in a complete halt of wave propagation in random media which can be achieved by increasing the randomness of the medium. This phenomenon was predicted fifty years ago in the framework of quantum (electronic) waves by Anderson [1] and its existence has been confirmed in recent years in experiments with matter [2] and classical waves [3–7].

While the localization of classical waves has been well understood by now [8], only in the last decade has light propagation in active random media been pursued intensively [9–16]. Due to the absence of a conservation law for photons, light may be absorbed or amplified in the medium while phase coherence is preserved. This interplay of absorption or amplification and localization has been studied by using the Helmholtz equation with an imaginary dielectric constant of an appropriate sign. Several interesting results have been found, such as the dual symmetry of absorption and amplification for the average transmitance and the localization length [14, 15], the sharpness of back scattering coherent peak and the statistics of super-reflectance and transmitance [9–13].

Quite recently, the possibility of synthesizing a new family of artificial optical materials that instead rely on balanced gain and loss regions has been suggested [17–24]. This class of optical structures deliberately exploits notions of parity ($\mathcal{P}$)and time ($\mathcal{T}$) symmetry [25–27] as a means to attain altogether new functionalities and optical characteristics [17]. In optics, $\mathcal{PT}$ symmetry demands that the complex refractive index obeys the condition $n(\vec{r}) = n^*(-\vec{r})$, in other words the real part of the refractive index should be an even function of position, whereas the imaginary part must be odd. $\mathcal{PT}$ symmetries are not only novel mathematical curiosities. In a series of recent experimental papers $\mathcal{PT}$ dynamics have been investigated and key predictions have been confirmed and demonstrated [19, 21, 28, 29]. These include among others, power oscillations [17, 19, 23], absorption enhanced transmition [21], double refraction and non-reciprocity of light propagation [17]. In the nonlinear domain, such pseudo-Hermitian non-reciprocal effects can be used to realize a new generation of on-chip isolators and circulators [22]. Other results within the framework of $\mathcal{PT}$-optics include the study of Bloch oscillations [30], the realization of coherent perfect laser absorbers [31] and nonlinear switching structures [32].

Work has also been done on disordered $\mathcal{PT}$ systems, with main focus on the spectral properties of the corresponding $\mathcal{PT}$-Hamiltonians [33, 34]. However, relatively little has been done concerning their transport properties. In this paper, we will examine the transmitance and reflectance through one-dimensional (1D) $\mathcal{PT}$-symmetric systems with random index of refraction (see Fig. 1). We show that the exponential decay rate of transmitance, which defines the inverse localization length ξ^{-1}, is associated with the harmonic sum of the localization length $\xi_0(W)$ of a passive system with the same degree of randomness and the amplification length $\xi_\gamma(0)$ of a periodic $\mathcal{PT}$-symmetric system with the same degree of gain/loss.

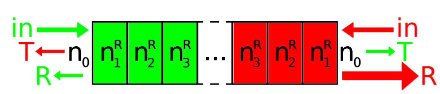

FIG. 1: (Color online) A one-dimensional $\mathcal{PT}$-symmetric multilayered random medium. The gain/loss refraction index profile is uniform (see Eq. (1)) with the loss side on the left (light green color) and the gain side (dark red color) on the right of the structure. The real part of the refraction index contrast n^{R} is random, uniformly distributed around n_0, and $n^{\mathrm{R}}(-z) = n^{\mathrm{R}}(z)$. For large enough system sizes (or strong disorder and/or large gain/loss) the system acts as a high performance absorber if the incident wave is entering from the lossy side of the structure (light green arrows), while it "super-reflects" if the incident wave enters the structure from the gain side (dark red arrows).

Furthermore, we find that the asymptotic value of the reflectance follows a single parameter scaling law which is dictated by the ratio $\Lambda = \xi_0(W)/\xi_\gamma(0)$. Finally, we show that while the transmition processes are reciprocal to left and right incident waves, the reflection is enhanced from one side and is inversely suppressed from the other, thus allowing such $\mathcal{PT}$-symmetric random media to act as unidirectional coherent absorbers (see Fig. 1).

*Mathematical Model–*We consider a one-dimensional ($1D$) active disorder sample having a random $\mathcal{PT}$-symmetric refractive index distribution $n(z) = n_0 + n^{\mathrm{R}}(z) + in^{\mathrm{I}}(z)$ in the interval $|z| < L/2$. The system is embedded in a homogeneous medium having a uniform refractive index n_0 for $|z| > L/2$ (see Fig. 1). Without loss of generality, below, we will assume that the refraction index n_0 outside the disordered medium is $n_0 = 1$. Here n^{R} represents the real index contrast and n^{I} the gain/loss spatial profile. In experimental realization in optics [17, 19], these amplitudes are small, e.g. $n^{\mathrm{R}}, n^{\mathrm{I}} \ll n_0$. For simplicity, we will assume that the sample is composed of an even number, L, of layers of uniform width d, each with a constant real refractive index $n^{\mathrm{R}}(z_j - d/2 \leq z \leq z_j + d/2) = n_j^{\mathrm{R}}$ given by a random variable with a uniform distribution between $(-W, +W)$ which satisfies the $\mathcal{PT}$-symmetric constraint $n^{\mathrm{R}}(z_j) = n^{\mathrm{R}}(-z_j)$. Specifically we will assume

$$\begin{aligned} n(z) &= n_0 + n^{\mathrm{R}}(z) + i\gamma \text{ for } -L/2 < z < 0 \\ &= n_0 + n^{\mathrm{R}}(z) - i\gamma \text{ for } 0 < z < L/2 \end{aligned} \quad (1)$$

where $\gamma \geq 0$ is a fixed gain/loss parameter n^{I}. Although the majority of our simulations below have been done for $n^{\mathrm{I}}(\pm z) = \mp\gamma$, we have also checked that our results apply for the case that $n^{\mathrm{I}}(\pm z) = \mp\gamma + \delta n^I$ where δn^I is a random variable given by a uniform distribution centered at zero. Since the qualitative features remain the same, we will not distinguish between these two cases. In this arrangement, a time-harmonic electric field of frequency ω obeys the Helmholtz equation:

$$\frac{\partial^2 E(z)}{\partial z^2} + \frac{\omega^2}{c^2} n^2(z) E(z) = 0 \ . \quad (2)$$

Eq. (2) admits the solutions $E_0^-(z) = E_f^- e^{ikz} + E_b^- e^{-ikz}$ for $z < -L/2$ and $E_0^+(z) = E_f^+ e^{ikz} + E_b^+ e^{-ikz}$, for $z > L/2$ where the wave-vector $k = n_0\omega/c$. The amplitudes of forward and backward propagating waves outside the disorder domain are related via the transfer matrix M:

$$\begin{pmatrix} E_f^+ \\ E_b^+ \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_f^- \\ E_b^- \end{pmatrix} \quad (3)$$

The transmission and reflection amplitudes for left (L) and right (R) incidence waves, can be obtained from the boundary conditions $E_b^+ = 0$ ($E_f^- = 0$) respectively, and are defined as $t_L \equiv \frac{E_f^+}{E_f^-}$, $r_L \equiv \frac{E_b^-}{E_f^-}$; ($t_R \equiv \frac{E_b^-}{E_b^+}$; $r_R \equiv \frac{E_f^+}{E_b^+}$). These can be expressed in terms of the transfer matrix elements as $t_L = t_R = t = \frac{1}{M_{22}}; r_L = -\frac{M_{21}}{M_{22}}; r_R = \frac{M_{12}}{M_{22}}$ [35, 36]. While the transmitance $T = |t|^2$ for left or right incidence is the same, this is not necessarily the case for the left and right reflectance $R_L = |r_L|^2$ and $R_R = |r_R|^2$ respectively. Furthermore, from the above relations one can deduce that $r_L r_R^* = (1 - |t|^2)$ [31, 37]. Thus one establishes the conservation relation [31, 37]:

$$\sqrt{R_L R_R} = |1 - T| \quad (4)$$

When considering Hermitian scattering systems, where $R_L = R_R$ and $T \leq 1$, this relation recovers the familiar $T + R = 1$. The non-Hermitian systems studied in this paper, however, exhibit non-identical reflectances, and Eq. (4) describes the connection between the two.

Below we investigate the scaling properties of transmitance T and left and right reflectances R_L, R_R from such optical structures, with respect to the disorder strength W, the gain/loss parameter γ, and the wave-vector of the incoming wave. We have used various random refraction index contrasts $W \in (0, 0.5)$, and gain/loss parameters $\gamma \in (0, 0.01)$ which is typical for optical media. We use slabs with $L = 10$ to $L = 10^4$ number of layers, each having width $d = 1$. The logarithmic averages $\langle \ln T \rangle$ and $\langle \ln R \rangle$ are performed over 10^4 disorder realizations.

*Transmittance–*The transport properties of passive (no gain or loss) disorder systems have been thoroughly studied. At large length-scale, such systems exhibit an exponential decay in transmittance (see middle black line in the inset of Fig. 2). The associated inverse decay rate $\xi_0(W)$, reflects the degree of randomness and it is defined as

$$1/\xi_0(W) \equiv -\lim_{L\to\infty} \langle \ln T \rangle / L \quad (5)$$

For 1D random media we have that $\xi_0(W) \sim 1/W^2$ [8].

The other limiting case of an ordered $\mathcal{PT}$-symmetric medium, can also be treated analytically. One can explicitly solve for the electric field inside the perfect $\mathcal{PT}$ layered structure subject to scattering boundary conditions. The resulting expression for the transmitance reads:

$$T = \frac{8\left(1+\gamma^2\right)^2}{\gamma^2\left(4+\gamma^2\right)\Big(\gamma^2\cosh(2kL\gamma)-\cos(2kL)\Big)+8+5\gamma^4-\gamma^6+4\gamma^2\Big(1+\cos(kL)\cosh(kL\gamma)-\gamma\left(2+\gamma^2\right)\sin(kL)\sinh(kL\gamma)\Big)} \quad (6)$$

For large L-values, the term involving $\cosh(2kL\gamma)$ becomes dominant and the transmitance decays exponentially as shown in the inset of Fig. 2 (see upper red line). For the experimentally relevant case $\gamma \ll 1$, the asymptotic decay in transmitance can be found from Eq. (6) to be

$$T_\infty \approx \frac{16 e^{-2k\gamma L}}{\gamma^4(4+\gamma^2)} \tag{7}$$

The corresponding decay rate of transmitance is

$$\frac{1}{\xi_\gamma(0)} \equiv -\lim_{L\to\infty} \frac{1}{L} \ln T \to 2k\gamma \tag{8}$$

which can serve as an operative definition of the so-called attenuation/amplification length $\xi_\gamma(0)$.

On the other hand, for system sizes L smaller than a critical length-scale L_c the transmitance remains approximately constant $T \approx 1$. Near the critical length $L \approx L_c$, large oscillations in the transmitance emerge (see upper red line in the inset of Fig.2) after which the transmitance decays according to the expression given by Eq. (7). The value of L_c can be evaluated approximately by the condition $T_\infty(L = L_c) = 1$ which leads to the following expression

$$L_c \approx \frac{1}{2k\gamma} \ln\Big(\frac{16}{\gamma^4(4+\gamma^2)}\Big). \tag{9}$$

The existence of a critical length-scale L_c is characteristic of gain media and is associated with the lasing threshold for which T diverges. Below this length stimulated emission enhances transmitance through the gain medium. On larger length scales stimulated emission reduces transmitance. The slopes of $\ln T$ at both sides of the maximum are approximately symmetric. In contrast, in the case of a $\mathcal{PT}$-symmetric refraction index the increase of the transmitance for $L < L_c$ which is due to the gain, is balanced by the equal amount of loss which is symmetrically arranged inside the medium. As a result $T \approx 1$ for $L < L_c$. Nevertheless, this gain/loss balance, is not able to smooth out the diverging behavior of T near the lasing threshold (see inset of Fig. 2).

Let us finally consider the case of $\mathcal{PT}$-symmetric disordered slab geometry. A representative behavior of the transmitance T as a function of the system size L is shown in the inset of Fig. 2 (see lower green line). To understand the exponential decay of T, one needs to consider the simplified geometry with index of refraction given by Eq. (1). For the case of Eq. (1) the transfer matrix of the total $\mathcal{PT}$-symmetric system is the product of the transfer matrix M_l associated with the lossy sub-system and M_g associated with the gain sub-system. The corresponding transmitance through the combined system is given by

$$T = \frac{|T_l T_g|}{|1 - r_l r_g|^2} \tag{10}$$

It is thus sufficient to know the scaling behavior of each of the terms on the rhs of Eq. (10) in order to predict the scaling behavior of T. These terms have been studied in Refs. [15], where it was found that both absorption and amplification lead to the same exponential decay of the transmitance which in both cases is enhanced with respect to a passive disordered medium by the strength of the gain (or loss) rate i.e. $\langle \ln T_{l,g}\rangle = -(2k\gamma + \xi_0(W)^{-1})L/2$. Somewhat counter-intuitively, the sample with amplification also exhibits exponentially decaying transmitance due to the enhanced internal reflections from the boundaries. Using the duality relation [15] $r_l r_g^* = 1$ for the reflection of an amplifying or attenuating medium (with the same rate of gain or loss respectively) applied for $L/2 > \xi_0(W)$ we get

$$\begin{aligned} \langle \ln T\rangle &= \langle \ln T_l\rangle + \langle \ln T_g\rangle - 2\langle \ln|1 - r_l r_g|\rangle \\ &= \langle \ln T_l\rangle + \langle \ln T_g\rangle - 2\langle \ln|2(1-\cos(2\theta))|\rangle \end{aligned} \tag{11}$$

where θ is the phase of the reflection amplitude r_l. Assuming that θ is a random variable uniformly distributed on the interval $[0, 2\pi]$ [16], we get that the last term after performing the average over the random variable θ is finite. Therefore we get:

$$\lim_{L\to\infty} \frac{\langle \ln T\rangle}{L} == -(2k\gamma + \xi_0(W)^{-1}) \tag{12}$$

From the above argument, we conclude that the localization length for a $\mathcal{PT}$-symmetric disorder medium is:

$$\xi_\gamma(W)^{-1} = \xi_\gamma(0)^{-1} + \xi_0(W)^{-1} \tag{13}$$

In Fig. 2, we show the results of our numerical simulations for a disordered $\mathcal{PT}$-symmetric sample. The extracted localization length nicely follows the scaling behavior indicated by our theoretical arguments.

Reflectance– We proceed with the analysis of the reflectances. In the case of random media with only gain or loss it was found in Ref. [15] that the reflectances of two *distinct* disordered systems, one with gain strength $-\gamma$ and the other with loss strength γ satisfy the following reciprocal relation between them:

$$R_{gain} R_{loss} = 1 \tag{14}$$

Specifically it was found that for gain media $R_{\rm gain} > 1$, while for lossy media we have via Eq. (14) the reciprocal behavior $R_{\rm loss} = R_{\rm gain}^{-1} < 1$. It is important to stress here that such systems do not distinguish between left and right incidence, that is $R_L = R_R$ for each of the cases.

On the other hand, $\mathcal{PT}$-symmetric systems distinguish the reflection between left or right incident wave, that is, $R_L \neq R_R$ in general (see previous discussion). This phenomenon has already been observed for periodic $\mathcal{PT}$-symmetric structures in Ref. [24]. Moreover, in the presence of random index of refraction, we have previously concluded that the transmitance is effectively diminished exponentially with a rate $1/\xi_\gamma(W)$ given by Eq. (13). Using the conservation relation Eq. (4) we get

$$R_{\rm R} R_{\rm L} \to 1. \tag{15}$$

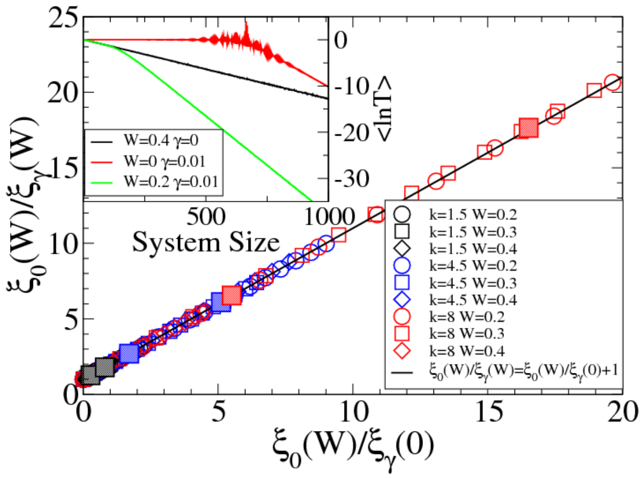

FIG. 2: (Color online) The numerically extracted localization length $\xi_\gamma(W)$ for various gain/loss parameter γ (not indicated in the figure) is plotted, rescaled with $\xi_0(W)$ versus the scaling parameter $\xi_0(W)/\xi_\gamma(0)$. The symbols and colors indicate different wavelengths k of the incident wave, and disorder strengths W. The black line indicates the theoretical prediction of Eq. (13). The meshed symbols correspond to some typical $\xi_\gamma(W)$, for the scenario where the imaginary part of the refraction index is $n^I = \gamma + \delta n^I$ where δn^I is a random variable uniformly distributed around zero. In the inset we report $\langle \ln T \rangle$ against the system size for: a system with only gain/loss and $n^R = n_0$ constant (upper red); a random layer medium with $n^{\rm I} = 0$ (middle black); and a $\mathcal{PT}$-symmetric disorder medium (lower green).

Although this relation is similar to Eq. (14), it should be emphasized once more that in the case of $\mathcal{PT}$-symmetric disorder media the medium behaves simultaneously as a gain medium (i.e. having $R_L > 1$) and as a lossy medium (i.e. it can enhance absorption of incoming coherent waves $R_R < 1$). The reciprocity of the left and right reflectances is clearly demonstrated for some representative cases in the inset of Fig. 3.

A natural question is associated with the scaling behavior of the asymptotic value of the reflectances $R^\infty_{L,R}$ as a function of the disorder strength W and the gain/loss parameter γ. We speculate that a one-parameter scaling law describes the asymptotic reflectance i.e.

$$R^\infty_{L,R}(\gamma, W) = f(\Lambda), \quad \text{where} \quad \Lambda = \xi_0(W)/\xi_\gamma(0) \quad (16)$$

Here, $R^\infty_{L,R}$ is the geometric mean of asymptotic reflectance, that is, $R^\infty_{L,R} = \exp(\langle \ln R^\infty_{L,R} \rangle)$. We have tested our hypothesis numerically. To this end, we have extracted $R^\infty_{L,R}$ from our data for various values of γ and W and plot them against the scaling variable Λ. The results are presented in the main part of Fig. 3. We find that for realistic values of the gain/loss parameter $\gamma \leq 10^{-2} - 10^{-3}$ the data nicely follow the one-parameter scaling hypothesis (16). As the scaling parameter Λ increases (either by decreasing W or by increasing the gain/loss parameter γ), the asymptotic value $R^\infty_{L,R}$ decreases/increases. Such a behavior allows us to use the proposed structure as a unidirectional coherent absorber that can increase absorption by tuning up the scaling parameter Λ. We want to mark that our structure, is different from the one suggested in Ref. [38], where it is shown that a disordered system with a single absorbing element causes coherent enhanced absorption if the phases of the input field are appropriately manipulated. Instead, we are addressing a different problem, where we have broadband absorption in one direction without a need for phase manipulation of the incoming wave.

Conclusions– We have investigated the transport properties of one-dimensional $\mathcal{PT}$-symmetric disordered layers. We have found that the localization length $\xi_\gamma(W)$, defined as the inverse decay rate of the transmitance, is smaller than the localization length of the passive disordered system $\xi_0(W)$ and from the absorption/amplification length $\xi_\gamma(0)$ of a periodic $\mathcal{PT}$-symmetric medium. At the same time the reflectance depends on the direction of the incident wave: while for incident waves entering the medium from the gain side it is enhanced, it is suppressed if the wave enters the medium from the lossy side. The reduction/enhancement of the reflectance is dictated by a one parameter scaling $\Lambda = \xi_0(W)/\xi_\gamma(0)$ and allows us to use such structures as unidirectional quasi-perfect coherent absorbers.

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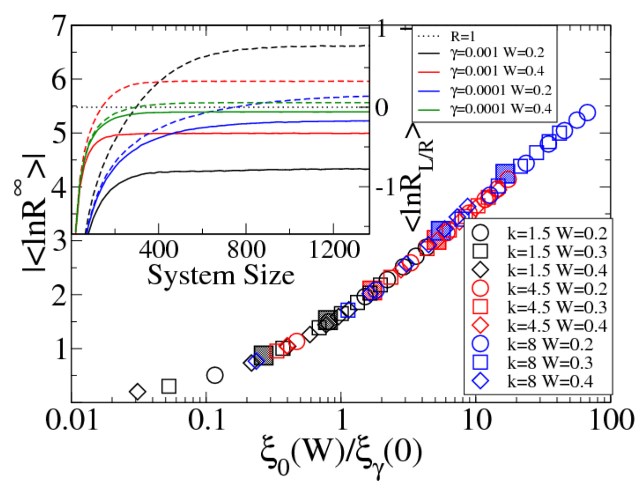

FIG. 3: (Color online) Inset: Typical reflectances $R_{\rm L,R}$ versus the system size (number of layers L). The main figure displays the asymptotic value of $|\log R^\infty|$ versus the scaled parameter Λ. For large values of Λ the $|\log R^\infty|$ increases indicating that $R^\infty_{\rm L}$ (i.e. reflectance for an incident wave entering the sample from the lossy side) diminishes. In this domain, the sample acts as a unidirectional absorber. We use the same symbol and color coding for our data as in Fig. 2.

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