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Irreducible Multi-Qutrit Correlations in Greenberger-Horne-Zeilinger Type States

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Following the idea of the continuity approach in [D. L. Zhou, Phys. Rev. Lett. 101, 180505 (2008)], we obtain the degrees of irreducible multi-party correlations in two families of n-qutrit Greenberger-Horne-Zeilinger type states. For the pure states in one of the families, the irreducible 2-party, n-party and $(n-m)$-party $(0 < m < n-2)$ correlations are nonzero, which is different from the n-qubit case. We also derive the correlation distributions in the n-qutrit maximal slice state, which can be uniquely determined by its $(n-1)$-qutrit reduced density matrices among pure states. It is proved that there is no irreducible n-qutrit correlation in the maximal slice state. This enlightens us to give a discussion about how to characterize the pure states with irreducible n-party correlation in arbitrarily high-dimensional systems by the way of the continuity approach.

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I. INTRODUCTION

Coherent superposition is the essential distinction between a quantum system and a classical one. This distinction becomes more significant in composite systems, which appears in the non-classical correlations in the quantum systems. Many concepts have been developed to describe these correlations, such as the entanglement [1] which depicts the nonseparability of the state of a composite quantum system. Another concept refers to the nonlocality is characterized by violation of a Bell inequality [2], which means that the local measurement outcomes of the state cannot be described by a local hidden variables model.

In the information-based viewpoint, the correlation in a quantum system can be viewed as the relationship of the whole system and its subsystems. Namely, it measures the degree a quantum state can be described by the reduced states of its subsystems. The total correlation [3] in a multipartite quantum system has been defined as the difference between the sum of the von Neumann entropies of all the subsystems and that of the whole system, while the so-called quantum discord [4–6], widely studied in very recent years, was considered to be the quantum part (opposite to the classical one) of the total correlation. In the current paper, we concerns us in another alternative classification, in which the total correlation in a multiparty system is divided into different levels, namely pairwise, triplewise and so forth.

Linden *et al.* [7] proposed the concept of irreducible n-party correlation in an n-partite quantum state. This concept is based on the principle of maximum entropy and describes how much more information in the n-party level than what is contained in the $(n-1)$-partite reduced states. A surprising result was given in the original work of Linden and his collaborators [7, 8] that almost all n-party pure states are determined by their reduced density matrices. In n-qubit case, the only pure states that can't be determined by their reduced density matrices are proved to be the generalized Greenber-Horne-Zeilinger (GHZ) states [9, 10]. This indicates that among n-qubit pure states, only in the case of the generalized GHZ states the irreducible n-party correlation has a nonzero value [11]. For the arbitrarily high-dimensional system, Feng *et al.* [12] introduced the generalized Schmidt decomposition (GSD) states and proved them to be the n-partite pure states undetermined *among pure states* by their reduced density matrices. It still remains an open question whether the GSD states identified in [12] are precisely the pure states undetermined by their reduced density matrices *among arbitrary states* (pure or mixed). In other words, it is under confirmation that, whether the irreducible n-party correlation in a n-partite non-GSD states is nonzero or not.

In Zhou's recent work [11], the concept of irreducible n-party correlation has been generalized to m-party $(2 \leq m \leq n)$ levels, where a classification of the total correlation in an n-partite state is constructed. For a given n-partite quantum state $\rho^{[n]}$, Zhou introduced a sequence of density matrices. The m-th one, $\tilde{\rho}_m^{[n]}$ $(1 \leq m \leq n)$, has the same m-party reduced density matrix as $\rho^{[n]}$, and the maximal value of von Neumann entropy, which is considered to contain the m-party level information of the given state without the higher level information. The degree of irreducible m-party correlation is defined as

$$C^{(m)}(\rho^{[n]}) = S(\tilde{\rho}_{m-1}^{[n]}) - S(\tilde{\rho}_m^{[n]}), \tag{1}$$

where $S(\sigma) = -\mathrm{Tr}(\sigma \ln \sigma)$ is the von Neumann entropy. For the states $\rho^{[n]}$ with maximal rank, $\tilde{\rho}_m^{[n]}$ is proved to have a *standard exponential form*

$$\tilde{\rho}_m^{[n]} = \mathrm{Exp}\big(Q^{[m]}\big), \tag{2}$$

where $Q^{[m]}$ is a sum of m-partite hermitian operators.

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Directly, $\tilde{\rho}^{[n]}_n = \rho^{[n]}$ is itself and $\tilde{\rho}^{[n]}_1 = \bigotimes_{i=1}^n \rho^{(i)}$ is the direct product of all the single partite reduced density matrices. To derive the degrees of the irreducible multiparty correlations in quantum states with nonmaximal ranks, Zhou presented a continuity approach based on the fact that a multipartite state without maximal rank can always be regarded as the limit of a series states with maximal rank, such $\rho^{[n]}(\gamma)|_{\gamma\to+\infty} = \rho^{[n]}$. By constructing the sequence of density matrices $\tilde{\rho}^{[n]}_m(\gamma)$, one can obtain the degree of irreducible m-party correlation of $\rho^{[n]}(\gamma)$

$$C^{(m)}(\rho^{[n]}(\gamma)) = S(\tilde{\rho}^{[n]}_{m-1}(\gamma)) - S(\tilde{\rho}^{[n]}_m(\gamma)). \tag{3}$$

Then $C^{(m)}(\rho^{[n]}) = C^{(m)}(\rho^{[n]}(\gamma))|_{\gamma\to+\infty}$ give the correlations of the state $\rho^{[n]}$.

In this approach, Zhou gave the correlation distributions of the n-qubit stabilizer states and the generalized GHZ states. It is worth noting that, in the n-qubit generalized GHZ states, which are the only pure states with nonzero n-qubit correlation [9, 10], only irreducible 2-party and n-party correlations have nonzero values. However, a systematic method to construct the standard exponential form density matrices in Eq. (2) for a given state with maximal rank has not been found. Consequently it is difficult to analytically obtain the correlation distributions in the states without maximal ranks. To the best of our knowledge, the known correlation distributions, both the analytical [11] and the numerical [13], in multipartite quantum states are restricted in n-qubit systems. The main purpose of this paper is to derive the degrees of irreducible multiparty correlations of some typical quantum states without the maximal rank in n-qutrit system. On one hand, through the analysis of some special examples, the difference between the correlation distributions in qubit systems and the ones in high-dimensional systems can be revealed. On the other hand, more analytical results could contribute to the construction a systematic method to calculate the degrees of irreducible multiparty correlations, at least for a family of states.

Our main results are given in the second section. As the first trial, we concerns us in two families of n-qutrit GHZ-like sates, in which the pure states belongs to the GSD states defined in [12]. The maximal slice (MS) states [14] can be viewed as another generalization of the original GHZ states. We introduce an n-qutrit version MS state, which is non-GSD according to the results in [12], and obtain its multiparty correlation distributions. Based on these results, in the last section we give a discussion about the feasibility to solve the open problem about the GSD states in the continuity approach.

II. MULTI-QUTRIT CORRELATIONS

To derive the degrees of irreducible multiparty correlations in the three families of n-qutrit states studied in this paper, we adopt Zhou's continuity approach with a little improvement. Namely, the results in the original work of Zhou [11] indicated that, the standard exponential form state $\tilde{\rho}^{[n]}_m(\gamma)$ contains the maximal von Neumann entropy among the states with the same m-party reduced density matrices, and this property is holden when its parameter $\gamma \to +\infty$. Therefore, for a given state, $\rho^{[n]}$, with nonmaximal rank, in stead of constructing the series states $\rho^{[n]}(\gamma)$ and corresponding $\tilde{\rho}^{[n]}_m(\gamma)$, we construct a sequence of states $\sigma^{[n]}_m(\gamma_m)$ in the standard exponential form (2) whose limit $\sigma^{[n]}_m|_{\gamma_m\to+\infty}$ has the same m-party reduced density matrices as $\rho^{[n]}$. Here, for different m, the states $\sigma^{[n]}_m(\gamma_m)$ are independent, and the same are true for their parameters γ_m. Then the degree of the irreducible m-party correlation in the state $\rho^{[n]}$ is given by

$$C^{(m)}(\rho^{[n]}) = S(\sigma^{[n]}_{m-1}|_{\gamma_{m-1}\to+\infty}) - S(\sigma^{[n]}_m|_{\gamma_m\to+\infty}). \tag{4}$$

A. First GHZ-type states

We introduce the n-qutrit states in the subspace $\{|0^{[n]}\rangle, |1^{[n]}\rangle, |2^{[n]}\rangle\}$ as the first family of GHZ-type states, where $|i^{[n]}\rangle = |ii...i\rangle$ with $i = 0, 1, 2$, denotes the direct product of the basis $|i\rangle$ for n qutrits. They can be expressed as

$$\mathcal{G} = \sum_{i=0,j=0}^{2,2} c_{ij}|i^{[n]}\rangle\langle j^{[n]}|, \tag{5}$$

with $c_{ij} = c^*_{ij}$ and the positive real numbers c_{ii} satisfying $\sum c_{ii} = 1$. One can write the diagonal elements in spherical coordinate as $(c_{00}, c_{11}, c_{22}) = (\sin^2\theta\cos^2\phi, \cos^2\theta, \sin^2\theta\sin^2\phi)$ with $\theta, \phi \in [0, \pi/4]$.

Theorem 1. The degrees of irreducible multiparty correlations in the n-qutrit GHZ-type state $\mathcal{G}$ in Eq. (5) are given by

$$\begin{aligned} C^{(2)} &= (n-1)\mathcal{H}_3(\theta,\phi), \\ C^{(n)} &= \mathcal{H}_3(\theta,\phi) - S(\mathcal{G}), \end{aligned} \tag{6}$$

and $C^{(m)} = 0$ for $m = 3, 4, ...n-1$. Here, $\mathcal{H}_3(\theta,\phi) = \mathcal{H}_2(\theta) + \sin^2\theta\mathcal{H}_2(\phi)$ denotes the trinary entropy of the probabilities $\{\cos^2\theta, \sin^2\theta\cos^2\phi, \sin^2\theta\sin^2\phi\}$, with $\mathcal{H}_2(\alpha) = -\cos^2\alpha\ln\cos^2\alpha - \sin^2\alpha\ln\sin^2\alpha$ being the binary entropy.

Proof. Let $Z_j = |0\rangle\langle 0| - |2\rangle\langle 2|$ be the spin-1 operator in z-axis of the j-th qutrit, the 2-partite operators defined as

$$Q_{ij} = \frac{2}{3}\big[\frac{1}{2} + \cos\frac{2\pi}{3}(Z_i - Z_j)\big] \tag{7}$$

satisfies $Q^2_{ij} = Q_{ij}$ and $\mathrm{Tr}_{i,j}Q_{ij} = 3$. We construct an

n-qutrit state

$$\sigma_g(\gamma) = \mathrm{Exp}\big(\eta + \gamma \sum_{j=2}^{n} Q_{1j} - \gamma_1 Z_1^2 + \gamma_2 Z_1\big), \qquad (8)$$

where $\tanh\gamma_2 = \cos 2\phi$ and $\exp\gamma_1 = 2\cosh\gamma_2 \cot^2\theta$, and the value of η is determined by the normalization condition $\mathrm{Tr}\sigma_g(\gamma) = 1$. Straightforward calculation gives

$$\sigma_g(\gamma) \;=\; \prod_{j=2}^{n}\Big(\frac{1}{e^\gamma + 2} + \frac{e^\gamma - 1}{e^\gamma + 2} Q_{1j}\Big) f(Z_1), \qquad (9)$$

with $f(Z_1) = \cos^2\theta(1 - Z_1^2) + \frac{1}{2}\sin^2\theta(Z_1^2 + \cos 2\phi Z_1)$. When γ approaches infinity the limit of $\sigma_g(\gamma)$ is nothing but the diagonal terms of $\mathcal{G}$, i.e., $\sigma_g|_{\gamma\to+\infty} = \mathcal{D}_g = \sum c_{ii}|i^{[n]}\rangle\langle i^{[n]}|$.

The state $\mathcal{D}_g$ has the same $(n-1)$-partite reduced matrices as the states $\mathcal{G}$, and there exists only irreducible 2-party correlation in $\sigma_g(\gamma)$. Therefore, one can take $\tilde{\mathcal{G}}_m = \mathcal{D}_g$ for $m = 2, 3, ...n-1$ and obtain the results in Eq. (6). □

The pure states in this family are always equivalent to the generalized GHZ states for n-qutrit system

$$|\mathcal{G}^p\rangle = \cos\theta|0^{[n]}\rangle + \sin\theta\cos\phi|1^{[n]}\rangle + \sin\theta\sin\phi|2^{[n]}\rangle \qquad (10)$$

under local unitary transformations, which belongs to the GSD states in [12] apparently. There are $n\mathcal{H}_3(\theta,\phi)$ correlations in $|\mathcal{G}^p\rangle$, $\mathcal{H}_3(\theta,\phi)$ of which is irreducible n-party correlation and the others belongs to the 2-party level. This distribution is the same as the generalized n-qubit GHZ states in [11]. When $\phi = 0$, $\mathcal{H}_3(\theta, 0) = \mathcal{H}_2(\theta)$, this result returns to the n-qubit case.

B. Second GHZ-type States

A generalization of the family of n-qutrit states $\mathcal{G}$ is the one in the subspace $\{|0^{[n]}\rangle, |1^{[n]}\rangle, |0^{[m]}2^{[n-m]}\rangle\}$ with m being a positive integer less than n, and $|0^{[m]}2^{[n-m]}\rangle = |0^{[m]}\rangle \otimes |2^{[n-m]}\rangle$ denoting the direct product of basis $|0\rangle$ for the first m qutrits and $|2\rangle$ for the others. Denoting the basis $(|\bar{0}\rangle, |\bar{1}\rangle, |\bar{2}\rangle) = (|0^{[n]}\rangle, |1^{[n]}\rangle, |0^{[m]}2^{[n-m]}\rangle)$, the states in this family can be written as

$$\mathcal{G}_2 = \sum_{i=0,j=0}^{2,2} c_{ij}|\bar{i}\rangle\langle\bar{j}|, \qquad (11)$$

with the same constraint on c_{ij} as Eq. (5), and the diagonal elements c_{ii} also can be expressed in the spherical coordinate θ and ϕ.

Theorem 2. The degrees of irreducible multiparty correlations in the second family of GHZ-type state $\mathcal{G}_2$ in Eq. (11) are given by

$$\begin{aligned} C^{(2)} &= m\mathcal{H}_2(\theta) + (n-m-1)\mathcal{H}_3(\theta,\phi), \\ C^{(n-m)} &= \mathcal{H}_3(\theta,\phi) - \mathcal{H}_3(\theta,\varphi), \\ C^{(n)} &= \mathcal{H}_3(\theta,\varphi) - S(\mathcal{G}_2), \end{aligned} \qquad (12)$$

and $C^{(k)} = 0$ for the other integer numbers $2 \le k \le n$, where the value of φ is given by $\cos^2 2\varphi = \cos^2 2\phi + 4|c_{02}|^2/\sin^4\theta$.

Proof. The quantum states $\mathcal{D}_2 = \sum c_{ii}|\bar{i}\rangle\langle\bar{i}|$ has the same k-partite reduced matrices as $\mathcal{G}_2$ for $k < n-m$. It can be proved to be the limit of a state in the form (2).

Let us construct a 2-patite operator $P_{ij} = (2Z_i^2 - 1)\lambda_j^{(3)}$ by using the spin operator Z_i and the third Gell-Mann matrix of the j-th qutrit, $\lambda_j^{(3)} = |0\rangle\langle 0| - |1\rangle\langle 1|$. It satisfies $P_{ij}^2 = {\lambda_j^{(3)}}^2$ and $P_{ij}^3 = P_{ij}$. Then, the basis $|\bar{0}\rangle$, $|\bar{1}\rangle$ and $|\bar{2}\rangle$ are the three eigenvectors of $\Omega = \sum_{j=1}^{m} P_{m+1,j} + \sum_{l=m+2}^{n} Q_{m+1,l}$, corresponding to the maximal eigenvalue $\omega_{max} = n-1$. Choosing the values of γ_1 and γ_2 the same as the ones in Eq. (8), and $\exp\eta = (2\cosh\gamma + 1)^{-m}(\exp\gamma + 2)^{-n+m+1}(\exp\gamma_1 + 2\cosh\gamma_2)^{-1}$, the quantum state with only irreducible 2-party correlation

$$\sigma_2(\gamma) = \mathrm{Exp}\big(\eta + \gamma\Omega - \gamma_1 Z_{m+1}^2 + \gamma_2 Z_{m+1}\big) \qquad (13)$$

has the limit $\sigma_2|_{\gamma\to+\infty} = \mathcal{D}_2$. Accordingly, we can choose $\tilde{\mathcal{G}}_{2,k} = \mathcal{D}_2$ for $k = 2, 3, ...n-m-1$.

To obtain $\tilde{\mathcal{G}}_{2,k}$ for the other values of k, we introduce three $(n-m)$-partite operators $\Sigma_1 = \prod_{j=m+1}^{n}\lambda_j^{(4)}$, $\Sigma_2 = \lambda_{m+1}^{(5)}\prod_{j=m+2}^{n}\lambda_j^{(4)}$ and $\Sigma_3 = Z_{m+1}\prod_{j=m+2}^{n} Z_j^2$, with $\lambda_j^{(4)} = |0\rangle\langle 2| + |2\rangle\langle 0|$ and $\lambda_j^{(5)} = -i|0\rangle\langle 2| + i|2\rangle\langle 0|$ being the fourth and fifth Gell-Mann matrices of the j-th qutrit. They satisfy the relations $\Sigma_\alpha\Sigma_\beta = i\epsilon_{\alpha\beta\gamma}\Sigma_\gamma$, $\Sigma_\alpha^2 = \prod_{j=m+1}^{n} Z_j^2$ and $[\Sigma_\alpha, \Omega] = 0$, where $\alpha, \beta, \gamma = 1, 2, 3$, and can be viewed as the Pauli matrices in the subspace $\{|\bar{0}\rangle, |\bar{2}\rangle\}$. Let $\tanh\gamma_2 = \cos 2\varphi$ and $\exp\gamma_1 = 2\cosh\gamma_2\cot^2\theta$, we can define an n-qutrit density matrix

$$\tau_2(\gamma) = \mathrm{Exp}\big(\eta + \gamma\Omega - \gamma_1\Sigma_r^2 + \gamma_2\Sigma_r\big), \qquad (14)$$

where $\Sigma_r = \cos\xi\Sigma_3 + \sin\xi\cos\zeta\Sigma_1 + \sin\xi\sin\zeta\Sigma_2$ with the parameters $\zeta = \frac{1}{2}\arg(c_{20}/c_{02})$ and $\xi = \arccos(\cos 2\phi/\cos 2\varphi)$, and η is determined by the normalization condition $\mathrm{Tr}\tau_2(\gamma) = 1$. It has no irreducible $(n-m+1)$-party or higher level correlation, and approaches the quantum state $\mathcal{B}_2 = \mathcal{D}_2 + c_{02}|\bar{0}\rangle\langle\bar{2}| + c_{20}|\bar{2}\rangle\langle\bar{0}|$ when the parameter $\gamma \to +\infty$. The n-qutrit state $\mathcal{G}_2$ has the same $(n-1)$-partite reduced density matrices as $\mathcal{B}_2$. Therefore, we can take $\tilde{\mathcal{G}}_{2,k} = \mathcal{B}_2$ for $k = n-m, n-m+1, ...n-1$. One can yield the results in Eq. (12) via a direct calculation. □

For the pure state in this family

$$|\mathcal{G}_2^p\rangle = \sin\theta\cos\phi|\bar{0}\rangle + \cos\theta|\bar{1}\rangle + \sin\theta\sin\phi|\bar{2}\rangle, \quad (15)$$

the variable $\varphi = 0$, the total correlation with the value $C^T = m\mathcal{H}_2(\theta) + (n-m)\mathcal{H}_3(\theta,\phi)$ is divided into three nonzero irreducible multi-qutrit correlations as $C^{(2)} = m\mathcal{H}_2(\theta) + (n-m-1)\mathcal{H}_3(\theta,\phi)$, $C^{(n-m)} = \sin^2\theta\mathcal{H}_2(\phi)$ and $C^{(n)} = \mathcal{H}_2(\theta)$. This result is different with the n-qubit case, which indicates there could exist nonzero $C^{(k)}(2 < k < n)$ in an n-qutrit pure states with the irreducible n-party correlation.

When $\phi = 0$, the state (15) becomes the n-qubit generalized GHZ state, and at the same time the above correlation distribution returns the corresponding one. When $\theta = \pi/2$, the state (15) equivalents to the direct product of m pure states $|0\rangle$ and a $(n-m)$-qubit generalized GHZ state

$$|\mathcal{G}_2^p\rangle_{qubit} = \cos\phi|0^{[n]}\rangle + \sin\phi|0^{[m]}1^{[n-m]}\rangle, \quad (16)$$

in which $C^{(2)} = (n-m-1)\mathcal{H}_2(\phi)$ and $C^{(n-m)} = \mathcal{H}_2(\phi)$, but $C^{(n)} = 0$.

C. MS states

The set of MS states [14] for qubit case is an important example to investigate the fundamental concepts in multipartite system [15]. We generalize the definition of MS states to n-qutrit system

$$|S\rangle = \frac{1}{\sqrt{3}}(|\underline{0}\rangle + |\underline{1}\rangle + |\underline{2}\rangle), \quad (17)$$

where $|\underline{i}\rangle = [\cos\alpha + \sin\alpha(X_1 + X_1^\dagger)/\sqrt{2}]|i^{[n]}\rangle$ with the operator of j-th qutrit $X_j = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 0|$ and $\alpha \in (0, \arctan\sqrt{2}]$. The operators X_j satisfy $X_j^2 = X_j^\dagger$, $X_j^{\dagger 2} = X_j$ and $X_j X_j^\dagger = X_j^\dagger X_j = 1$.

Theorem 3. In the n-qutrit MS state (17), only irreducible 2-party and $(n-1)$-party correlations exist as

$$\begin{aligned} C^{(2)} &= \mathcal{H}_3(\chi, \pi/4) + (n-2)\ln 3, \\ C^{(n-1)} &= \ln 3, \end{aligned} \quad (18)$$

where χ satisfies $\cos^2\chi = \frac{1}{6}(3 - \cos 2\alpha + 2\sqrt{2}\sin 2\alpha)$, which is determined by the eigenvalue of the reduced density matrix $\rho_s^{(1)}$ for the first qutrit.

Proof. The operator $Q = \sum_{j=3}^n Q_{2j}$ has 9 eigenvectors as $|i\rangle \otimes |j^{[n-1]}\rangle$ with $i, j = 0, 1, 2$, corresponding to its maximal eigenvalues $q_{max} = n-2$. We construct an operator $X = \cos\beta Q_{12} + \frac{1}{2}\sin\beta(X_1 + X_1^\dagger + \prod_{j=2}^n X_j + \prod_{j=2}^n X_j^\dagger)$ commuting with Q. Thus, the eigenvector of $Q + X$ with the maximal eigenvalue is the one of X in the subspace $\{|i\rangle \otimes |j^{[n-1]}\rangle\}$. Choosing the value of β satisfying $\cot\beta = \sqrt{2}(\cot\alpha - \tan\alpha) + 1$, one can check that the MS state (17) is the unique eigenvector corresponding to the maximal eigenvalues (UEME) of $Q + X$. Therefore, the MS state can be viewed as the limit of a state without irreducible n-party correlation

$$\rho_s = |S\rangle\langle S| = \lim_{\gamma\to+\infty} \mathrm{Exp}\big(\eta + \gamma Q + \gamma X\big), \quad (19)$$

where $\eta = -\ln\mathrm{Tr}[\mathrm{Exp}(\gamma Q + \gamma X)]$, which leads to $\tilde{\rho}_{s,n-1} = \rho_s$.

The two-partite operator $R_{12} = [\cos\alpha + \sin\alpha(X_1 + X_1^\dagger)/\sqrt{2}]Q_{12}[\cos\alpha + \sin\alpha(X_1 + X_1^\dagger)/\sqrt{2}]$ shares the similar properties with Q_{ij}, $R_{12}^2 = R_{12}$ and $\mathrm{Tr}_{1,2}R_{12} = 3$. The maximal eigenvalue of $Q + R_{12}$ is triple degenerate with the eigenvectors $|\underline{i}\rangle$. Consequently, the quantum state with only irreducible 2-party correlation

$$\sigma_s(\gamma) = \mathrm{Exp}(\eta + \gamma Q + \gamma R_{12}) \quad (20)$$

has the limit $\sigma_s|_{\gamma\to\to+\infty} = \mathcal{D}_s = \sum_{i=0}^2 \frac{1}{3}|\underline{i}\rangle\langle\underline{i}|$, where the value of η is determined by the normalization condition $\mathrm{Tr}\sigma_s(\gamma) = 1$. Thus, we can take $\tilde{\rho}_{s,m} = \mathcal{D}_s$ for $m = 2, 3, ...n-2$ and obtain the results in Eq. (18). □

According to the results by Feng *et al.* [12], it is easy to identify the n-qutrit MS state $|S\rangle$ can be determined by its $(n-1)$-partite reduced density matrices among n-qutrit pure states. Our results show there is no irreducible n-qutrit correlation in the state $|S\rangle$, which indicates it can be determined by its $(n-1)$-partite reduced density matrices among arbitrary n-qutrit states (pure or mixed).

The same scheme can be used to deal with the n-qubit MS state

$$|S\rangle_{qubit} = (\cos\alpha + \sin\alpha\sigma_1^x)\frac{1}{\sqrt{2}}(|0^{[n]}\rangle + |1^{[n]}\rangle), \quad (21)$$

where $\sigma_j^{x,y,z}$ denote the Pauli operators for the j-th qubit and $\alpha \in (0, \pi/4]$. Replacing X_j and Q_{ij} by $(\sigma_j^x + i\sigma_j^y)/2$ and $(1 + \sigma_i^z\sigma_j^z)/2$, one can construct the operators corresponding to Q, X and R_{ij} in the states (19) and (20), and obtain the nonzero correlations

$$\begin{aligned} C^{(2)} &= \mathcal{H}_2(\xi) + (n-2)\ln 2, \\ C^{(n-1)} &= \ln 2, \end{aligned} \quad (22)$$

with $\xi = \pi/4 - \alpha$. The results indicate our generalization of the MS state to the n-qutrit system in Eq. (17) has the similar correlation distribution with the qubit case.

III. CONCLUSION AND DISCUSSION

Following the idea of Zhou's continuity approach, for an n-qutrit quantum state $\rho^{[n]}$ without maximal rank, we construct $\tilde{\rho}_m^{[n]}$ as the limit of a series n-qutrit states in the standard exponential form (2). In this way, we obtain the degrees of irreducible multiparty correlations in three families n-qutrit states, which can be viewed as three generalizations of the original GHZ state. The distribution of the total correlations in the generalized GHZ state (10) is the same as the n-qubit case. Whereas, in the n-qutrit pure state (15), there exist three nonzero irreducible correlations which are $C^{(2)}, C^{(n)}$ and $C^{(n-m)}(0 < m < n-2)$. By contrast, only $C^{(2)}$ and $C^{(n)}$ are nonzero in the n-qubit generalize GHZ state which is the only pure n-qubit state with irreducible n-party correlation. This indicates the classification of the total correlations for a multipartite pure state in high-dimensional system would be more complicated. Also, an interesting question is raised, which kind of the irreducible multiparty correlations can simultaneously be nonzero in a pure states.

To prove the absence of irreducible n-qutrit correlation in the MS state (17), we construct an operator $Q + X$ which is a sum of $(n-1)$-partite operators and with $|S\rangle$ being its UEMS. The conclusion is proved by Eq. (19) in the continuity approach. This indicates the open problem about the GSD states is equivalent to the following one in the sense of limit: Whether the GSD state is precisely the pure states which can't be viewed as the UEME of an operator $Q^{[n-1]}$, a sum of $(n-1)$-partite operators. It is straightforward to prove the sufficiency part that, there exists no $Q^{[n-1]}$ whose UEME being a GSD state. In the results of [12], for an n-partite GSD state $|\psi\rangle$, there exist two projectors $P_1 = \bigotimes_{j=1}^{n} P_j^{(1)}$ and $P_2 = \bigotimes_{j=1}^{n} P_j^{(2)}$, such that $|\psi\rangle = P_1|\psi\rangle + P_2|\psi\rangle$. The projectors $P_j^{(1)}$ and $P_j^{(2)}$ for the j-th partite satisfy $P_j^{(1)}|\psi\rangle \neq 0$, $P_j^{(2)}|\psi\rangle \neq 0$ and $P_j^{(1)} \perp P_j^{(2)}$. For any $Q^{[n-1]}$, $\langle\psi|Q^{[n-1]}|\psi\rangle = \langle\psi'|Q^{[n-1]}|\psi'\rangle$, where $|\psi'\rangle = P_1|\psi\rangle - P_2|\psi\rangle$. Then, the necessity part of this question is left as, how to construct a sum of $(n-1)$-partite operators, $Q^{[n-1]}$, whose UEMS is the given non-GSD state. We hope to find an universal generalization of the construction in Theorem 3 to an arbitrary non-GSD state in our subsequent investigation.

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[1] A. Einstein, B. Podosky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, Physics **1**, 195 (1964).
[3] S. Wantanabe, IBM Journal of research and development **4**, 66 (1960).
[4] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[5] L. Henderson and V. Vedral, Journal of Physics A: Mathematical and General **34**, 6899 (2001).
[6] A. Datta, A. Shaji, and C. Caves, Physical review letters **100**, 50502 (2008).
[7] N. Linden, S. Popescu, and W. Wootters, Phys. Rev. Lett. **89**, 207901 (2002).
[8] N. Linden and W. Wootters, Phys. Rev. Lett. **89**, 277906 (2002).
[9] S. N. Walck and D. W. Lyons, Phys. Rev. Lett. **100**, 050501 (2008).
[10] S. N. Walck and D. W. Lyons, Phys. Rev. A **79**, 032326 (2009).
[11] D. L. Zhou, Phys. Rev. Lett. **101**, 180505 (2008).
[12] Y. Feng, R. Duan, and M. Ying, Quantum Information & Computation **9**, 0997 (2009).
[13] D. L. Zhou, preprint arXiv:0909.3700 (2009).
[14] H. A. Carteret and A. Sudbery, J. Phys. A **33**, 4981 (2000).
[15] S. Ghose, N. Sinclair, S. Debnath, P. Rungta, and R. Stock, Phys. Rev. Lett. **102**, 250404 (2009).