Synthesis of arbitrary SU(3) transformations of atomic qutrits

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Synthesis of arbitrary SU(3) transformations of atomic qutrits

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Several scenarios are proposed and analyzed for engineering of arbitrary preselected SU(3) transformations of laser-driven atomic qutrits. Two of the most natural implementations of qutrits are considered, in which the three qutrit states are coupled to each other via two-photon transitions through either (i) a common state in a tripod linkage pattern, or (ii) two upper states in an M-shaped linkage chain. The SU(3) transformation for the tripod qutrit can be realized by 3 Givens SU(2) rotations, which require 9 consecutive interaction steps. Alternatively, because under certain conditions the propagator of the tripod system reduces to the Householder reflection operator, any SU(3) transformation can be constructed physically by 3 Householder reflections, each of which is implemented in a single interaction step. As an example, the discrete Fourier transform can be synthesized by 7 consecutive interaction steps with Givens rotations or, alternatively, by only a single Householder reflection and a phase gate. For the M-qutrit, the propagator is given by coupled Householder reflections and it cannot be reduced to Givens rotations or independent Householder reflections. By using these coupled Householder reflections it is shown that an arbitrary SU(3) transformation of the M-qutrit can be realized with just two fields in at most 3 interaction steps; the discrete Fourier transform, in particular, requires only 2 interaction steps.

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I. INTRODUCTION

The theory of quantum information processing is largely based upon *qubits* — two-state quantum systems — because of their conceptual simplicity and the relative ease of their physical implementation in various quantum systems [1]. A vital part of quantum information processing is the ability for complete control of the qubits, the most general SU(2) transformation of which is described by three real parameters: one mixing angle θ and two phases α and β. Its most popular implementation uses three sequential physical steps: a phase gate, a rotation gate and another phase gate [2].

A quantum computer built of *qutrits* — realized physically by three-state quantum systems — promises some advantages over the qubit quantum computer. In addition to the immediate exponential increase of the computational Hilbert space qutrits offer new types of quantum protocols [3, 4], more secure and efficient quantum communications [5], new types of entanglement [6], larger violations of nonlocality via Bell tests [7], and optimization of the Hilbert-space dimensionality [8]. While most of the research on qutrits has been focused on photonic qutrits, a qutrit quantum computer with trapped ions was proposed by Klimov *et al.* [9] who described how conditional gates and the quantum Fourier transform can be implemented in analogy to the circuit model of the qubit quantum computer.

For the complete control of a qutrit it is essential to have the ability to construct any preselected SU(3) transformation of it. The extension of the physical control techniques from qubits to qutrits is nontrivial because the coherent operations in the 3D Hilbert space of a qutrit are far more demanding than in the 2D Hilbert space of a qubit. The most general SU(3) transformation of a qutrit is described by 8 independent real parameters, compared to only 3 parameters for a SU(2) transformation of a qubit.

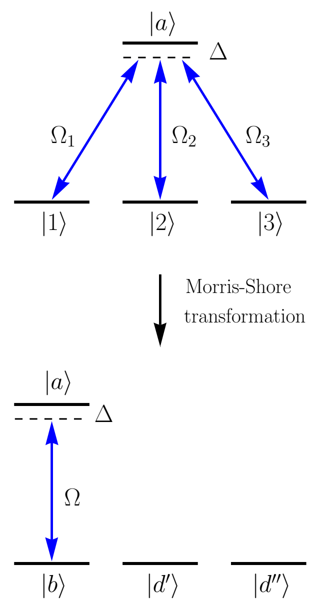

FIG. 1: Qutrit in a tripod linkage: linkage patterns in the original basis (top) and in the Morris-Shore basis (bottom). The qutrit is formed of states $|1\rangle$, $|2\rangle$ and $|3\rangle$. The bright state $|b\rangle$ and the two dark state $|d'\rangle$ and $|d''\rangle$ are linear superpositions of the ground sublevels (i.e. the qutrit states).

Here I describe how the most general SU(3) transformation of a qutrit can be constructed physically in two of the most natural realizations of an atomic qutrit: in a quantum system with a tripod linkage pattern and in a quantum system with a chainwise M-shaped linkage pattern. The qutrit states $|1\rangle$, $|2\rangle$ and $|3\rangle$ are represented by the lower manifolds of states in these systems. These

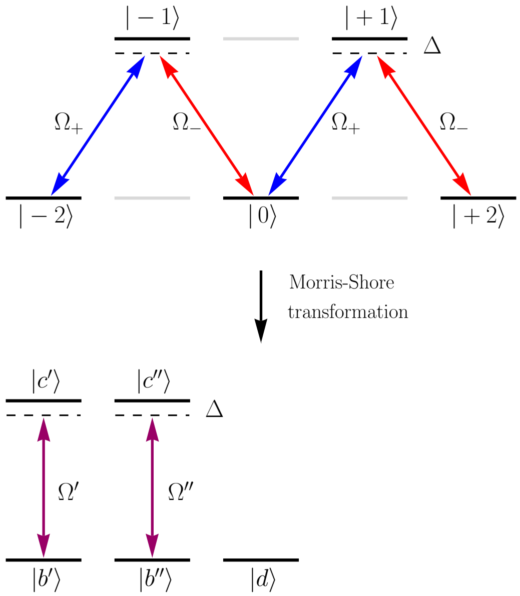

FIG. 2: Qutrit in an M system: linkage patterns in the original basis (top) and in the Morris-Shore basis (bottom). The qutrit is formed of the magnetic sublevels $|-2\rangle$, $|0\rangle$ and $|+2\rangle$ of the lower level. The qutrit states are coupled with two-photon Raman transitions via the upper sublevels $|-1\rangle$ and $|+1\rangle$ by an elliptically polarized field, which is a superposition of right (σ^+) and left (σ^-) circularly polarized fields. Ω_+ and Ω_- the the Rabi frequency "units" associated with the σ^+ and σ^- fields; the actual couplings are given by Ω_+ (or Ω_-) times the corresponding Clebsch-Gordan coefficient $\xi_{m_g}^{m_e}$, cf. Sec. IV. The two bright states $|b'\rangle$ and $|b''\rangle$ and the dark state $|d\rangle$ in the Morris-Shore basis are linear superpositions of the ground sublevels (i.e. the qutrit states), while the two new upper states $|c'\rangle$ and $|c''\rangle$ are linear superpositions of the excited sublevels.

systems arise naturally in the optical transition between the magnetic sublevels of two levels with angular momenta J_g and J_e. The tripod system, shown in Fig. 1, is formed of three lower states $|1\rangle$, $|2\rangle$ and $|3\rangle$ coupled to each other with two-photon transitions via a common upper state $|a\rangle$. Such a tripod system can be formed by the $m_g = -1, 0, 1$ sublevels of a $J_g = 1$ level coupled to the only $m_e = 0$ sublevel of a $J_e = 0$ level with σ^+, π, and σ^- polarized light, respectively; the tripod linkage, however, can emerge also in other physical implementations. The M system, shown in Fig. 2, is formed by the $m_g = -2, 0, 2$ sublevels (which form the qutrit) of a $J_g = 2$ level coupled to the $m_e = -1$ and 1 sublevels of a $J_e = 1$ (or $J_e = 2$) level with σ^+ and σ^- polarized light.

In general, the implementations of SU(N) transformations of N-state systems usually use sequences of SU(2) operations, i.e., transformations acting at each instance of time upon only two of the N states; these are known as Givens rotations. The general SU(N) transformation of a qunit requires $\mathcal{O}(N^2)$ such Givens operations [10]. For a qutrit ($N = 3$), three Givens rotations are needed to construct SU(3), which require 9 interaction steps in total. It has been shown recently that a general SU(N) transformation can be implemented much more efficiently, in only $N - 1$ interaction steps, by using Householder reflections (HR) [11],

$$\mathbf{M}(\chi;\varphi) = \mathbf{I} + \left(e^{i\varphi} - 1\right) |\chi\rangle\langle\chi|, \tag{1}$$

where $|\chi\rangle$ is a normalized complex N-dimensional vector, φ is a real phase and $\mathbf{I}$ is the identity operator. HR is a very powerful unitary transformation, which has many applications in classical data analysis [12]. Each unitary matrix can be represented as a product of $N - 1$ HR matrices and a diagonal matrix (a phase gate) and hence, any SU(N) transformation can be constructed in this manner. In contrast to the construction of SU(N) transformations by Givens rotations, here each HR acts simultaneously upon many states: upon all N states in the first step, $N-1$ states in the second, etc. This allows one to greatly reduce the number of steps, from $\mathcal{O}(N^2)$ with Givens rotations to only $\mathcal{O}(N)$ with HRs.

Recently, it has been discovered that the HR transformation occurs naturally as the propagator in the degenerate manifold of an N-pod quantum system, which consists of N degenerate states coupled to a single common upper state by N simultaneous pulsed fields [13]. This remarkable property of the N-pod system has been used subsequently to design techniques for engineering of arbitrary unitary transformations in a single atom [11] and in an ensemble of trapped ions in a linear Paul trap [14]. It has been demonstrated that just a single HR suffices in an important special case — the transition between two arbitrary N-dimensional superpositions of quantum states [15]. Furthermore, because the HR transformation is the key operation in the quantum search algorithm of Grover [16], it has been shown that the N-pod system can run this algorithm naturally, without gates and circuits [17]; this implementation in a classical database can be modified to non-classical [18] and fully scalable quantum databases [19].

The tripod qutrit, as a special case of the N-pod system, allows for such a natural implementation of the HR transformation, and hence, for efficient construction of SU(3) in just three physical steps. The explicit construction, which requires the simultaneous addressing of the three qutrit states, is presented in Sec. III B. In some implementations, it may be more convenient to use Givens rotations because they require the addressing of only two of the qutrit states at each step; the explicit synthesis of SU(3) for a tripod qutrit by Givens rotations is described in detail in Sec. III A. Moreover, the mathematical SU(3) parameters are explicitly linked to the physically controlled parameters: Rabi frequencies, relative phases and detunings.

The M-qutrit system is a special case of a more general physical system, in which the upper state of the N-pod is replaced by another manifold of degenerate states. It has been shown recently that the propagator in each of the degenerate manifolds is given by a product of HRs with orthogonal vectors [20]; however, no decomposition of SU(N) is known in terms of such coupled HRs. The propagator of the M-qutrit is a product of two coupled HRs. The decomposition of SU(3) by this coupled-HR operator is a far more demanding mathematical task than for the tripod qutrit because coupled HRs cannot be reduced to independent Givens rotations or Householder reflections. Fortunately, the recipe for the experimenter, which is worked out in detail in Sec. IV, appears far less complex: it requires a proper control of ellipticity, polarization, pulse area and detuning of the driving field. Moreover, the proposed implementation for the M qutrit requires just two fields, left and right circularly polarized, and three physical steps, whereas for the tripod qutrit three fields are needed.

This paper is organized as follows. The basic theoretical background, including the manipulation of qubits and two parametrizations of SU(3), are presented in Sec. II. The construction of arbitrary SU(3) transformations for qutrits in tripod systems is described in Sec. III and for qutrits in M-systems in Sec. IV. The conclusions are summarized in Sec. V.

II. BACKGROUND

A. SU(2) transformation of a qubit

The toolbox for qubit transformations in quantum information processing is well established. The most general SU(2) transformation of a qubit,

$$\mathbf{U}(\theta,\xi,\eta) = \begin{bmatrix} e^{i\xi}\cos\theta & -e^{-i\eta}\sin\theta \\ e^{i\eta}\sin\theta & e^{-i\xi}\cos\theta \end{bmatrix}, \tag{2}$$

can be realized by three gates: phase gates $\mathbf{\Phi}(\xi\pm\eta)$ and a rotation gate $\mathbf{R}(\theta)$ [2],

$$\mathbf{U}(\theta,\xi,\eta) = \mathbf{\Phi}(\xi-\eta)\mathbf{R}(\theta)\mathbf{\Phi}(\xi+\eta), \tag{3}$$

where

$$\mathbf{\Phi}(\phi) = \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix}, \quad \mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{4}$$

The SU(2) transformation can be realized more efficiently, with only two gates, a HR $\mathbf{M}(\chi,\varphi)$ and a phase gate [11],

$$\mathbf{U}(\theta,\xi,\eta) = e^{i\varphi/2}\mathbf{M}(\chi;-\varphi)\mathbf{\Phi}(-\varphi), \tag{5}$$

where

$$|\chi\rangle = \frac{[e^{i(\xi+\eta)}\cos\theta - 1, e^{i(\eta-\xi)}\sin\theta]^T}{\sqrt{2[1-\cos\theta\cos(\xi+\eta)]}}, \tag{6a}$$

$$\varphi = \pi - 2\arg[1 - e^{i(\xi+\eta)}\cos\theta]. \tag{6b}$$

Since $\det\mathbf{M}(\chi,-\varphi) = e^{-i\varphi}$ one finds $\det\mathbf{U}(\theta,\xi,\eta) = 1$ due to the phase factor $e^{i\varphi/2}$. Because the Householder implementation (5) requires only two physical steps it is superior to the implementation (3), which needs three physical steps.

B. Parametrizations of SU(3)

1. *Parametrization of SU(3) by generalized Euler angles*

A general SU(3) matrix $\mathbf{U}$ is parametrized by 8 real parameters. There exist several parametrizations derived from various mathematical arguments. I shall use the intuitive geometric parametrization of SU(3) by Bronzan [21] in terms of 3 Euler angles θ_j ($0 \leqq \theta_j \leqq \pi/2$; $j = 1,2,3$) and 5 phases ϕ_k ($0 \leqq \phi_k \leqq 2\pi$; $k = 1,2,3,4,5$),

$$\mathbf{U} = \begin{bmatrix} e^{i\phi_1}c_1c_2 & e^{i\phi_3}s_1 & e^{i\phi_4}c_1s_2 \\ e^{-i\phi_4-i\phi_5}s_2s_3 - e^{i\phi_1+i\phi_2-i\phi_3}s_1c_2c_3 & e^{i\phi_2}c_1c_3 & -e^{-i\phi_1-i\phi_5}c_2s_3 - e^{i\phi_2-i\phi_3+i\phi_4}s_1s_2c_3 \\ -e^{-i\phi_2-i\phi_4}s_2c_3 - e^{i\phi_1-i\phi_3+i\phi_5}s_1c_2s_3 & e^{i\phi_5}c_1s_3 & e^{-i\phi_1-i\phi_2}c_2c_3 - e^{-i\phi_3+i\phi_4+i\phi_5}s_1s_2s_3 \end{bmatrix}, \tag{7}$$

where $c_k = \cos\theta_k$ and $s_k = \sin\theta_k$. There exist other equivalent parametrizations [22].

2. *Householder parametrization of SU(3)*

An alternative, very simple parametrization of SU(3), which is scalable to an arbitrary SU(N) in a straightforward manner, is by the product of two Householder reflections [11, 14],

$$\mathbf{U} = \mathbf{M}(\chi_1;\varphi_1)\mathbf{M}(\chi_2;\varphi_2)\mathbf{\Phi}_3(-\varphi_1-\varphi_2), \tag{8}$$

where $\mathbf{\Phi}_3(\varphi) = e^{i\varphi}|3\rangle\langle 3|$ is a 3D phase gate, i.e. the diagonal matrix $\mathbf{\Phi}_3(\varphi) = \operatorname{diag}(1,1,e^{i\varphi})$. Here χ_1 is a three-component normalized complex vector; because the overall phase of χ_1 is irrelevant this vector brings four real parameters. The vector χ_2 is another three-component normalized complex vector but with a null element; hence this vector brings only two real parameters. With the two phases φ_1 and φ_2 the total number of real independent parameters in Eq. (8) is eight, exactly as needed. The phase gate $\mathbf{\Phi}_3(-\varphi_1-\varphi_2)$ serves to compensate the HR phases so that $\det\mathrm{SU}(3) = 1$.

3. Householder parametrization of SU(N)

The Householder factorization of an arbitrary SU(N) matrix $\mathbf{U}$ proceeds similarly [11],

$$\mathbf{U} = \mathbf{M}(\chi_1;\varphi_1)\mathbf{M}(\chi_2;\varphi_2)\cdots\mathbf{M}(\chi_{N-1};\varphi_{N-1})\mathbf{\Phi}_N(-\varphi_N), \tag{9}$$

with $\varphi_N = \sum_{k=1}^{N-1}\varphi_k$, where $\mathbf{\Phi}_N(-\varphi_N) = e^{-i\varphi_N}|N\rangle\langle N|$ is an N-dimensional phase gate. Here χ_k is an N-component normalized complex vector with $N+1-k$ nonzero components and $k-1$ null components. The normalization and the irrelevant global phase take away two real parameters and leave $2(N-k)$ independent real parameters in this vector. The vectors χ_k contain altogether $\sum_{k=1}^{N-1}2(N-k) = N(N-1)$ real parameters; with the addition of the $N-1$ phases φ_k there are N^2-1 real independent parameters in Eq. (9), as needed for a most general SU(N) matrix.

III. QUTRIT IN A TRIPOD LINKAGE

A mathematically very convenient implementation of a qutrit is the manifold of three degenerate lower states $|1\rangle$, $|2\rangle$, and $|3\rangle$, forming the qutrit, coupled to each other by two-photon Raman processes via an ancilla upper state $|a\rangle$, thereby forming a tripod linkage pattern, as illustrated in Fig. 1 (top) [23]. The Hamiltonian describing this tripod system reads

$$\mathbf{H}(t) = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & 0 & \Omega_1(t) \\ 0 & 0 & 0 & \Omega_2(t) \\ 0 & 0 & 0 & \Omega_3(t) \\ \Omega_1(t)^* & \Omega_2(t)^* & \Omega_3(t)^* & 2\Delta \end{bmatrix}, \tag{10}$$

where $\Omega_k(t) = |\Omega_k(t)|\,e^{i\alpha_k}$ is the complex Rabi frequency of the coupling of the $|k\rangle \leftrightarrow |a\rangle$ transition, and Δ is the single-photon detuning of state $|a\rangle$. It is obvious from this Hamiltonian that two-photon resonance conditions are assumed in the Raman transition between each pair of qutrit states. The coherent dynamics of the tripod propagator is determined by the Schrödinger equation,

$$i\hbar\partial_t\mathbf{U}(t) = \mathbf{H}(t)\mathbf{U}(t), \tag{11}$$

subject to the initial condition at time t_i: $\mathbf{U}(t_i) = \mathbf{I}$, the identity matrix. If the Rabi frequencies of the three coupling fields coincide in time, which I assume hereafter, the dynamics of this qutrit system is reducible by the Morris-Shore transformation [24] to a two-state system, as illustrated in Fig. 1. The solution of the tripod system can be derived from the two-state solution; it is discussed in detail elsewhere [13]. I shall use this earlier solution for the construction of arbitrary SU(3) transformations of the qutrit.

A. Synthesis of SU(3) by Givens rotations

1. Givens decomposition

It is straightforward to verify that the SU(3) matrix (7) can be decomposed as a product of three SU(2) transformations,

$$\mathbf{U} = \mathbf{R}_{23}(\theta_3,\phi_2-\phi_5,0)\mathbf{R}_{12}(\theta_1,-\phi_5,\phi_3+\pi) \times \mathbf{R}_{31}(\theta_2,\phi_1+\phi_5,\phi_4+\phi_5), \tag{12}$$

where

$$\mathbf{R}_{23}(\theta,\xi,\eta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\xi}\cos\theta & -e^{i\eta}\sin\theta \\ 0 & e^{-i\eta}\sin\theta & e^{-i\xi}\cos\theta \end{bmatrix}, \tag{13a}$$

$$\mathbf{R}_{31}(\theta,\xi,\eta) = \begin{bmatrix} e^{i\xi}\cos\theta & 0 & e^{i\eta}\sin\theta \\ 0 & 1 & 0 \\ -e^{-i\eta}\sin\theta & 0 & e^{-i\xi}\cos\theta \end{bmatrix}, \tag{13b}$$

$$\mathbf{R}_{12}(\theta,\xi,\eta) = \begin{bmatrix} e^{i\xi}\cos\theta & -e^{i\eta}\sin\theta & 0 \\ e^{-i\eta}\sin\theta & e^{-i\xi}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{13c}$$

Each of these Givens rotations $\mathbf{R}_{jk}$ can be realized physically by addressing only two of the qutrit states $|j\rangle$ and $|k\rangle$ in three steps — a phase gate, a rotation, and another phase gate, as described in Sec. II. Hence, 9 consecutive steps are required to construct an arbitrary SU(3) transformation of the qutrit by Givens rotations.

2. Example: Discrete Fourier transform

The discrete Fourier transform (DFT) is the key subroutine in many quantum algorithms [2], most notably in Shor's quantum factoring [25]. The DFT may be viewed also as an analog of the Hadamard transform of a qubit because, when acting upon any qutrit state, it creates an equal superposition of all qutrit states. The DFT of a qutrit has the matrix elements $F_{jk} = e^{2i\pi(j-1)(k-1)/3}/\sqrt{3}$, or explicitly,

$$\mathbf{F} = \tfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2i\pi/3} & e^{-2i\pi/3} \\ 1 & e^{-2i\pi/3} & e^{2i\pi/3} \end{bmatrix}. \tag{14}$$

It is U(3) but not SU(3) because $\det\mathbf{F} = -i$; however, if multiplied by the phase factor $e^{i\pi/6}$ it becomes SU(3). The DFT can be decomposed by three Givens rotations

$$\mathbf{F} = e^{-i\pi/6}\,\mathbf{R}_{23}\left(\frac{\pi}{4},\frac{\pi}{3},\pi\right)\mathbf{R}_{12}\left(\cot^{-1}\sqrt{2},-\frac{\pi}{2},-\frac{5\pi}{6}\right) \times \mathbf{R}_{31}\left(\frac{\pi}{4},\frac{2\pi}{3},\frac{2\pi}{3}\right). \tag{15}$$

(This decomposition is superior to the one given in Ref. [9], which requires an additional three-dimensional phase gate.) Hence the DFT can be constructed with 3 rotations and 5 phase gates [one phase gate reduces to an identity because of the equal angles in $\mathbf{R}_{31}$ in Eq. (15)].

The decomposition of SU(3) to Givens rotations is not unique. For example, the decomposition (15) is obtained by first nullifying the element F_{32} of $\mathbf{F}$, then F_{21} and finally F_{13}. Alternatively, one can, for instance, proceed by nullifying first the element F_{21}, then F_{31}, and then F_{32}; this gives the decomposition

$$\mathbf{F} = e^{-i\pi/6}\,\mathbf{R}_{12}\left(\frac{\pi}{4},0,0\right)\mathbf{R}_{31}\left(\cot^{-1}\sqrt{2},\frac{\pi}{6},\frac{5\pi}{6}\right) \times \mathbf{R}_{23}\left(\frac{3\pi}{4},\pi,\frac{\pi}{3}\right); \tag{16}$$

which can be obtained in 7 steps by 3 rotations and 4 phase gates [two phase gates vanish because of the zero angles in $\mathbf{R}_{12}$ in Eq. (15)].

B. Synthesis of SU(3) by Householder reflections

1. Householder decomposition of SU(3)

The decomposition (9) of any SU(3) matrix into a product of two HRs provides a much faster recipe for construction of arbitrary SU(3) transformations. Given an SU(3) matrix $\mathbf{U}$, the HR vectors $|\chi_k\rangle$ and phases φ_k $(k = 1, 2)$ can be found as follows [11].

The vector $|\chi_1\rangle$ and the phase φ_1 of the first HR read

$$|\chi_1\rangle = \frac{|u_1\rangle - |e_1\rangle}{\sqrt{2(1-\operatorname{Re} u_{11})}}, \quad \varphi_1 = \pi - 2\arg u_{11}, \tag{17}$$

where $|u_1\rangle = (u_{11}, u_{21}, u_{31})^T$ is the first column of $\mathbf{U}$ and $|e_1\rangle = (1,0,0)^T$. The action of $\mathbf{M}(\chi_1; -\varphi_1)$ on $\mathbf{U}$ produces the matrix

$$\mathbf{U}' = \mathbf{M}(\chi_1; -\varphi_1)\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u'_{22} & u'_{23} \\ 0 & u'_{32} & u'_{33} \end{bmatrix}, \tag{18}$$

from where the vector $|\chi_2\rangle$ and the phase φ_2 of the second HR are derived,

$$|\chi_2\rangle = \frac{|u'_2\rangle - |e_2\rangle}{\sqrt{2(1-\operatorname{Re} u'_{22})}}, \quad \varphi_2 = \pi - 2\arg u'_{22}, \tag{19}$$

where $|u'_2\rangle = (0, u'_{22}, u'_{32})^T$ is the second column of $\mathbf{U}'$ and $|e_2\rangle = (0,1,0)^T$. The combined action of the HRs $\mathbf{M}(\chi_1; -\varphi_1)$ and $\mathbf{M}(\chi_2; -\varphi_2)$ on $\mathbf{U}$ produces the matrix

$$\mathbf{M}(\chi_2; -\varphi_2)\mathbf{M}(\chi_1; -\varphi_1)\mathbf{U} = \mathbf{\Phi}_3(-\varphi_1 - \varphi_2), \tag{20}$$

which gives $\mathbf{U}$ immediately in the form (8) by noticing that $\mathbf{M}^{-1}(\chi;\varphi) = \mathbf{M}^{\dagger}(\chi;\varphi) = \mathbf{M}(\chi;-\varphi)$.

2. Implementation

The convenience of the tripod implementation of a qutrit derives from the fact that the tripod system is decomposed by the Morris-Shore (MS) transformation [24] into a set involving two decoupled dark states and a two-state system composed of the bright coherent superposition of qutrit states and the upper state [13, 23], as illustrated in Fig. 1 (bottom). The conditions for this decomposition are that the three coupling fields have the same time dependence and are equally detuned from the common upper state. The coupling in the MS two-state system is the root-mean-square (rms) of the original coupling, $\Omega = \sqrt{|\Omega_1|^2 + |\Omega_2|^2 + |\Omega_3|^2}$, and the detuning Δ is the original one. If the transition probability in the MS two-state system is zero then the propagator in the tripod manifold is exactly the HR operator needed for the synthesis of SU(3) [11, 13],

$$\mathbf{U} = \mathbf{M}(\chi;\varphi), \tag{21}$$

where the HR vector is the bright ground state, $|\chi\rangle = |b\rangle$; its components are the complex Rabi frequencies,

$$|b\rangle = [|\Omega_1|\,e^{i\alpha_1}, |\Omega_2|\,e^{i\alpha_2}, |\Omega_3|\,e^{i\alpha_3}]^T/\Omega. \tag{22}$$

This allows one to produce any desired Householder vector by adjusting the couplings and their relative phases.

The Householder phase φ depends on the Rabi frequency and the detuning; in the asymptotic limit of very large detuning it is given by the ac Stark shift,

$$\varphi \approx \int_{-\infty}^{\infty} \frac{\Omega(t)^2}{4\Delta(t)}\,dt, \quad (|\Delta| \gg \Omega). \tag{23}$$

For smoothly shaped $\Omega(t)$ (e.g. Gaussian), a more accurate formula which is valid also for $|\Delta| < \Omega$ reads [26]

$$\varphi \approx \int_{-\infty}^{\infty} \frac{\Delta(t)}{2}\left[\sqrt{\frac{\Omega(t)^2}{\Delta(t)^2} + 1} - 1\right] dt, \tag{24}$$

provided $|\Delta| \gg 1/T$, where T is the pulse width. Even a more accurate expression for this phase exists which uses the superadiabatic approximation [26]. Hence any desired HR phase can be produced by a suitably chosen detuning Δ.

These properties of the tripod system remain valid in the general case of N degenerate states coupled to each other via an upper state, forming an N-pod linkage pattern. This allows one to construct SU(N) in a similar fashion: the HR vectors by properly adjusting the magnitudes and the relative phases of the couplings and the HR phases by properly choosing the detuning.

The HR decomposition (8) of SU(3) into a product of two HRs and a phase gate, and the direct correspondence of the HR vectors and phases to the interaction parameters described above, allow one to produce an arbitrary preselected SU(3) transform of a tripod qutrit in only 3 physical steps. For the first reflection $\mathbf{M}(\chi_1;\varphi_1)$

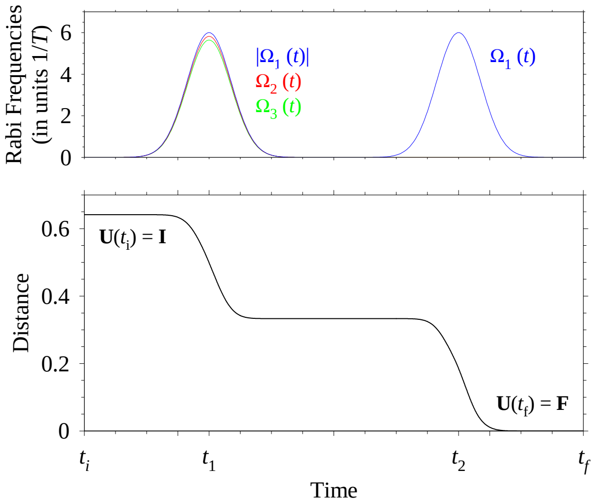

FIG. 3: Numerical simulation of two-step synthesis of DFT of a tripod qutrit according to Eq. (26) for Gaussian pulse shapes, $\Omega^{(1)}(t) = \Omega_0^{(1)} e^{-(t-t_1)^2/T^2}$ and $\Omega^{(2)}(t) = \Omega_0^{(2)} e^{-(t-t_2)^2/T^2}$, where $\Omega_0^{(1)} = 6/T$, $\Delta^{(1)} = 15.06/T$ in the first step and $\Omega_0^{(2)} = 6/T$, $\Delta^{(2)} = -4.45/T$ in the second step (approximate numerically derived values), with T being the characteristic pulse width. *Upper frame:* Magnitudes of the Rabi frequencies of the three fields $\Omega_1(t)$, $\Omega_2(t)$ and $\Omega_3(t)$. In the first step at time t_1 the three Rabi frequencies are equal (they are made slightly different for visualization), but $\Omega_1(t)$ is phase shifted, Eq. (26), while in the second step at time t_2 only $\Omega_1(t)$ is applied, while $\Omega_2(t) = \Omega_3(t) = 0$. *Lower frame:* Distance $D = \|\mathbf{U}(t) - \mathbf{F}\|$ between the propagator $\mathbf{U}(t)$ and the DFT (14).

all three couplings are applied, for the second reflection $\mathbf{M}(\chi_2; \varphi_2)$ only the couplings for the transitions $|2\rangle \leftrightarrow |a\rangle$ and $|3\rangle \leftrightarrow |a\rangle$ are applied, and in the phase gate $\mathbf{\Phi}_3$ only state $|3\rangle$ is involved.

It is clear that the HR decomposition of SU(3) described above is not unique, for one can choose different columns or rows at each step of the diagonalization of $\mathbf{U}$.

3. *Example: Discrete Fourier transform*

It is readily verified that the DFT (14) can be constructed by a single HR and a phase gate,

$$\mathbf{F} = i\mathbf{\Phi}_1(-2\pi/3)\mathbf{M}(\chi; 2\pi/3), \tag{25}$$

with $|\chi\rangle = (e^{-2i\pi/3}, 1, 1)/\sqrt{3}$. These steps can be constructed by the following interaction sets:

$$[\Omega_1(t), \Omega_2(t), \Omega_3(t)] = [\Omega^{(1)}(t)\, e^{-2i\pi/3}, \Omega^{(1)}(t), \Omega^{(1)}(t)], \tag{26a}$$

$$[\Omega_1(t), \Omega_2(t), \Omega_3(t)] = [\Omega^{(2)}(t), 0, 0], \tag{26b}$$

where the values of the rms Rabi frequencies in the two steps, $\Omega^{(1)}(t)$ and $\Omega^{(2)}(t)$, and the respective detunings $\Delta^{(1)}$ and $\Delta^{(2)}$ are obtained numerically from the conditions to produce the HR phase $2\pi/3$ in the first step and the phase gate phase $-2\pi/3$ in the second step. There exist infinitely many sets of values of the interaction parameters which satisfy these conditions; these values depend on the pulse shapes. An example for construction of SU(3) of a tripod qutrit with Gaussian pulses is presented in Fig. 3, which shows the evolution of the distance between the time-dependent propagator $\mathbf{U}(t)$ and the desired DFT (14). There are two steep declines of this distance, one around the first step, when the HR is constructed, and another one around the second step, when the phase gate $\mathbf{\Phi}_1(-2\pi/3)$ is applied. Ultimately, the distance vanishes, which indicates the synthesis of the desired DFT (14).

Because between the two steps the population resides in the qutrit states, the possible decay from the upper state $|a\rangle$ with a rate Γ is only relevant during each pulse, i.e. one must have $T \ll 1/\Gamma$. Of course, possible decay can be suppressed by using large enough detuning Δ.

C. Summary

To summarize this section, a general SU(3) transformation of a tripod qutrit can be constructed by 3 Givens SU(2) rotations, each of which is decomposed into a rotation and two phase gates, thereby adding up to 9 physical steps. In particular, the DFT is decomposed into 3 Givens rotations, which can be constructed by at least 7 physical steps in total: 3 rotations and 4 phase gates. Alternatively, an arbitrary SU(3) transformation can be synthesized by only two Householder reflections and a phase gate, that is by only 3 physical steps. The DFT itself is constructed by a single HR and a phase gate, i.e. by just two physical steps.

IV. QUTRIT IN AN M-SHAPED LINKAGE

Another natural implementation of an atomic qutrit is with the magnetic sublevels $m_g = -2, 0, 2$ of a level with an angular momentum $J_g = 2$. The three sublevels are coupled to each other by two-photon transitions via the magnetic sublevels $m_e = -1$ and $m_e = 1$ of an exited level with an angular momentum $J_e = 1$ or $J_e = 2$ by two laser fields with right (σ^+) and left (σ^-) circular polarizations. The coupling linkage pattern is reminiscent of the letter "M" as illustrated in Fig. 2, hence the term M-system. In this $J_g \leftrightarrow J_e$ transition, these laser fields create also another Λ-linkage between the magnetic sublevels $m_g = -1$, $m_e = 0$ and $m_g = 1$ (shown by light-grey lines in Fig. 2); however, this Λ-system is decoupled from the M-system and will be ignored.

A. Description

If all couplings have the same time dependence and the detunings between the laser frequencies and the corresponding transition frequencies are the same then the M-shaped chain can be decomposed by the Morris-Shore transformation [20, 24, 27] into a pair of independent two-state systems and a dark state $|d\rangle$, as shown in Fig. 2. In each of the MS two-state systems, the lower state $|b'\rangle$ or $|b''\rangle$ is a (bright) coherent superposition of ground sublevels and the upper state is a coherent superposition of upper sublevels only. The dark state itself is a superposition of ground sublevels. The bright MS states are defined as the eigenstates of the matrix $\mathbf{V}\mathbf{V}^\dagger$ with nonzero eigenvalues $(\Omega')^2$ and $(\Omega'')^2$, while the dark state $|d\rangle$ is the null-eigenvalue of $\mathbf{V}\mathbf{V}^\dagger$. Similarly, the upper MS states are the eigenstates of $\mathbf{V}^\dagger\mathbf{V}$. Here the matrix $\mathbf{V}$ contains the couplings between the three lower and two upper states (ordered in ascending order of m_g and m_e) in the original M-system (cf. Fig. 2) [20, 24, 27],

$$\mathbf{V} = \begin{bmatrix} \xi_{-2}^{-1}\Omega_+ e^{i\beta_+} & 0 \\ \xi_0^{-1}\Omega_- e^{-i\beta_-} & \xi_0^1\Omega_+ e^{i\beta_+} \\ 0 & \xi_2^1\Omega_- e^{-i\beta_-} \end{bmatrix}, \tag{27}$$

where $\xi_{m_g}^{m_e}$ are the Clebsch-Gordan coefficients for the respective transitions. For the transition $J_g = 2 \leftrightarrow J_e = 1$ they are $\xi_{-2}^{-1} = \xi_2^1 = \sqrt{\frac{3}{5}}$, $\xi_0^{-1} = \xi_0^1 = \sqrt{\frac{1}{10}}$, while for $J_g = 2 \leftrightarrow J_e = 2$ they are $\xi_{-2}^{-1} = -\xi_2^1 = -\sqrt{\frac{1}{3}}$, $\xi_0^{-1} = -\xi_0^1 = \sqrt{\frac{1}{2}}$. The two real functions $\Omega_+(t)$ and $\Omega_-(t)$ are time-dependent (pulse-shaped) Rabi frequency "units" for the couplings induced by the σ^+ and σ^- polarized laser fields; they are supposed to share the same time dependence but are allowed to have different peak values. β_+ and β_- are the phases of the σ^+ and σ^- fields.

The couplings in the interaction matrix (27) can be produced by a single elliptically polarized laser pulse, which can be represented as a superposition of two circularly polarized σ^+ and σ^- pulses. The electric field of the elliptically polarized pulse in the complex representation $E(t) = E_x(t) + iE_y(t)$ is given by $E(t) = E_+(t)e^{-i\omega t+i\beta_+} + E_-(t)e^{i\omega t+i\beta_-}$ [28, 29]. The polarization ellipticity is

$$\varepsilon = \frac{\Omega_+^2 - \Omega_-^2}{\Omega_+^2 + \Omega_-^2} = \frac{E_+^2 - E_-^2}{E_+^2 + E_-^2} \tag{28}$$

and the angle of rotation of the polarization ellipse is $\frac{1}{2}\beta$, with $\beta = \beta_+ + \beta_-$. Ω_+ and Ω_- are parametrized in terms of ε as $\Omega_\pm = \Omega\sqrt{\frac{1}{2}(1 \pm \varepsilon)}$ where $\Omega = \sqrt{\Omega_+^2 + \Omega_-^2}$. Ellipticity $\varepsilon = \pm 1$ corresponds to $\sigma^\pm$ polarization and $\varepsilon = 0$ to linear polarization.

The dark and bright states read [27]

$$|d\rangle = d_-(\varepsilon)e^{i\beta}|-2\rangle + d_0(\varepsilon)|0\rangle + d_+(\varepsilon)e^{-i\beta}|2\rangle, \tag{29a}$$

$$|b'\rangle = b'_-(\varepsilon)e^{i\beta}|-2\rangle + b'_0(\varepsilon)|0\rangle + b'_+(\varepsilon)e^{-i\beta}|2\rangle, \tag{29b}$$

$$|b''\rangle = b''_-(\varepsilon)e^{i\beta}|-2\rangle + b''_0(\varepsilon)|0\rangle + b''_+(\varepsilon)e^{-i\beta}|2\rangle, \tag{29c}$$

where the parameters of these new basis states are given in the Tables I and II. The MS couplings are Ω' and Ω'',

TABLE I: Coefficients of the bright and dark states, Eqs. (29), for the transition $J_g = 2 \leftrightarrow J_e = 1$ [27].

	$J_g = 2 \leftrightarrow J_e = 1$
$d_-(\varepsilon)$	$\nu_d\,(1-\varepsilon)$
$d_0(\varepsilon)$	$-\nu_d\sqrt{6\,(1-\varepsilon^2)}$
$d_+(\varepsilon)$	$\nu_d\,(1+\varepsilon)$
$[\nu_d(\varepsilon)]^{-2}$	$4\left(2-\varepsilon^2\right)$
$b'_-(\varepsilon)$	$-\frac{1}{2}\nu'_b\,(1+\varepsilon)\left(1-6\varepsilon-\sqrt{1+24\varepsilon^2}\right)$
$b'_0(\varepsilon)$	$\nu'_b\varepsilon\sqrt{6\,(1-\varepsilon^2)}$
$b'_+(\varepsilon)$	$\frac{1}{2}\nu'_b\,(1-\varepsilon)\left(1+6\varepsilon-\sqrt{1+24\varepsilon^2}\right)$
$[\nu'_b(\varepsilon)]^{-2}$	$\sqrt{1+24\varepsilon^2}\left[\left(1+\varepsilon^2\right)\sqrt{1+24\varepsilon^2}+11\varepsilon^2-1\right]$
$b''_-(\varepsilon)$	$-\frac{1}{2}\nu''_b\,(1+\varepsilon)\left(1-6\varepsilon+\sqrt{1+24\varepsilon^2}\right)$
$b''_0(\varepsilon)$	$\nu''_b\varepsilon\sqrt{6\,(1-\varepsilon^2)}$
$b''_+(\varepsilon)$	$\frac{1}{2}\nu''_b\,(1-\varepsilon)\left(1+6\varepsilon+\sqrt{1+24\varepsilon^2}\right)$
$[\nu''_b(\varepsilon)]^{-2}$	$\sqrt{1+24\varepsilon^2}\left[\left(1+\varepsilon^2\right)\sqrt{1+24\varepsilon^2}-11\varepsilon^2+1\right]$

TABLE II: Coefficients of the bright and dark states, Eqs. (29), for the transition $J_g = 2 \leftrightarrow J_e = 2$ [27].

	$J_g = 2 \leftrightarrow J_e = 2$
$d_-(\varepsilon)$	$\nu_d\sqrt{3}\,(1-\varepsilon)$
$d_0(\varepsilon)$	$\nu_d\sqrt{2\,(1-\varepsilon^2)}$
$d_+(\varepsilon)$	$\nu_d\sqrt{3}\,(1+\varepsilon)$
$[\nu_d(\varepsilon)]^{-2}$	$4\left(2+\varepsilon^2\right)$
$b'_-(\varepsilon)$	$\frac{1}{2}\nu'_b\,(1+\varepsilon)\left(3-2\varepsilon-\sqrt{9-8\varepsilon^2}\right)$
$b'_0(\varepsilon)$	$\nu'_b\varepsilon\sqrt{6\,(1-\varepsilon^2)}$
$b'_+(\varepsilon)$	$-\frac{1}{2}\nu'_b\,(1-\varepsilon)\left(3+2\varepsilon-\sqrt{9-8\varepsilon^2}\right)$
$[\nu'_b(\varepsilon)]^{-2}$	$\sqrt{9-8\varepsilon^2}\left[\left(1+\varepsilon^2\right)\sqrt{9-8\varepsilon^2}+\varepsilon^2-3\right]$
$b''_-(\varepsilon)$	$\frac{1}{2}\nu''_b\,(1+\varepsilon)\left(3-2\varepsilon+\sqrt{9-8\varepsilon^2}\right)$
$b''_0(\varepsilon)$	$\nu''_b\varepsilon\sqrt{6\,(1-\varepsilon^2)}$
$b''_+(\varepsilon)$	$-\frac{1}{2}\nu''_b\,(1-\varepsilon)\left(3+2\varepsilon+\sqrt{9-8\varepsilon^2}\right)$
$[\nu''_b(\varepsilon)]^{-2}$	$\sqrt{9-8\varepsilon^2}\left[\left(1+\varepsilon^2\right)\sqrt{9-8\varepsilon^2}-\varepsilon^2+3\right]$

where $(\Omega')^2$ and $(\Omega'')^2$ are the eigenvalues of $\mathbf{V}\mathbf{V}^\dagger$ [27],

$$(\Omega^{\prime,\prime\prime})^2 = \begin{cases} \frac{1}{20}\left(7 \pm \sqrt{1+24\varepsilon^2}\right)\Omega^2, & (J_g = 2 \leftrightarrow J_e = 1), \\ \frac{1}{12}\left(5 \pm \sqrt{9-8\varepsilon^2}\right)\Omega^2, & (J_g = 2 \leftrightarrow J_e = 2). \end{cases} \tag{30}$$

B. Propagator

I assume that the transition probabilities in both MS two-state systems vanish; hence the MS states $|b'\rangle$ and $|b''\rangle$ acquire only phases φ' and φ''. Then the propagator in the lower (qutrit) manifold of states is given by a product of HRs [20],

$$\mathbf{U}_M = \mathbf{M}(b'; \varphi')\mathbf{M}(b''; \varphi''). \tag{31}$$

Note that the HR vectors are the MS bright ground states $|b'\rangle$ or $|b''\rangle$, and that $\det \mathbf{U}_M = e^{i(\varphi'+\varphi'')}$. If the detuning Δ from the upper states is large enough the HR phases

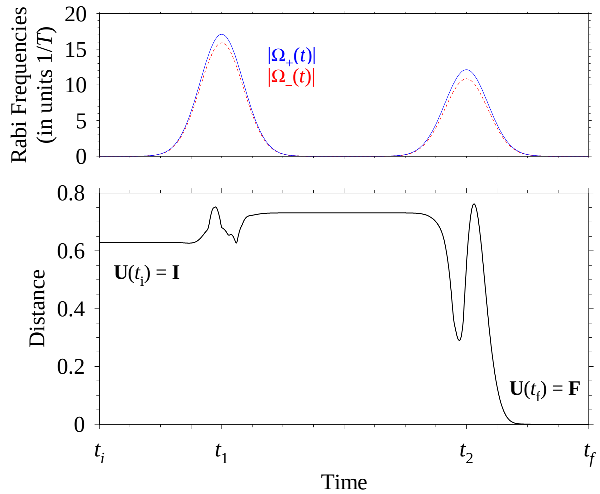

FIG. 4: Numerical simulation of two-step synthesis of DFT of an M-qutrit according to Eq. (31) for the transition $J_g = 2 \leftrightarrow J_e = 1$ for Gaussian pulse shapes, $\Omega(t) = \Omega_0\, e^{-(t-t_1)^2/T^2}$ in the first step and $\Omega(t) = \Omega_0\, e^{-(t-t_2)^2/T^2}$ in the second. The parameters for the two steps are $\varepsilon = 0.073$, $\beta = 1.257\pi$, $\varphi' = 0.452\pi$, $\varphi'' = 1.247\pi$ in the first step, and $\varepsilon = 0.112$, $\beta = 0.675\pi$, $\varphi' = 0.621\pi$, $\varphi'' = 1.344\pi$ in the second step (approximate numerically derived values). The HR phases φ' and φ'' are produced with the following detunings and peak rms Rabi frequencies: $\Delta = -18.88/T$, $\Omega_0 = 46.68/T$ in the first step and $\Delta = -7.51/T$, $\Omega_0 = 32.55/T$ in the second step. The corresponding σ^+ and σ^- Rabi frequencies calculated from here are $\Omega_+ = 34.20/T$, $\Omega_- = 31.78/T$ in the first step and $\Omega_+ = 24.27/T$, $\Omega_- = 21.69/T$ in the second step. *Upper frame:* Magnitudes of the Rabi frequencies $\Omega_+(t)$ and $\Omega_-(t)$. *Lower frame:* Distance $D = \|\mathbf{U}(t) - \mathbf{F}\|$ between the propagator $\mathbf{U}(t)$ and the DFT (14). Initially the propagator is the identity, $\mathbf{U}(t_i) = \mathbf{I}$ and in the end it is the DFT, $\mathbf{U}(t_f) = \mathbf{F}$.

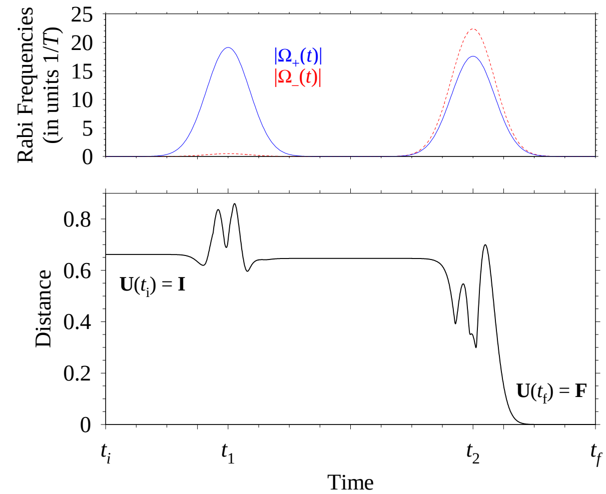

FIG. 5: The same as Fig. 4 but for the $J_g = 2 \leftrightarrow J_e = 2$ transition. The parameters for the two steps are $\varepsilon = 0.084$, $\beta = 1.632\pi$, $\varphi' = 0.987\pi$, $\varphi'' = 1.490\pi$ in the first step, and $\varepsilon = -0.999$, $\beta = 1.558\pi$, $\varphi' = 1.282\pi$, $\varphi'' = 0.551\pi$ in the second step (approximate numerically derived values). The HR phases φ' and φ'' are produced with the following detunings and peak rms Rabi frequencies: $\Delta = 27.66/T$, $\Omega_0 = 51.91/T$ in the first step and $\Delta = -13.70/T$, $\Omega_0 = 44.74/T$ in the second step. The corresponding σ^+ and σ^- Rabi frequencies calculated from here are $\Omega_+ = 38.21/T$, $\Omega_- = 35.13/T$ in the first step and $\Omega_+ = 0.97/T$, $\Omega_- = 44.73/T$ in the second step.

φ' and φ'' are given by expressions similar to Eq. (23) (for $|\Delta| \gg \Omega^{\prime,\prime\prime}$) and Eq. (24) (for $|\Delta| \gg 1/T$), with Ω replaced by Ω' or Ω''.

Unlike the tripod qutrit, here the propagator is expressed by two coupled HRs, with vectors $|b'\rangle$ or $|b''\rangle$ that are orthogonal to each other, $\langle b'|b''\rangle = 0$; moreover, the components of these vectors have two degrees of freedom only, ε and β. Two more degrees of freedom are introduced by the rms Rabi frequency Ω and the detuning Δ, which determine the HR phases φ' and φ''. Hence the propagator (31) contains four independent parameters: $\varepsilon, \beta, \Omega, \Delta$. This operator is a far less convenient tool for decomposition, and hence for synthesis of SU(3) than the single HR found in the tripod qutrit. Because it does not appear possible to decompose an arbitrary SU(3) matrix by this operator analytically, this decomposition has to be done numerically. As discussed above, an arbitrary SU(3) matrix is described by 8 real parameters, Eq. (7). Because $\det \mathbf{M}(\chi;\varphi) = e^{i\varphi}$ it is not sufficient to use just two interaction steps, which would introduce 8 independent parameters and hence the SU(3) requirement $\det \mathbf{U} = 1$ cannot be guaranteed. Therefore three interaction steps are needed, which introduce 12 independent parameters $\varepsilon_k, \beta_k, \Omega_k, \Delta_k$ ($k = 1, 2, 3$). Because the resulting system of algebraic equations for these parameters is underdetermined, any given SU(3) matrix $\mathbf{U}$ can be synthesized by many sets of such parameters and additional restrictions on their values may be imposed if needed.

C. Example: DFT

As an example, I consider the construction of DFT, Eq. (14): this can be achieved in just two interaction steps. The synthesis of DFT for the qutrit in the $J_g = 2 \leftrightarrow J_e = 1$ transition by two elliptically polarized Gaussian laser pulses is demonstrated in Fig. 4. The distance between the propagator $\mathbf{U}(t)$ and the desired DFT vanishes after the second set of fields, which signals the synthesis of DFT. Because $\det \mathbf{U}_M = e^{i(\varphi'+\varphi'')}$, after the two steps the accumulated phase is $\varphi_1' + \varphi_1'' + \varphi_2' + \varphi_2''$; therefore this phase and the DFT phase $-\pi/2$ (coming from $\det \mathbf{F} = e^{-i\pi/2}$) are factored out when calculating the distance D. There exist several other sets of solutions for ε, β, φ' and φ'', apart from these listed in the caption of Fig. 4. For each of these sets, there are infinitely many solutions for the interaction parameters Δ and Ω; these are different for different pulse shapes.

Similar synthesis of DFT but for the qutrit in the $J_g = 2 \leftrightarrow J_e = 2$ transition is demonstrated in Fig. 5. The distance between the propagator $\mathbf{U}(t)$ and the desired DFT vanishes in the end, which indicates the synthesis of DFT. As for the $J_g = 2 \leftrightarrow J_e = 1$ transition, there exist several other sets of solutions for ε, β, φ' and φ'', and for each such set there are infinitely many solutions for the interaction parameters Δ and Ω.

As for the tripod qutrit, between the two steps in Figs. 4 and 5 the population resides in the qutrit states; therefore the possible decay from the ancillary upper states $|a_1\rangle$ and $|a_2\rangle$ with a rate Γ is only relevant during each pulse, i.e. one must have $T \ll 1/\Gamma$. Such a decay can be further suppressed, if needed, by increasing the detuning Δ in each step.

V. CONCLUSIONS

The increasing number of applications of the qutrit quantum computer require a pool of techniques for complete control of the qutrit dynamics, that is the ability to construct any arbitrary preselected SU(3) transformation. In this paper, several techniques for synthesizing arbitrary SU(3) transformations have been proposed for two physical implementations of qutrits, in systems with tripod and M linkage patterns, which occur naturally in laser-driven transition between magnetic sublevels for appropriate angular momenta. For the tripod qutrit two implementations are applicable using Givens SU(2) rotations and Householder reflections. The technique using Householder reflections significantly outperforms the one using Givens rotations for two reasons: (i) mathematically, the decomposition of a general SU(N) matrix demands $N(N-1)/2$ Givens rotations but only $N-1$ Householder reflections; (ii) physically, each Givens rotation is realized by 3 interaction steps, one rotation and two phase gates, while a Householder reflection can be realized in a single interaction step. This double speedup makes the implementations based on Householder reflections much faster than those using Givens rotations. The DFT of a tripod qutrit, in particular, can be constructed in 7 physical steps by Givens rotations and in only 2 steps with Householder reflections. In M-qutrits, any preselected SU(3) transformation can be constructed by at most 3 steps by coupled Householder reflections; the DFT, in particular, is synthesized in just 2 steps.

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