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Phys. Rev. A **84**, 052312 — Published 11 November 2011

DOI: 10.1103/PhysRevA.84.052312

Efficient multipartite entanglement purification with entanglement link from subspace

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We present an efficient multipartite entanglement purification protocol (MEPP) for N-photon systems in a Greenberger-Horne-Zeilinger state with parity-check detectors. It contains two parts. One is the conventional MEPP with which the parties can obtain a high-fidelity N-photon ensemble directly, similar to the MEPP with controlled-not gates. The other is our recycling MEPP in which the entanglement link is used to produce some N-photon entangled systems from entangled N'-photon subsystems ($2 \leq N' < N$) coming from the instances which are just discarded in all existing conventional MEPPs. The entangled N'-photon subsystems are obtained efficiently by measuring the photons with potential bit-flip errors. With these two parts, the present MEPP has a higher efficiency than all other conventional MEPPs.

PACS numbers: 03.67.Pp, 03.65.Ud, 03.67.Hk

I. INTRODUCTION

Entanglement is an important quantum resource for quantum communication and computation [1]. The powerful speedup of quantum computation resorts to multipartite entanglement [1]. Some branches in quantum communication should require entanglement to set up the quantum channel. For example, quantum teleportation [2] requires a maximally entangled photon pair to set up the quantum channel for the teleportation of an unknown single-qubit state. Some quantum key distribution (QKD) protocols work with maximally entangled photon systems [3–7]. Moreover, people should resort to quantum repeaters for a long-distance QKD or a QKD network as the quantum signals (i.e., single photons [8, 9], weak pulses [10–13], or entangled photons [3–7]) can only be transmitted over a fiber or a free space not more than several hundreds kilometers with current technology (for example, an experimental demonstration of free-space decoy QKD over 144 km was obtained in Ref. [14]). In a quantum repeater, entanglement is required to connect two neighboring nodes. The high capacity of quantum dense coding [15–17] should also resort to maximally entangled photon systems. Quantum secret sharing (QSS) [18–20] and quantum state sharing (QSTS) [21–24] require the parties in quantum communication possess maximally entangled multi-photon systems. However, entangled photon systems inevitably suffer from channel noise when the entangled photons propagate away from each other. For instance, the thermal fluctuation, vibration, the imperfection of an optical fiber, and the birefringence effects will inevitably affect the polarization of photons. In general, the interaction will make an entangled system be in a less entangled state or even in a mixed state. The decoherence of entanglement in quantum systems will affect quantum communication largely. It will decrease the security of QKD, QSS, and QSTS protocols if a maximally entangled state transmitted over a noisy channel becomes a mixed entangled state as a vicious eavesdropper can exploit the decoherence to hide her illegal action. The non-maximally entangled quantum channel will decrease the fidelity of quantum teleportation and quantum dense coding.

Entanglement purification [25–33] is a useful tool for the parties in quantum communication to obtain some maximally entangled photon pairs from a set of less-entangled photon pairs with the help of local operations and classical communications. In 1996, Bennett *et al.* [25] proposed the first entanglement purification protocol (EPP) to purify a Werner state [34], resorting to quantum controlled-not (CNOT) gates. Subsequently, Deutsh *et al.* [26] optimized this EPP with two additional specific unitary operations. In 2001, Pan *et al.* [27] introduced an EPP with linear optical elements and an ideal entanglement source by sacrificing a half of the efficiency. In 2002, Simon and Pan [28] proposed an EPP with a currently available parametric down-conversion (PDC) source. In 2008, an efficient EPP [29] based on a PDC source was proposed with cross-Kerr nonlinearity. It has the same efficiency as the EPP by Bennett *et al.* with perfect CNOT gates. In 2010, the concept of deterministic entanglement purification was proposed [30] for two-photon entangled systems, which is far different from the conventional entanglement purification protocols (CEPPs) [25–29] as the former works in a deterministic way, while the latter works in a probabilistic way. In 2010, we introduced a two-step deterministic entanglement purification protocol (DEPP) [30] for entangled photon pairs, resorting to hyerentanglement. Subsequently, a one-step DEPP [31, 32] was proposed, only resorting to the spatial entanglement of a practical PDC source and linear optical elements. In essence, both the CEPPs [25–29] and the DEPPs [30–33] are based on entanglement transfer. The CEPPs are based on the entanglement transfer between different entangled photon systems, while the DEPPs are based on the transfer between different degrees of freedom of an entangled photon system itself.

By far, there have been several interesting EPPs [25–

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32] focusing on entangled two-photon systems, while there are only two EPPs for multipartite photon systems [35, 36] and an EPP for multipartite electronic systems [37] with charge detection. In 1998, Murao *et al.* [35] proposed a multipartite entanglement purification protocol (MEPP) to purify multipartite quantum systems in a Greenberger-Horne-Zeilinger (GHZ) with CNOT gates, following some ideas in the EPP by Bennett *et al.* [25]. In 2009, a MEPP based on cross-Kerr nonlinearities was proposed [36]. In this protocol, the cross-Kerr nonlinearity is used to construct a nondestructive quantum nondemolition detector (QND) [38] which has the functions of both a parity-check gate and a photon-number detector. With QNDs, the parties can obtain some high-fidelity GHZ states from an ensemble in a mixed entangled state by performing this MEPP repeatedly. In both these two MEPPs, the original fidelity before the MEPPs is required to be larger than $1/2$ and a lot of entangled quantum resource is discarded. So does the MEPP for electronic systems [37].

In this article, we will present an efficient MEPP for N-photon systems in a GHZ state. It contains two parts. One is our conventional MEPP with which the parties can obtain a high-fidelity N-photon ensemble directly, similar to the conventional MEPPs with controlled-not gates [35], but it doubles the efficiency of the MEPP with cross-Kerr nonlinearity in Ref. [36] and the MEPP for electronic systems [37]. The other is our recycling MEPP in which the entanglement link is used to produce some N-photon entangled systems from subspaces. That is, the parties in quantum communication first distil some entangled N'-photon subsystems ($2 \leq N' < N$) from the cross-combinations which are just the discarded instances in the conventional MEPPs [35–37], and then they produce some N-photon entangled systems with entanglement link. It is interesting to show that the entangled N'-photon subsystems are obtained efficiently by measuring the potential photons with bit-flip errors in the two cross-combinations of two N-photon states. We discuss the entanglement link in detail for the three-photon systems. Moreover, the present MEPP works by replacing parity-check detectors with CNOT gates. With these two parts, the present MEPP has a higher efficiency than all other conventional MEPPs [35–37].

This article is organized as follows: we discuss our conventional three-photon entanglement purification for bit-flip errors in Sec. II B. In Sec. II C 1, we give the detail for the two-photon entanglement purification from three-photon systems with potential bit-flip errors. That is, how can the parties obtain a high-fidelity entangled two-photon subsystem from the cross-combinations of two three-photon systems which are just discarded in other conventional MEPPs [35–37]. In Sec. II C 2, we give a way for three-photon entanglement production from two-photon subsystems with entanglement link. The differences of the efficiency and the fidelity between the present MEPP and the conventional MEPPs are shown in Sec. II D. In Sec. III, a conventional three-photon entanglement purification for phase-flip errors is given. A discussion and a summary are given in Sec. IV. In Appendix A, we exploit four-photon systems as an example to describe the principle of the present MEPP for N-photon systems.

II. HIGH-EFFICIENCY THREE-PHOTON ENTANGLEMENT PURIFICATION FOR BIT-FLIP ERRORS WITH ENTANGLEMENT LINK

A. Parity-check detector based on cross-Kerr nonlinearity

Cross-Kerr nonlinearity is a powerful tool for us to construct QNDs [38, 39]. The cross-kerr nonlinearity has been used to prepare CNOT gates [38] and complete a local Bell-state analysis [39, 40]. Also it can be used to fulfill the quantum entanglement purification protocols [29, 30, 36, 41] and the entanglement concentration protocol [42]. The Hamiltonian of the cross-Kerr nonlinearity is $H_{ck} = \hbar\chi a_s^+ a_s a_p^+ a_p$ [38, 39]. Here a_s^+ and a_p^+ are the creation operations, and a_s and a_p are the destruction operations. χ is the coupling strength of the nonlinearity, which is decided by the property of nonlinear material. Suppose a signal state $|\Psi\rangle_s = c_0|0\rangle_s + c_1|1\rangle_s$ ($|0\rangle_s$ and $|1\rangle_s$ denote that there are no photon and one photon respectively in this state) and a coherent probe beam in the state $|\alpha\rangle$ couple with a cross-Kerr nonlinearity medium, the whole system evolves as:

$$\begin{aligned} U_{ck}|\Psi\rangle_s|\alpha\rangle_p &= e^{iH_{ck}t/\hbar}[c_0|0\rangle_s + c_1|1\rangle_s]|\alpha\rangle_p \\ &= c_0|0\rangle_s|\alpha\rangle_p + c_1|1\rangle_s|\alpha e^{i\theta}\rangle_p, \end{aligned} \quad (1)$$

where $\theta = \chi t$ and t is the interaction time. The coherent beam picks up a phase shift θ directly proportional to the number of the photons in the Fock state $|\Psi\rangle_s$, which can be read out with a general homodyne-heterodyne measurement. So one can exactly check the number of photons in the Fock state but not destroy them. We will exploit this feature to construct our QND in our EPP, instead of the CNOT gates in the CEPPs [25, 26, 35].

The principle of our QND is shown in Fig.1, similar to those in Refs. [38, 40]. It is composed of two cross-Kerr nonlinearities (ck_1 and ck_2), four polarization beam splitters (PBSs), a coherent beam $|\alpha\rangle_p$, and an X quadrature measurement. b_1 and b_2 represent the up spatial mode and the down spatial mode, respectively. Each polarization beam splitter (PBS) is used to pass through the horizontal polarization photons $|H\rangle$ and reflect the vertical polarization photons $|V\rangle$. The cross-Kerr nonlinearity will make the coherent beam $|\alpha\rangle_p$ pick up a phase shift θ or $-\theta$ if there is a photon in the mode. The probe beam $|\alpha\rangle_p$ will pick up a phase shift θ or $-\theta$ if the state of the two photons injected into the two spatial modes b_1 and b_2 is $|HH\rangle_{b_1b_2}$ or $|VV\rangle_{b_1b_2}$, respectively; otherwise it picks up a phase shift 0 when the state of the two photons injected into the two spatial modes b_1 and b_2 is $|VH\rangle_{b_1b_2}$

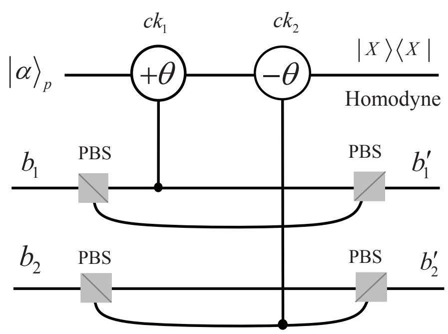

FIG. 1: The principle of the nondestructive quantum nondemolition detector (QND) in the present entanglement purification protocol. PBS represents a polarizing beam splitter which transmits the $|H\rangle$ polarization photons and reflect the $|V\rangle$ polarization photons. $\pm\theta$ represent two cross-Kerr nonlinear media which introduce the phase shifts $\pm\theta$ when there is a photon passing through the media.

or $|HV\rangle_{b_1b_2}$. That is, when the parity of the two photons is even, the coherent beam $|\alpha\rangle_p$ will pick up a phase shift θ or $-\theta$; otherwise it will pick up a phase shift 0. Each party of quantum communication can determine the parity of his two photons with an X quadrature measurement in which the the states $|\alpha e^{\pm i\theta}\rangle_p$ cannot be distinguished [38, 43]. With this QND, each party can distinguish superpositions and mixtures of $|HH\rangle$ and $|VV\rangle$ from $|HV\rangle$ and $|VH\rangle$.

B. Conventional three-photon entanglement purification for bit-flip errors

For three-photon systems, there are eight GHZ states for polarization degree of freedom. They can be written as follows

$$
\begin{aligned}
|\Phi_0^{\pm}\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|HHH\rangle \pm |VVV\rangle)_{ABC}, \\
|\Phi_1^{\pm}\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|VHH\rangle \pm |HVV\rangle)_{ABC}, \\
|\Phi_2^{\pm}\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|HVH\rangle \pm |VHV\rangle)_{ABC}, \\
|\Phi_3^{\pm}\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|HHV\rangle \pm |VVH\rangle)_{ABC}. \qquad (2)
\end{aligned}
$$

Here the subscripts A, B, and C represent the photons (qubits) sent to the parties Alice, Bob, and Charlie, respectively. Suppose that the original GHZ state transmitted among the three parties is $|\Phi_0^+\rangle_{ABC}$. If a bit-flip error takes place on the first qubit in this GHZ state after it is transmitted over a noisy channel, the three-photon system is in the state $|\Phi_1^+\rangle_{ABC}$. $|\Phi_2^+\rangle_{ABC}$ and $|\Phi_3^+\rangle_{ABC}$ represent the instances that a bit-flip error takes place on the second qubit and the third qubit, respectively. If $|\Phi_0^+\rangle$ becomes $|\Phi_0^-\rangle$, there is a phase-flip error. Sometimes, both a bit-flip error and a phase-flip error will take place on a three-photon quantum system transmitted over a noisy channel. The task for purifying three-photon entangled systems requires to correct both bit-flip errors and phase-flip errors on the quantum system. We first discuss the principle of the present MEPP for purifying the bit-flip errors and discuss it for phase-flip errors in next section.

Suppose that Alice, Bob and Charlie share a three-qubit ensemble ρ after the transmission of qubits over noisy channels, i.e.,

$$
\begin{aligned}
\rho &= F_0|\Phi_0^+\rangle\langle\Phi_0^+| + F_1|\Phi_1^+\rangle\langle\Phi_1^+| \\
&+ F_2|\Phi_2^+\rangle\langle\Phi_2^+| + F_3|\Phi_3^+\rangle\langle\Phi_3^+|. \qquad (3)
\end{aligned}
$$

Here $F_0 = \langle\Phi_0^+|\rho|\Phi_0^+\rangle$ is the fidelity of the quantum systems transmitted over noisy channels, and

$$F_0 + F_1 + F_2 + F_3 = 1. \qquad (4)$$

The density matrix ρ means that there is a bit-flip error on the first qubit, the second qubit, and the third qubit of the quantum system with a probability of F_1, F_2, and F_3, respectively. For obtaining some high-fidelity entangled three-photon systems, the three parties divide their quantum systems in the ensemble ρ into many groups and each group is composed of a pair of three-photon quantum systems. We label each group with $A_1B_1C_1A_2B_2C_2$ (i.e., the two three-photon quantum systems $A_1B_1C_1$ and $A_2B_2C_2$).

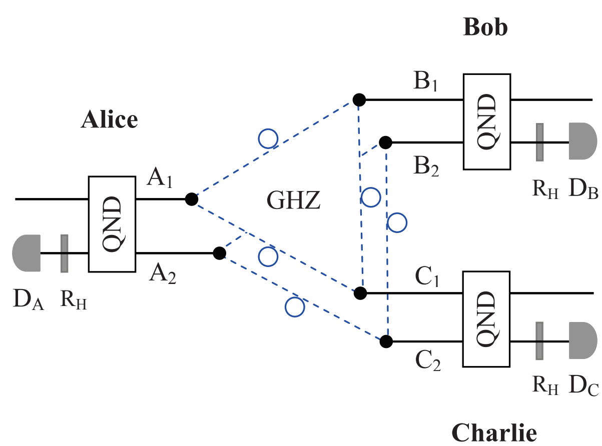

FIG. 2: (Color online) The principle of our conventional three-photon entanglement purification scheme for bit-flip errors with QNDs. The wave plate R_H represents a Hadamard operation and it is used to transform the polarization states $|H\rangle$ and $|V\rangle$ into $1/\sqrt{2}(|H\rangle + |V\rangle)$ and $1/\sqrt{2}(|H\rangle - |V\rangle)$, respectively. D_A, D_B, and D_C represent the single-photon measurements with the basis $Z = \{|H\rangle, |V\rangle\}$ done by Alice, Bob, and Charlie, respectively. The circles with blue virtual lines represent long-distance optical-fiber channels. The dots with black real lines represent qubits.

The principle of our conventional three-photon EPP for bit-flip errors is shown in Fig.2. The state of the system composed of the two three-photon subsystems $A_1B_1C_1$ and $A_2B_2C_2$ can be viewed as the mixture of the sixteen pure states, i.e., $|\Phi_i^+\rangle \otimes |\Phi_j^+\rangle$ with a probability of F_iF_j ($i,j = 0,1,2,3$). For each group, Alice takes her two photons A_1 and A_2 to pass through the setup shown in Fig.2. The photon A_1 enters the up spatial mode and the photon A_2 enters the down spatial mode. So do Bob and Charlie. After the QNDs, the three parties compare the parities of their photons. They keep the groups for which all the three parties obtain an even parity or an odd parity. These instances correspond to the identity-combinations $|\Phi_i^+\rangle_{A_1B_1C_1} \otimes |\Phi_i^+\rangle_{A_2B_2C_2}$ ($i = 0,1,2,3$). That is, the parties only distill some high-fidelity three-photon entangled systems from the identity-combinations in our conventional MEPP, similar to all existing EPPs [25–29, 35–37].

When all the three parties obtain an even parity, the quantum system $A_1B_1C_1A_2B_2C_2$ is in a new mixed state which is composed of the four states

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}(|HHH\rangle_{A_1B_1C_1}|HHH\rangle_{A_2B_2C_2} + |VVV\rangle_{A_1B_1C_1}|VVV\rangle_{A_2B_2C_2}), \tag{5}$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|VHH\rangle_{A_1B_1C_1}|VHH\rangle_{A_2B_2C_2} + |HVV\rangle_{A_1B_1C_1}|HVV\rangle_{A_2B_2C_2}), \tag{6}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|HVH\rangle_{A_1B_1C_1}|HVH\rangle_{A_2B_2C_2} + |VHV\rangle_{A_1B_1C_1}|VHV\rangle_{A_2B_2C_2}), \tag{7}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|HHV\rangle_{A_1B_1C_1}|HHV\rangle_{A_2B_2C_2} + |VVH\rangle_{A_1B_1C_1}|VVH\rangle_{A_2B_2C_2}) \tag{8}$$

with a probability of $\frac{1}{2}F_0^2$, $\frac{1}{2}F_1^2$, $\frac{1}{2}F_2^2$, and $\frac{1}{2}F_3^2$, respectively.

When all the three parties obtain an odd parity, the quantum system $A_1B_1C_1A_2B_2C_2$ is in another mixed state which is composed of the four states

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|HHH\rangle_{A_1B_1C_1}|VVV\rangle_{A_2B_2C_2} + |VVV\rangle_{A_1B_1C_1}|HHH\rangle_{A_2B_2C_2}), \tag{9}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|VHH\rangle_{A_1B_1C_1}|HVV\rangle_{A_2B_2C_2} + |HVV\rangle_{A_1B_1C_1}|VHH\rangle_{A_2B_2C_2}), \tag{10}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|HVH\rangle_{A_1B_1C_1}|VHV\rangle_{A_2B_2C_2} + |VHV\rangle_{A_1B_1C_1}|HVH\rangle_{A_2B_2C_2}), \tag{11}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|HHV\rangle_{A_1B_1C_1}|VVH\rangle_{A_2B_2C_2} + |VVH\rangle_{A_1B_1C_1}|HHV\rangle_{A_2B_2C_2}) \tag{12}$$

with the probabilities of $\frac{1}{2}F_0^2$, $\frac{1}{2}F_1^2$, $\frac{1}{2}F_2^2$, and $\frac{1}{2}F_3^2$, respectively. With a bit-flip operation on each of the three qubits $A_2B_2C_2$, Alice, Bob, and Charlie obtain the same outcomes as the case in which they all obtain an even parity. That is, the states $|\psi_i\rangle$ can be transformed into the states $|\phi_i\rangle$ ($i = 0,1,2,3$), respectively. By this way, Alice, Bob, and Charlie obtain the states $|\phi_i\rangle$ with the probabilities F_i^2 as they can obtain the similar outcomes when they all get whether an even parity or an odd parity with QNDs. We only discuss the case that the system is in the states $|\phi_i\rangle$ with the probabilities F_i^2 below.

After a Hadamard operation (i.e., the wave plate R_H in Fig.2) on the polarization of each photon in the down spatial mode, Alice, Bob, and Charlie measure the photons $A_2B_2C_2$ with the basis $\{|H\rangle, |V\rangle\}$. The outcomes will divide the instances into two groups. In the first group, the number of the outcomes $|V\rangle$ is even. In this time, Alice, Bob, and Charlie obtain the states $|\Phi_0^+\rangle_{A_1B_1C_1}$, $|\Phi_1^+\rangle_{A_1B_1C_1}$, $|\Phi_2^+\rangle_{A_1B_1C_1}$, and $|\Phi_3^+\rangle_{A_1B_1C_1}$ with the probabilities $\frac{1}{2}F_0^2$, $\frac{1}{2}F_1^2$, $\frac{1}{2}F_2^2$, and $\frac{1}{2}F_3^2$, respectively. In the second group, the number of the outcomes $|V\rangle$ is odd and the three parties obtain the states $|\Phi_0^-\rangle_{A_1B_1C_1}$, $|\Phi_1^-\rangle_{A_1B_1C_1}$, $|\Phi_2^-\rangle_{A_1B_1C_1}$, and $|\Phi_3^-\rangle_{A_1B_1C_1}$ with the probabilities $\frac{1}{2}F_0^2$, $\frac{1}{2}F_1^2$, $\frac{1}{2}F_2^2$, and $\frac{1}{2}F_3^2$, respectively. Alice, Bob, and Charlie can transform the states $|\Phi_0^-\rangle$, $|\Phi_1^-\rangle$, $|\Phi_2^-\rangle$, and $|\Phi_3^-\rangle$ into the states $|\Phi_0^+\rangle$, $|\Phi_1^+\rangle$, $|\Phi_2^+\rangle$, and $|\Phi_3^+\rangle$ with a phase-flip operation $\sigma_z = |H\rangle\langle H| - |V\rangle\langle V|$ on the first photon A_1, respectively. That is, by keeping the instances in which all the three parties obtain the same parity and measuring the photons through the down spatial modes, Alice, Bob, and Charlie can obtain a new ensemble in the state

$$\rho' = F_0'|\Phi_0^+\rangle\langle\Phi_0^+| + F_1'|\Phi_1^+\rangle\langle\Phi_1^+| + F_2'|\Phi_2^+\rangle\langle\Phi_2^+| + F_3'|\Phi_3^+\rangle\langle\Phi_3^+|, \tag{13}$$

where

$$\begin{aligned}
F_0' &= \frac{F_0^2}{F_0^2+F_1^2+F_2^2+(1-F_0-F_1-F_2)^2},\\
F_1' &= \frac{F_1^2}{F_0^2+F_1^2+F_2^2+(1-F_0-F_1-F_2)^2},\\
F_2' &= \frac{F_2^2}{F_0^2+F_1^2+F_2^2+(1-F_0-F_1-F_2)^2},\\
F_3' &= \frac{(1-F_0-F_1-F_2)^2}{F_0^2+F_1^2+F_2^2+(1-F_0-F_1-F_2)^2}.
\end{aligned} \tag{14}$$

The fidelity $F_0' > F_0$ if F_0 satisfies the relation

$$F_0 > \frac{1}{4}\{3-2F_1-2F_2 - \sqrt{1+4(F_1+F_2)-12(F_1^2+F_2^2)-8F_1F_2}\}. \tag{15}$$

With three symmetric noisy channels, the fidelity of the state $|\Phi_0^+\rangle$ will be improved by this conventional three-photon EPP if its original fidelity $F_0 > 1/4$.

In fact, our conventional three-photon EPP is similar to the MEPP in Ref. [35]. We use some QNDs, instead of perfect CNOT gates in Ref. [35], to complete the purification of bit-flip errors and we give a general case for this purification.

C. Recycling three-photon entanglement purification for bit-flip errors from subspaces

In our conventional three-photon entanglement purification for bit-flip errors, the three parties in quantum communication discard the instances in which Alice, Bob, and Charlie obtain different parities, i.e., the cross-combinations $|\Phi_i^+\rangle \otimes |\Phi_j^+\rangle$ $(i \neq j \in \{0,1,2,3\})$, similar to all conventional EPPs [25–29, 35–37]. That is, when the system composed of six photons is in the state $|\Phi_i^+\rangle \otimes |\Phi_j^+\rangle$ $(i \neq j \in \{0,1,2,3\})$ which take place with a probability of F_iF_j, Alice, Bob, and Charlie discard the system in the conventional three-photon EPPs because the probabilities of the states $|\Phi_i^+\rangle_{A_1B_1C_1} \otimes |\Phi_j^+\rangle_{A_2B_2C_2}$ and $|\Phi_j^+\rangle_{A_1B_1C_1} \otimes |\Phi_i^+\rangle_{A_2B_2C_2}$ are the same one F_iF_j and the three parties cannot determine the state of the three-photon system $A_1B_1C_1$ after they measure the photons $A_2B_2C_2$ in this time. However, this system can be used to distil a high-fidelity two-photon entangled state. With a set of high-fidelity two-photon systems, Alice, Bob, and Charlie can produce a subset of high-fidelity three-photon systems. We call this part of our MEPP as the recycling MEPP as the parties should distil three-photon systems from the cross-combinations $|\Phi_i^+\rangle \otimes |\Phi_j^+\rangle$ $(i \neq j \in \{0,1,2,3\})$ which are just discarded in all other conventional MEPPs. Our recycling MEPP will increase the efficiency (the yield) of our three-photon EPP largely.

1. *Two-photon entanglement purification for bit-flip errors from three-photon systems*

We only discuss the principle of our recycling MEPP in the case that the system is in the state $|\Phi_0^+\rangle_{A_1B_1C_1} \otimes |\Phi_2^+\rangle_{A_2B_2C_2}$ or $|\Phi_2^+\rangle_{A_1B_1C_1} \otimes |\Phi_0^+\rangle_{A_2B_2C_2}$ below and the other cases are similar to it with or without a little modification.

The cross-combinations $|\Pi\rangle_1 \equiv |\Phi_0^+\rangle_{A_1B_1C_1} \otimes |\Phi_2^+\rangle_{A_2B_2C_2}$ and $|\Pi\rangle_2 \equiv |\Phi_2^+\rangle_{A_1B_1C_1} \otimes |\Phi_0^+\rangle_{A_2B_2C_2}$ can be rewritten as

$$\begin{aligned} |\Pi\rangle_1 &= \frac{1}{2}(|HHH\rangle_{A_1B_1C_1}|VHV\rangle_{A_2B_2C_2} \\ &\quad + |VVV\rangle_{A_1B_1C_1}|HVH\rangle_{A_2B_2C_2} \\ &\quad + |HHH\rangle_{A_1B_1C_1}|HVH\rangle_{A_2B_2C_2} \\ &\quad + |VVV\rangle_{A_1B_1C_1}|VHV\rangle_{A_2B_2C_2}), \end{aligned} \tag{16}$$

$$\begin{aligned} |\Pi\rangle_2 &= \frac{1}{2}(|VHV\rangle_{A_1B_1C_1}|HHH\rangle_{A_2B_2C_2} \\ &\quad + |HVH\rangle_{A_1B_1C_1}|VVV\rangle_{A_2B_2C_2} \\ &\quad + |HVH\rangle_{A_1B_1C_1}|HHH\rangle_{A_2B_2C_2} \\ &\quad + |VHV\rangle_{A_1B_1C_1}|VVV\rangle_{A_2B_2C_2}). \end{aligned} \tag{17}$$

That is, if the outcomes of the parity-check measurements obtained by Alice, Bob, and Charlie are odd, even, and odd, respectively, the six-photon system is in the state

$$\begin{aligned} |\Omega\rangle_1 &\equiv \frac{1}{\sqrt{2}}(|HHH\rangle_{A_1B_1C_1}|VHV\rangle_{A_2B_2C_2} \\ &+ \; |VVV\rangle_{A_1B_1C_1}|HVH\rangle_{A_2B_2C_2}) \end{aligned} \tag{18}$$

or

$$\begin{aligned} |\Omega\rangle_2 &\equiv \frac{1}{\sqrt{2}}(|VHV\rangle_{A_1B_1C_1}|HHH\rangle_{A_2B_2C_2} \\ &+ \; |HVH\rangle_{A_1B_1C_1}|VVV\rangle_{A_2B_2C_2}), \end{aligned} \tag{19}$$

which takes place with the probability of F_0F_1. If they are even, odd, and even, respectively, the six-photon system is in the state

$$\begin{aligned} |\Omega\rangle_3 &\equiv \frac{1}{\sqrt{2}}(|HHH\rangle_{A_1B_1C_1}|HVH\rangle_{A_2B_2C_2} \\ &+ \; |VVV\rangle_{A_1B_1C_1}|VHV\rangle_{A_2B_2C_2}) \end{aligned} \tag{20}$$

or

$$\begin{aligned} |\Omega\rangle_4 &\equiv \frac{1}{\sqrt{2}}(|HVH\rangle_{A_1B_1C_1}|HHH\rangle_{A_2B_2C_2} \\ &+ \; |VHV\rangle_{A_1B_1C_1}|VVV\rangle_{A_2B_2C_2}), \end{aligned} \tag{21}$$

which takes place with the probability of F_0F_1 yet.

The states $|\Omega\rangle_1$ can be rewritten as follows:

$$\begin{aligned} |\Omega\rangle_1 &= \frac{1}{2\sqrt{2}}\{(|HH\rangle_{A_1C_1} + |VV\rangle_{A_1C_1})(|++++\rangle \\ &\quad + |++--\rangle - |+-+-\rangle \\ &\quad - |+--+\rangle - |-++-\rangle \\ &\quad - |-+-+\rangle + |--++\rangle \\ &\quad + |----\rangle)_{A_2C_2B_1B_2} \\ &\quad + (|HH\rangle_{A_1C_1} - |VV\rangle_{A_1C_1})(|+++-\rangle \\ &\quad + |++-+\rangle - |+-++\rangle \\ &\quad - |+---\rangle - |-+++\rangle \\ &\quad - |-+--\rangle + |--+-\rangle \\ &\quad + |---+\rangle)_{A_2C_2B_1B_2}, \end{aligned} \tag{22}$$

where

$$|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle),$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle). \quad (23)$$

That is, Alice, Bob, and Charlie can distil a high-fidelity two-photon entangled state $|\phi^+\rangle_{A_1C_1} = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{A_1C_1}$ from the six-photon state $|\Omega\rangle_1$. In detail, Alice and Charlie measure their photons A_2 and C_2, respectively, and Bob measures his two photons B_1 and B_2 with the measuring basis $X \equiv \{|+\rangle, |-\rangle\}$. Alice and Charlie obtain the two-photon entangled state $|\phi^+\rangle_{A_1C_1}$ from the six-photon state $|\Omega\rangle_1$ when the number of the outcomes $|-\rangle$ is even. When the number of the outcomes $|-\rangle$ is odd, Alice and Charlie obtain the two-photon entangled state $|\phi^-\rangle_{A_1C_1} = \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle)_{A_1C_1}$ and they can transform the state $|\phi^-\rangle_{A_1C_1}$ into the state $|\phi^+\rangle_{A_1C_1}$ by performing a phase-flip operation σ_z on the photon C_1.

For the other three states $|\Omega\rangle_i$ $(i = 2, 3, 4)$, Alice, Bob, and Charlie can also obtain the two-photon entangled state $|\phi^+\rangle_{A_1C_1}$ with the same principle. That is, Alice, Bob, and Charlie can obtain the two-photon maximally entangled state $|\phi^+\rangle_{A_1C_1} = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{A_1C_1}$ from the states $|\Phi_0^+\rangle_{A_1B_1C_1} \otimes |\Phi_2^+\rangle_{A_2B_2C_2}$ and $|\Phi_2^+\rangle_{A_1B_1C_1} \otimes |\Phi_0^+\rangle_{A_2B_2C_2}$ with the probability of $2F_0F_2$.

In the same way, Alice, Bob, and Charlie can obtain the two-photon maximally entangled states $|\phi^+\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{A_1B_1}$ from the cross-combinations $|\Phi_0^+\rangle_{A_1B_1C_1} \otimes |\Phi_3^+\rangle_{A_2B_2C_2}$ and $|\Phi_3^+\rangle_{A_1B_1C_1} \otimes |\Phi_0^+\rangle_{A_2B_2C_2}$ with the probability of $2F_0F_3$. Also, they can obtain the $|\phi^+\rangle_{B_1C_1} = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{B_1C_1}$ from $|\Phi_0^+\rangle_{A_1B_1C_1} \otimes |\Phi_1^+\rangle_{A_2B_2C_2}$ and $|\Phi_1^+\rangle_{A_1B_1C_1} \otimes |\Phi_0^+\rangle_{A_2B_2C_2}$ with the probability of $2F_0F_1$.

Certainly, there is a probability that the two three-photon systems take place a bit-flip error on two different photons, such as the states $|\Phi_1^+\rangle_{A_1B_1C_1} \otimes |\Phi_2^+\rangle_{A_2B_2C_2}$, $|\Phi_2^+\rangle_{A_1B_1C_1} \otimes |\Phi_1^+\rangle_{A_2B_2C_2}$, $|\Phi_1^+\rangle_{A_1B_1C_1} \otimes |\Phi_3^+\rangle_{A_2B_2C_2}$, $|\Phi_3^+\rangle_{A_1B_1C_1} \otimes |\Phi_1^+\rangle_{A_2B_2C_2}$, $|\Phi_2^+\rangle_{A_1B_1C_1} \otimes |\Phi_3^+\rangle_{A_2B_2C_2}$, and $|\Phi_3^+\rangle_{A_1B_1C_1} \otimes |\Phi_2^+\rangle_{A_2B_2C_2}$. With the same process as the case that there is only one three-photon system taking place a bit-flip error (i.e., the states $|\Phi_0^+\rangle_{A_1B_1C_1} \otimes |\Phi_i^+\rangle_{A_2B_2C_2}$ and $|\Phi_i^+\rangle_{A_1B_1C_1} \otimes |\Phi_0^+\rangle_{A_2B_2C_2}$ $(i = 1, 2, 3)$), Alice, Bob, and Charlie will obtain the states with a bit-flip error, such as $|\psi^+\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)_{A_1B_1}$, $|\psi^+\rangle_{A_1C_1} = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)_{A_1C_1}$, and $|\psi^+\rangle_{B_1C_1} = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)_{B_1C_1}$ with the probabilities $2F_1F_2$, $2F_1F_3$, and $2F_2F_3$, respectively. The relation between the states of the two-photon systems and the cross-combinations is shown in Table I.

TABLE I: The states of the two-photon systems obtained from cross-combinations and their probabilities.

cross-combinations	$\lvert\Phi_0^+\rangle \otimes \lvert\Phi_2^+\rangle$ $\lvert\Phi_2^+\rangle \otimes \lvert\Phi_0^+\rangle$	$\lvert\Phi_0^+\rangle \otimes \lvert\Phi_1^+\rangle$ $\lvert\Phi_1^+\rangle \otimes \lvert\Phi_0^+\rangle$	$\lvert\Phi_0^+\rangle \otimes \lvert\Phi_3^+\rangle$ $\lvert\Phi_3^+\rangle \otimes \lvert\Phi_0^+\rangle$	$\lvert\Phi_1^+\rangle \otimes \lvert\Phi_2^+\rangle$ $\lvert\Phi_2^+\rangle \otimes \lvert\Phi_1^+\rangle$	$\lvert\Phi_1^+\rangle \otimes \lvert\Phi_3^+\rangle$ $\lvert\Phi_3^+\rangle \otimes \lvert\Phi_1^+\rangle$	$\lvert\Phi_2^+\rangle \otimes \lvert\Phi_3^+\rangle$ $\lvert\Phi_3^+\rangle \otimes \lvert\Phi_2^+\rangle$
two-photon states	$\lvert\phi^+\rangle_{A_1C_1}$	$\lvert\phi^+\rangle_{B_1C_1}$	$\lvert\phi^+\rangle_{A_1B_1}$	$\lvert\psi^+\rangle_{A_1B_1}$	$\lvert\psi^+\rangle_{A_1C_1}$	$\lvert\psi^+\rangle_{B_1C_1}$
probabilities	$2F_0F_2$	$2F_0F_1$	$2F_0F_3$	$2F_1F_2$	$2F_1F_3$	$2F_2F_3$

With Table I, one can see that the two-photon systems shared by two of the three parties can be described with the following density matrices (without conventionalization):

$$\rho_{AB} = 2F_0F_3|\phi^+\rangle_{AB}\langle\phi^+| + 2F_1F_2|\psi^+\rangle_{AB}\langle\psi^+|,$$
$$\rho_{AC} = 2F_0F_2|\phi^+\rangle_{AC}\langle\phi^+| + 2F_1F_3|\psi^+\rangle_{AC}\langle\psi^+|,$$
$$\rho_{BC} = 2F_0F_1|\phi^+\rangle_{BC}\langle\phi^+| + 2F_2F_3|\psi^+\rangle_{BC}\langle\psi^+|. \quad (24)$$

Suppose the fidelities $F_1 = F_2 = F_3$ and $F_0 > F_1$. One can see that the fidelity of two-photon systems is larger than that of the original three-photon systems transmitted. For example, $F(|\phi^+\rangle_{AB}) = \frac{2F_0F_3}{2F_0F_3+2F_1F_2} = \frac{F_0}{F_0+F_1} > F_0$ as $F_0 + F_1 < 1$.

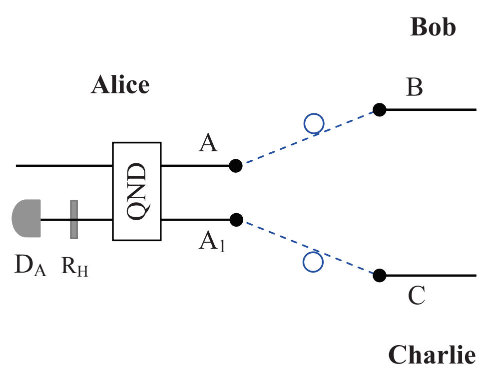

FIG. 3: (Color online) The principle of the entanglement link for producing a three-photon entangled system from two two-photon entangled systems with a QND.

2. Three-photon entanglement production from two-photon systems with entanglement link

As the three photons in the original system are symmetric to each other, we use the states ρ_{AB} and ρ_{AC} as an example to describe the principle of three-photon entanglement production from two-photon systems with entanglement link and assume that $F_1=F_2=F_3$, shown in Fig.3. The density matrices in Eq.(24) become

$$\begin{aligned}\rho^b_{AB} &= F^b_0|\phi^+\rangle_{AB}\langle\phi^+|+F^b_1|\psi^+\rangle_{AB}\langle\psi^+|,\\ \rho^b_{AC} &= F^b_0|\phi^+\rangle_{AC}\langle\phi^+|+F^b_1|\psi^+\rangle_{AC}\langle\psi^+|,\\ \rho^b_{BC} &= F^b_0|\phi^+\rangle_{BC}\langle\phi^+|+F^b_1|\psi^+\rangle_{BC}\langle\psi^+|,\end{aligned} \quad (25)$$

where

$$\begin{aligned}F^b_0 &= \frac{F_0}{F_0+F_1},\\ F^b_1 &= \frac{F_1}{F_0+F_1}.\end{aligned} \quad (26)$$

The system composed of the four photons A, B, A_1, and C is in the state $\rho^b_{AB}\otimes\rho^b_{A_1C}$ which can be viewed as the mixture of the four pure states $|\Gamma\rangle_0\equiv|\phi^+\rangle_{AB}\otimes|\phi^+\rangle_{A_1C}$, $|\Gamma\rangle_1\equiv|\phi^+\rangle_{AB}\otimes|\psi^+\rangle_{A_1C}$, $|\Gamma\rangle_2\equiv|\psi^+\rangle_{AB}\otimes|\phi^+\rangle_{A_1C}$, and $|\Gamma\rangle_3\equiv|\psi^+\rangle_{AB}\otimes|\psi^+\rangle_{A_1C}$ with the probabilities $F^b_0F^b_0$, $F^b_0F^b_1$, $F^b_1F^b_0$, and $F^b_1F^b_1$, respectively. After the QND, Alice, Bob, and Charlie will divide the instances into two groups. One is the case that the outcome of the parity-check measurement is even and the other is odd. When Alice obtains an even parity, the four-photon system is in the states $\frac{1}{\sqrt{2}}(|HHHH\rangle+|VVVV\rangle)_{ABA_1C}$, $\frac{1}{\sqrt{2}}(|HHHV\rangle+|VVVH\rangle)_{ABA_1C}$, $\frac{1}{\sqrt{2}}(|HVHH\rangle+|VHVV\rangle)_{ABA_1C}$, and $\frac{1}{\sqrt{2}}(|HVHV\rangle+|VHVH\rangle)_{ABA_1C}$ with the probabilities $\frac{1}{2}F^s_0F^s_0$, $\frac{1}{2}F^b_0F^b_1$, $\frac{1}{2}F^b_1F^b_0$, and $\frac{1}{2}F^b_1F^b_1$, respectively. After Alice performs a Hadamard operation on the photon A_1 and measures it with the basis $Z=\{|H\rangle,|V\rangle\}$, Alice, Bob, and Charlie will obtain a three-photon entangled system in the states $|\Phi^+_0\rangle$, $|\Phi^+_3\rangle$, $|\Phi^+_2\rangle$, and $|\Phi^+_1\rangle$ with the probabilities $\frac{1}{2}F^b_0F^b_0$, $\frac{1}{2}F^b_0F^b_1$, $\frac{1}{2}F^b_1F^b_0$, and $\frac{1}{2}F^b_1F^b_1$, respectively. These outcomes will be obtained with (if the outcome of the measurement on the photon A_1 by Alice is $|V\rangle_{A_1}$) or without ($|H\rangle_{A_1}$) a phase-flip operation on the photon A.

When Alice obtains an odd parity with her QND, the four-photon system is in the states $\frac{1}{\sqrt{2}}(|HHVV\rangle+|VVHH\rangle)_{ABA_1C}$, $\frac{1}{\sqrt{2}}(|HHVH\rangle+|VVHV\rangle)_{ABA_1C}$, $\frac{1}{\sqrt{2}}(|HVVV\rangle+|VHHH\rangle)_{ABA_1C}$, and $\frac{1}{\sqrt{2}}(|HVVH\rangle+|VHHV\rangle)_{ABA_1C}$ with the probabilities $\frac{1}{2}F^b_0F^b_0$, $\frac{1}{2}F^b_0F^b_1$, $\frac{1}{2}F^b_1F^b_0$, and $\frac{1}{2}F^b_1F^b_1$, respectively. Alice, Bob, and Charlie can obtain the same outcomes as the case with an even parity by performing a bit-flip operation on the photons A_1 and C independently. That is, with entanglement link, Alice, Bob, and Charlie can obtain a new ensemble for three-photon systems in the state

$$\begin{aligned}\rho^t_{a_1bc} &= F^t_0|\Phi^+_0\rangle\langle\Phi^+_0|+F^t_1|\Phi^+_1\rangle\langle\Phi^+_1|\\ &+ F^t_2|\Phi^+_2\rangle\langle\Phi^+_2|+F^t_3|\Phi^+_3\rangle\langle\Phi^+_3|.\end{aligned} \quad (27)$$

Here

$$\begin{aligned}F^t_0 &= \frac{F^2_0}{(F_0+F_1)^2},\\ F^t_1 &= \frac{F^2_1}{(F_0+F_1)^2},\\ F^t_2 &= F^t_3=\frac{F_0F_1}{(F_0+F_1)^2}.\end{aligned} \quad (28)$$

$F^t_0>F_0$ when $F_0>\frac{1}{4}$, which means that the three parties can obtain a high-fidelity three-photon entangled system from two two-photon entangled subsystems if and only if the original fidelity of the three-photon systems transmitted over noisy channels is larger than $\frac{1}{4}$ (this is just the condition for the conventional three-photon entanglement purification over three symmetric noisy channels).

D. Efficiency and fidelity of the present three-photon entanglement purification protocol

Here the efficiency of an EPP Y is defined as the probability (i.e., the yield) that the parties can obtain a high-fidelity entangled multi-photon system from a pair of multi-photon systems transmitted over a noisy channel without loss. The efficiency of the present three-photon EPP Y_e depends on the parameters F_1, F_2, and F_3. For simpleness, we only discuss the case with the parameters $F_1=F_2=F_3=\frac{1-F_0}{3}$ below.

With our conventional three-photon EPP only, the efficiency of the three-photon EPP Y_c is

$$Y_c = F^2_0+F^2_1+F^2_2+F^2_3=\frac{1-2F_0+4F^2_0}{3}. \quad (29)$$

It is the probability that the pair of three-photon systems are in the identity-combinations $|\Phi^+_i\rangle_{A_1B_1C_1}\otimes|\Phi^+_i\rangle_{A_2B_2C_2}$ $(i=0,1,2,3)$. Y_c is just the maximal value of the efficiency in all existing conventional MEPPs for three-qubit systems [35–37].

As each cross-combination $|\Phi^+_i\rangle\otimes|\Phi^+_j\rangle$ $(i\neq j\in\{0,1,2,3\})$ will lead the three parties to obtain an entangled two-photon pair, the probability that the three parties obtain two-photon pairs from a pair of three-photon systems $P_{3\rightarrow 2}$ is

$$\begin{aligned}P_{3\rightarrow 2} &= \sum^3_{j\neq l=0}F_jF_l\\ &= F_0(F_1+F_2+F_3)+F_1(F_0+F_2+F_3)\\ &+ F_2(F_0+F_1+F_3)+F_3(F_0+F_1+F_2)\\ &= \frac{2+2F_0-4F^2_0}{3}.\end{aligned} \quad (30)$$

That is, the efficiency that the three parties obtain three-photon entangled systems from two-photon entangled systems with entanglement link $Y_{2\to3}$ is

$$Y_{2\to3} = \frac{1}{2}P_{3\to2} = \frac{1+F_0-2F_0^2}{3} \quad (31)$$

because they can obtain a three-photon system from two two-photon systems.

Taking three-photon entanglement production with entanglement link into account, the efficiency of the present MEPP Y_e is

$$Y_e \;=\; Y_c + Y_{2\to3} = \frac{2-F_0+2F_0^2}{3}. \quad (32)$$

The efficiency of the present MEPP and the maximal value of that from conventional MEPPs for three-qubit systems are shown in Fig.4(a). It is clear that the present MEPP is more efficient than the conventional MEPPs, especially in the case that the original fidelity F_0 is not big.

The fidelity of our conventional MEPP is

$$F_c \;=\; \frac{F_0^2}{F_0^2+F_1^2+F_2^2+F_3^2} = \frac{3F_0^2}{1-2F_0+4F_0^2}. \quad (33)$$

The fidelity of the two-photon systems obtained from cross-combinations is

$$F_2 \;=\; F_0^b = \frac{F_0}{(F_0+F_1)} = \frac{3F_0}{1+2F_0}. \quad (34)$$

The fidelity of the three-photon systems obtained from two-photon systems with entanglement link is

$$F_{2\to3} \;=\; F_0^t = \frac{F_0^2}{(F_0+F_1)^2} = \frac{9F_0^2}{1+4F_0+4F_0^2}. \quad (35)$$

The average fidelity of the present MEPP F_e can be calculated as

$$\begin{aligned} F_e \;&=\; (F_cY_c + F_{2\to3}Y_{2\to3})/Y_e \\ &=\; \frac{3F_0^2(4+7F_0-2F_0^2)}{(1+2F_0)^2(2-F_0+2F_0^2)}. \end{aligned} \quad (36)$$

The relation of the four fidelities F_c, F_2, $F_{2\to3}$, and F_e is shown in Fig.4(b). The line for the original fidelity F_0 is used for a clear comparison.

From Fig.4, one can see that the yield $Y_{2\to3}$ is larger than Y_c when $F_0 < \frac{1}{2}$. Y_e is far larger than Y_c, which means that the present MEPP has a larger efficiency than that in conventional MEPPs. On the other hand, the fidelity $F_{2\to3}$ is smaller than F_c although they both are larger than the original fidelity F_0 when $F_0 > \frac{1}{4}$. F_2 is larger than F_c when $F_0 < \frac{1}{2}$ and it is smaller than F_c when $F_0 > \frac{1}{2}$.

In a practical purification, the three parties need not mix the three-photon systems obtained by our conventional MEPP and our recycling MEPP. They can purify them independently in the next round. Also, they can first purify two-photon systems with the fidelity F_2 and then produce high-fidelity three-photon systems with entanglement link.

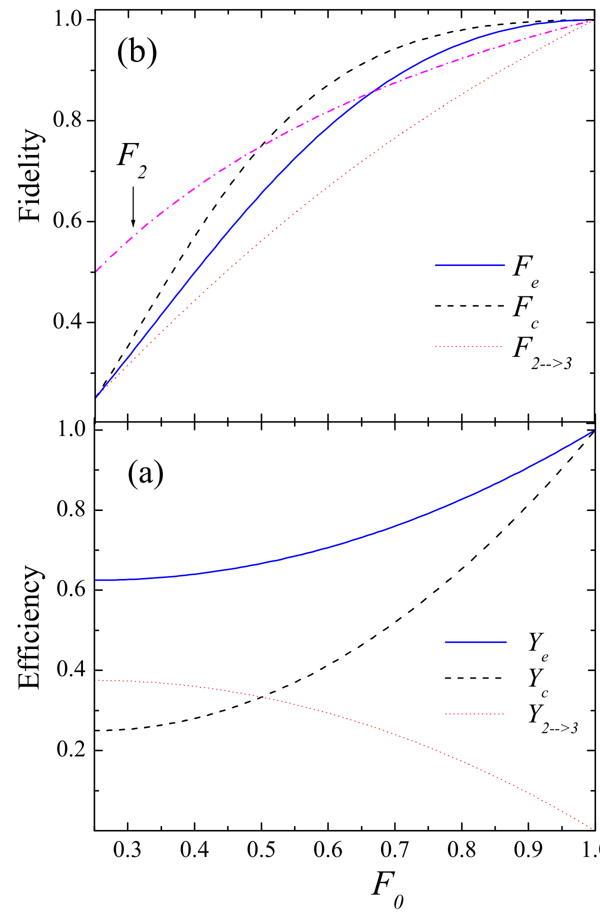

FIG. 4: (Color online) (a) The efficiency of the present MEPP Y_e and that of the conventional MEPP Y_c (it is just the maximal value of efficiency from conventional MEPPs) for three-photon systems under a symmetric noise ($F_1 = F - 2 = F_3$) are shown with a blue solid line and a black dash line, respectively. Here $Y_{2\to3}$ is the yield that the three parties can obtain three-photon systems from two-photon systems with entanglement link. (b) The fidelity of the present MEPP F_e and that of the conventional MEPP F_c. Here F_2 and $F_{2\to3}$ is the fidelities of the two-photon systems obtained from the cross-combinations and that of the three-photon systems obtained from two-photon systems with entanglement link, respectively. F_0 is just the original fidelity of three-photon systems before entanglement purification.

III. CONVENTIONAL THREE-PHOTON ENTANGLEMENT PURIFICATION FOR PHASE-FLIP ERRORS

In the process for purifying bit-flip errors, the relative probabilities of the states $|\Phi_1^\pm\rangle$, $|\Phi_2^\pm\rangle$, and $|\Phi_3^\pm\rangle$ are decreased. However, the relative probability of the state $|\Phi_0^-\rangle$ is not changed, compared with that of the state $|\Phi_0^+\rangle$. The task of the entanglement purification for phase-flip errors is in principle to complete the process with which the parties can depress the relative probability of the state $|\Phi_0^-\rangle$. Same as the entanglement purification in two-photon systems, a phase-flip error cannot be corrected directly in three-photon systems, different from a bit-flip error, but it can be transformed into a bit-flip error with a Hadamard operation on each photon. That is, after a Hadamard operation on each photon, the states $|\Phi_0^+\rangle$ and $|\Phi_0^-\rangle$ shown in Eq.(2) are transformed into the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$, respectively. Here

$$|\Psi^+\rangle = \frac{1}{2}(|HHH\rangle + |HVV\rangle + |VHV\rangle + |VVH\rangle),$$
$$|\Psi^-\rangle = \frac{1}{2}(|HHV\rangle + |HVH\rangle + |VHH\rangle + |VVV\rangle). \quad (37)$$

From Eq.(37), one can see that the transformation between phase-flip errors and bit-flip errors in three-photon GHZ states is more complex than that in Bell states [25–32]. The three parties cannot use the equipment shown in Fig.2 to purify the states in Eq.(37) directly. That is, we cannot exploit simply Hadamard operations to complete the transformation between phase-flip errors and bit-flip errors in three-photon GHZ states perfectly, different from Bell states. Fortunately, the number of the polarization state $|V\rangle$ is different in these two three-photon states. That is, the number of $|V\rangle$ is even in the state $|\Psi^+\rangle$, while it is odd in the state $|\Psi^-\rangle$. With this feature, the relative probability of the state $|\Psi^-\rangle$ will be depressed with QNDs.

Now, let us use a pair of partner states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ as an example to describe the principle of the purification for phase-flip errors. The density matrix of an ensemble over noisy channels with only phase-flip errors can be written as

$$\rho' = p_0|\Psi^+\rangle\langle\Psi^+| + p_1|\Psi^-\rangle\langle\Psi^-|. \quad (38)$$

Here p_0 and p_1 represent the probabilities of the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ in the ensemble ρ', respectively, and $p_0 + p_1 = 1$. For each pair of the entangled quantum systems picked out from this ensemble, say $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$, their state can be viewed as the mixture of four pure states, i.e., $|\Psi^+\rangle\otimes|\Psi^+\rangle$, $|\Psi^+\rangle\otimes|\Psi^-\rangle$, $|\Psi^-\rangle\otimes|\Psi^+\rangle$, and $|\Psi^-\rangle\otimes|\Psi^-\rangle$ with the probabilities p_0^2, p_0p_1, p_1p_0, and p_1^2, respectively.

The relation of the outcomes of the parity-check measurements by Alice, Bob, and Charlie, the states of the quantum system composed of the six photons $A_1'B_1'C_1'A_2'B_2'C_2'$, and their probabilities is shown in Table II. Here

$$|\varphi\rangle_0 = \frac{1}{2}(|HHHHHH\rangle + |HVVHVV\rangle + |VHVVHV\rangle + |VVHVVH\rangle), \quad (39)$$

$$|\varphi'\rangle_0 = \frac{1}{2}(|HHVHHV\rangle + |HVHHVH\rangle + |VHHVHH\rangle + |VVVVVV\rangle), \quad (40)$$

$$|\varphi\rangle_1 = \frac{1}{2}(|HHHHVV\rangle + |HVVHHH\rangle + |VHVVVH\rangle + |VVHVHV\rangle), \quad (41)$$

$$|\varphi'\rangle_1 = \frac{1}{2}(|HHVHVH\rangle + |HVHHHV\rangle + |VHHVVV\rangle + |VVVVHH\rangle), \quad (42)$$

$$|\varphi\rangle_2 = \frac{1}{2}(|HHHVHV\rangle + |HVVVVH\rangle + |VHVHHH\rangle + |VVHHVV\rangle), \quad (43)$$

$$|\varphi'\rangle_2 = \frac{1}{2}(|HHVVHH\rangle + |HVHVVV\rangle + |VHHHHV\rangle + |VVVHVH\rangle), \quad (44)$$

$$|\varphi\rangle_3 = \frac{1}{2}(|HHHVVH\rangle + |HVVVHV\rangle + |VHVHVV\rangle + |VVHHHH\rangle), \quad (45)$$

$$|\varphi'\rangle_3 = \frac{1}{2}(|HHVVVV\rangle + |HVHVHH\rangle + |VHHHVH\rangle + |VVVHHV\rangle). \quad (46)$$

U in Table II represents the unitary operations with which Alice, Bob, and Charlie can transform the six-photon states $|\varphi\rangle_i$ and $|\varphi'\rangle_i$ $(i = 1, 2, 3)$ into the states $|\varphi\rangle_0$ and $|\varphi'\rangle_0$, respectively. I and σ_x represent the identity operation and the bit-flip operation, respectively.

One can see that the three parties will obtain the states $|\varphi\rangle_i$ and $|\varphi'\rangle_i$ $(i = 1, 2, 3)$ with the probabilities $\frac{1}{4}p_0^2$ and $\frac{1}{4}p_1^2$, respectively, if the number of odd parities is even. Moreover, the states $|\varphi\rangle_i$ and $|\varphi'\rangle_i$ can be transformed into the states $|\varphi\rangle_0$ and $|\varphi'\rangle_0$ with two local unitary operations, respectively. That is, if the number of odd parities is even, Alice, Bob and Charlie can obtain a six-photon ensemble in which the probabilities of the states $|\varphi\rangle$ and $|\varphi'\rangle$ are $\frac{p_0^2}{p_0^2+p_1^2}$ and $\frac{p_1^2}{p_0^2+p_1^2}$, respectively.

TABLE II: The relation between the outcomes of parity-check measurements and the six-photon states.

outcomes	states	probabilities	U
even, even, even	$|\varphi\rangle_0$ $|\varphi'\rangle_0$	$\frac{1}{4}p_0^2$ $\frac{1}{4}p_1^2$	$I^{A_1'}\otimes I^{B_2'}\otimes I^{C_2'}$
even, odd, odd	$|\varphi\rangle_1$ $|\varphi'\rangle_1$	$\frac{1}{4}p_0^2$ $\frac{1}{4}p_1^2$	$I^{A_1'}\otimes \sigma_x^{B_2'}\otimes \sigma_x^{C_2'}$
odd, even, odd	$|\varphi\rangle_2$ $|\varphi'\rangle_2$	$\frac{1}{4}p_0^2$ $\frac{1}{4}p_1^2$	$\sigma_x^{A_1'}\otimes I^{B_2'}\otimes \sigma_x^{C_2'}$
odd, odd, even	$|\varphi\rangle_3$ $|\varphi'\rangle_3$	$\frac{1}{4}p_0^2$ $\frac{1}{4}p_1^2$	$\sigma_x^{A_1'}\otimes \sigma_x^{B_2'}\otimes I^{C_2'}$

Certainly, Alice, Bob, and Charlie can obtain a three-photon ensemble in the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ with the probabilities $\frac{p_0^2}{p_0^2+p_1^2}$ and $\frac{p_1^2}{p_0^2+p_1^2}$, respectively. They can obtain this ensemble by performing or not a local single-photon phase-flip operation on the three-photon systems after they measure the three photons $A_2'B_2'C_2'$

with the basis $X = \{|\pm\rangle\}$ in each six-photon system $A_1'B_1'C_1'A_2'B_2'C_2'$. For example, if they obtain the outcomes of the single-photon measurements $|++-\rangle_{A_2'B_2'C_2'}$ or $|--+\rangle_{A_2'B_2'C_2'}$, they need only perform a phase-flip operation $\sigma_z = |H\rangle\langle H| - |V\rangle\langle V|$ on the photon C_1'. If they obtain the outcomes $|+--+\rangle_{A_2'B_2'C_2'}$ or $|-++-\rangle_{A_2'B_2'C_2'}$, they need only perform a phase-flip operation $\sigma_z = |H\rangle\langle H| - |V\rangle\langle V|$ on the photon B_1'. If the outcomes are $|+--\rangle_{A_2'B_2'C_2'}$ or $|-++\rangle_{A_2'B_2'C_2'}$, they need only perform a phase-flip operation $\sigma_z = |H\rangle\langle H| - |V\rangle\langle V|$ on the photon A_1'.

With a Hadamard operation on each photon in a three-photon system kept, the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ become the states $|\Phi_0^+\rangle$ and $|\Phi_0^-\rangle$, respectively. That is, by performing the conventional three-photon entanglement purification for phase-flip errors, the three parties can obtain a new ensemble in the states $|\Phi_0^+\rangle$ and $|\Phi_0^-\rangle$ with the probabilities $\frac{p_0^2}{p_0^2+p_1^2}$ and $\frac{p_1^2}{p_0^2+p_1^2}$, respectively. The fidelity of the state $|\Phi_0^+\rangle$ is $p_0' = \frac{p_0^2}{p_0^2+p_1^2} = \frac{p_0^2}{p_0^2+(1-p_0)^2}$. $p_0' > p_0$ when $p_0 > 1/2$.

The cross-combinations $|\Psi^+\rangle \otimes |\Psi^-\rangle$ and $|\Psi^-\rangle \otimes |\Psi^+\rangle$ will lead the number of odd parities in the outcomes of their parity-check measurements to be odd and they are discarded as these instances will lead the three parties to obtain the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ with the same probability p_0p_1. That is, these instances cannot improve the fidelity of the entangled state $|\Psi^+\rangle$ as a phase-flip error in a GHZ state is an unlocal (global) error, which is different from the case with some bit-flip errors (a bit-flip error in a GHZ state can be considered as a local error). The three parties in quantum communication can only obtain a two-photon ensemble without entanglement if they want to distil two-photon subsystems from the cross-combinations with a phase-flip error.

IV. DISCUSSION AND SUMMARY

The present MEPP works for N-photon systems in GHZ states, as shown in detail in Appendix A.

It is not difficult to show that the present MEPP works by replacing the parity-check detectors (QNDs) with CNOT gates. We can use three-photon systems as an example to describe the principle of entanglement purification with entanglement link from two-photon subsystems, shown in Fig.5. The principle of the conventional three-photon EPP for bit-flip errors and phase-flip errors with CNOT gates is the same as that in the first MEPP by Murao *et al.* [35], shown in Fig.5(a). However, our MEPP can also distil some high-fidelity two-photon subsystems from the cross-combinations which are just discarded in Ref. [35]. For the cross-combinations $|\Phi_0^+\rangle_{A'B'C'} \otimes |\Phi_2^+\rangle_{A''B''C''}$ and $|\Phi_2^+\rangle_{A'B'C'} \otimes |\Phi_0^+\rangle_{A''B''C''}$, Alice, Bob, and Charlie obtain the three-photon states $\frac{1}{\sqrt{2}}(|HHH\rangle + |VVV\rangle)_{A'B'C'}$ and $\frac{1}{\sqrt{2}}(|HVH\rangle + |VHV\rangle)_{A'B'C'}$ with the same probability F_0F_2 after they perform their CNOT operations on their two photons and measure the three photons $A''B''C''$ with the basis Z. The outcomes of the single-photon measurements by Alice, Bob, and Charlie are $|HVH\rangle_{A''B''C''}$ or $|VHV\rangle_{A''B''C''}$. After Bob performs a Hadamard operation on the photon B' and measures it with the basis Z, Alice and Charlie will obtain a two-photon entangled state $\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{A'C'}$ with or without a phase-flip operation which depends on the fact that the outcome of the single-photon measurement by Bob on the photon B' is $|V\rangle_{B'}$ or $|H\rangle_{B'}$, respectively. For the cross-combinations $|\Phi_1^+\rangle_{A'B'C'} \otimes |\Phi_3^+\rangle_{A''B''C''}$ and $|\Phi_3^+\rangle_{A'B'C'} \otimes |\Phi_1^+\rangle_{A''B''C''}$, Alice, Bob, and Charlie also obtain the outcomes $|HVH\rangle_{A''B''C''}$ or $|VHV\rangle_{A''B''C''}$. In this time, they obtain the three-photon states $\frac{1}{\sqrt{2}}(|VHH\rangle + |HVV\rangle)_{A'B'C'}$ and $\frac{1}{\sqrt{2}}(|VHH\rangle + |HVV\rangle)_{A'B'C'}$ with the same probability F_1F_3. After Bob performs a Hadamard operation on the photon B' and measures it with the basis Z, Alice and Charlie obtain a two-photon entangled state $\frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)_{A'C'}$ with or without a phase-flip operation. That is, when Alice, Bob, and Charlie obtain the outcomes $|HVH\rangle_{A''B''C''}$ or $|VHV\rangle_{A''B''C''}$ after they perform their CNOT operations on their two photons and measure the three photons $A''B''C''$ with the basis Z, they can obtain a two-photon ensemble in the states $\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{A'C'}$ and $\frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)_{A'C'}$ with the probabilities F_0F_2 and F_1F_3, respectively. This is just the case when Alice, Bob, and Charlie obtain their parities (odd, even, odd) or (even, odd, even) with QNDs discussed in Sec.II C 1. So do the other cases. The principle of entanglement link with CNOT gates is similar to that with QNDs. That is, the present MEPP works by replacing QNDs with CNOT gates. However, when the parties replace their QNDs with polarizing beam splitters, the present MEPP does not work because the cross-combinations cannot be exploited to distil N'-photon subsystems as there are at least two photons which cannot be distinguished.

Compared with the conventional MEPPs [35–37], the present MEPP contains two parts. One is our conventional MEPP. Its principle is the same as other conventional MEPPs [35–37] although its efficiency is double as those with QNDs in Refs. [36, 37] because our conventional MEPP takes all the instances in which all the parties obtain either an even parity or an odd parity into account for obtaining high-fidelity multipartite entangled systems, not only the instances in which all parties obtain an even parity as those in Refs. [36, 37]. The other part is our recycling MEPP in which entanglement link is used to produce a multipartite entangled system from some subsystems with QNDs. In essence, the parties distil some multipartite entangled systems from the instances which are discard in the conventional MEPPs [35–37], which makes the present MEPP have a higher efficiency than others.

In the process for describing the principle of our ef-

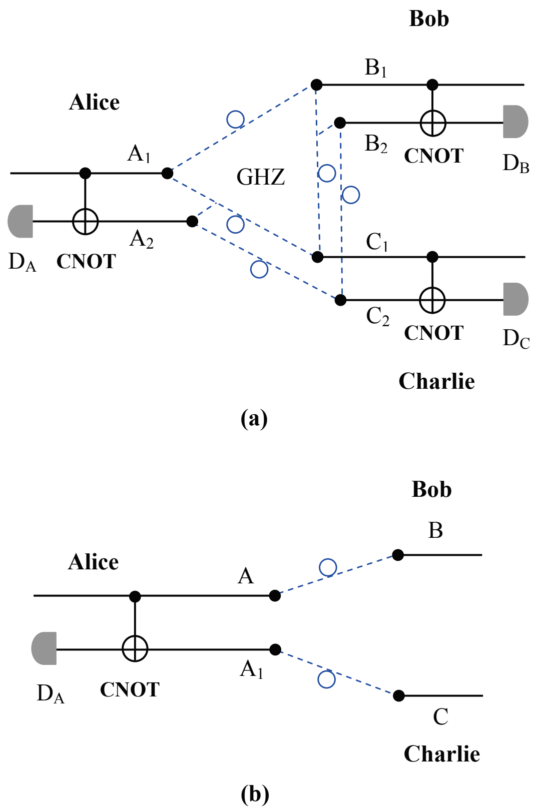

FIG. 5: (Color online) The principle of the present multipartite entanglement purification protocol with CNOT gates. (a) The conventional entanglement purification for three-photon systems with CNOT gates; (b) The principle for producing a high-fidelity three-photon entangled system with entanglement link from two two-photon subsystems with a CNOT gate.

ficient MEPP with entanglement link from subspaces, we mainly exploit the cross-Kerr nonlinearity to construct the parity-check detector (i.e., QND). We should acknowledge that the implementation of a clean cross-Kerr nonlinearity is still quite difficult in experiment, especially with natural cross-Kerr nonlinearities. It may be feasible in principle in the present MEPP. On one hand, some works have been studied on cross-Kerr nonlinearity [44–53]. For example, Kok *et al.* [44] showed that operating in the optical single-photon regime, the Kerr phase shift is only $\tau \approx 10^{-18}$. With electromagnetically induced transparent materials, cross-Kerr nonlinearities of $\tau \approx 10^{-5}$ can be obtained [45]. Also, Hofmann *et al.* [46] showed that a phase shift of π can be achieved with a single two-level atom in a one-sided cavity. In 2010, Wittmann *et al.* [47] investigated quantum measurement strategies capable of discrimination two coherent states using a homodyne detector and a photon number resolving detector. On the other hand, our QND does not require a large nonlinearity and it works for small values of the cross-Kerr coupling, which makes it possible [38, 43].

Although we only exploit the QND based on a cross-Kerr nonlinearity to explain the principle of our efficient MEPP, other elements can also be used to construct QNDs, such as quantum dots in optical cavities, as shown in Refs.[54–58]. We use the polarization degree of freedom of photon systems as an example to describe the principle of our efficient MEPP with entanglement link from subspaces. Obviously, it works for other degrees of freedom of quantum systems.

Same as all existing EPPs [25–33, 35–37], the present MEPP does not take the loss of photons in transmission over an optical fiber into account. In the principle of the present scheme, we assume that the parties in quantum communication deal with their entanglement purification under an ideal condition. That is, there is no photon loss in transmission. In a practical transmission of photons over an optical fiber, the losses should be considered, which will decrease the efficiency of EPPs [25–33, 35–37], especially those based on CNOT gates or linear optical elements. With cross-Kerr nonlinear media, the parties can construct photon-number detectors (PNDs) to distinguish the number of photons in the two spatial modes, shown in Fig.6. With PNDs, the present MEPP works efficiently with a photon-loss channel as well because the parties can in principle determine the number of the photons in each spatial mode before they perform their MEPP. In detail, the different phase shifts of the coherent beam $|\alpha\rangle_p$ represent the different numbers of photons passing through the two spatial modes.

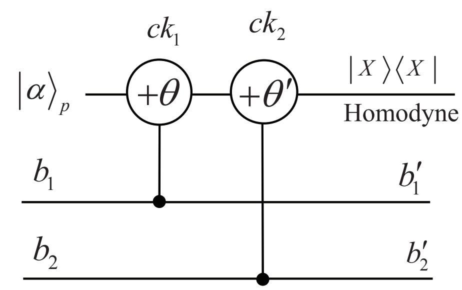

FIG. 6: The principle of PNDs with cross-Kerr nonlinear media. $+\theta$ and $+\theta'$ represent two different cross-Kerr nonlinear media which introduce the phase shifts $+\theta$ and $+\theta'$ when there is a photon passing through the media, respectively.

To summary, we have proposed an efficient MEPP for N-photon systems in a GHZ state. It contains two parts. One is our conventional MEPP with which the parties can obtain a high-fidelity N-photon ensemble directly. The other is our recycling MEPP in which entanglement link is used to produce some N-photon entangled systems from subspaces. Our conventional MEPP is similar to the conventional MEPPs with perfect CNOT gates [35], but it doubles the efficiency of the MEPP with QNDs based on cross-Kerr nonlinearity in Ref. [36] and the MEPP for electronic systems [37]. In the present MEPP, the parties can obtain not only high-fidelity N-photon entangled systems directly but also high-fidelity N'-photon entangled subsystems ($2 \leq N' < N$). With entanglement link, the parties can produce some high-fidelity N-photon entangled systems from N'-photon entangled subsystems. In the entanglement purification for phase-flip errors, the parties can only perform the conventional process as a phase-flip error in GHZ states is an unlocal one. The present MEPP has higher efficiency than the conventional MEPPs [35–37] as the cross-combinations, which are just the discarded instances in the latter, can be used to distil high-fidelity N'-photon entangled subsystems. We discuss the principle of our MEPP with the entanglement link in detail for the three-photon. The result can be generalized to the case with N-photon systems. Moreover, the present MEPP works by replacing parity-check detectors with CNOT gates or replacing the polarization degree of freedom of N-photon systems with others.

ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation of China under Grant Nos. 10974020 and 11174039, A Foundation for the Author of National Excellent Doctoral Dissertation of P. R. China under Grant No. 200723, Beijing Natural Science Foundation under Grant No. 1082008, and the Fundamental Research Funds for the Central Universities.

Appendix A: High-efficiency multipartite entanglement purification with entanglement link

1. Conventional N-photon entanglement purification for bit-flip errors and phase-flip errors

The GHZ state of a multipartite entangled system composed of N two-level particles can be described as

$$|\Phi_0^+\rangle_N = \frac{1}{\sqrt{2}}(|HH\cdots H\rangle + |VV\cdots V\rangle)_{A,B,\cdots,Z}. \quad \text{(A1)}$$

Here the subscripts A, B, $\cdots$ and Z represent the photons sent to the parties Alice, Bob, $\cdots$, and Zach, respectively. Certainly, there are another $2^N - 1$ GHZ states for an N-qubit system and can be written as

$$|\Phi_{ij\cdots k}^+\rangle_N = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle + |\overline{i}\overline{j}\cdots \bar{k}\rangle)_{AB\cdots C} \quad \text{(A2)}$$

and

$$|\Phi_{ij\cdots k}^-\rangle_N = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle - |\overline{i}\overline{j}\cdots \bar{k}\rangle)_{AB\cdots C}. \quad \text{(A3)}$$

Here $\bar{i} = 1 - i$, $\bar{j} = 1 - j$, $\bar{k} = 1 - k$, and $i, j, k \in \{0, 1\}$. $|0\rangle \equiv |H\rangle$ and $|1\rangle \equiv |V\rangle$.

For correcting the bit-flip errors in multipartite entangled quantum systems, we can also divide the whole entanglement purification into two steps. One is the conventional multipartite entanglement purification and the other is the entanglement purification with entanglement link from subspaces. The conventional entanglement purification for multipartite entangled quantum systems with bit-flip errors is similar to that for three-photon entangled quantum systems. We should only increase the number of the QNDs and the Hadamard operations shown in Fig. 2. Let us use a simple example to describe the principle of the conventional entanglement purification for N-photon systems. That is, let us assume that the ensemble of photon systems after the transmission over a noisy channel is in the state

$$\begin{aligned}\rho_N \;=\; & f_0|\Phi_0^+\rangle_N\langle\Phi_0^+| + \cdots + f_{ij\cdots k}|\Phi_{ij\cdots k}^+\rangle_N\langle\Phi_{ij\cdots k}^+| \\ & + \cdots + f_{2^{N-1}-1}|\Phi_{2^{N-1}-1}^+\rangle_N\langle\Phi_{2^{N-1}-1}^+|. \quad \text{(A4)}\end{aligned}$$

Here $f_{ij\cdots k}$ presents the probability that an N-photon system is in the state $|\Phi_{ij\cdots k}^+\rangle_N$ and

$$f_0 + \cdots + f_{ij\cdots k} + \cdots + f_{2^{N-1}-1} = 1. \quad \text{(A5)}$$

In conventional multipartite entanglement purification for each two N-photon systems, the parties in quantum communication will keep the identity-combinations $|\Phi_0^+\rangle_N \otimes |\Phi_0^+\rangle_N$, $\cdots$, $|\Phi_{ij\cdots k}^+\rangle_N \otimes |\Phi_{ij\cdots k}^+\rangle_N$, $\cdots$, and $|\Phi_{2^{N-1}-1}^+\rangle_N \otimes |\Phi_{2^{N-1}-1}^+\rangle_N$ with the probabilities f_0^2, $\cdots$, $f_{ij\cdots k}^2$, $\cdots$, and $f_{2^{N-1}-1}^2$, respectively. That is, they keep the instances in which they all obtain the even parity and those in which they all obtain the odd parity with their QNDs.

When all the parties obtain the even parity, the system with $2N$ photons is in the state $|\phi_{ij\cdots k}\rangle_{2N} = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle|ij\cdots k\rangle + |\overline{i}\overline{j}\cdots\bar{k}\rangle|\overline{i}\overline{j}\cdots\bar{k}\rangle)_{A_1,B_1,\cdots,Z_1,A_2,B_2,\cdots,Z_2}$ with the probability $\frac{1}{2}f_{ij\cdots k}^2$. When they all obtain the odd parity, the system is in the state $|\psi_{ij\cdots k}\rangle_{2N} = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle|\overline{i}\overline{j}\cdots\bar{k}\rangle + |\overline{i}\overline{j}\cdots\bar{k}\rangle|ij\cdots k\rangle)_{A_1,B_1,\cdots,Z_1,A_2,B_2,\cdots,Z_2}$ with the probability $\frac{1}{2}f_{ij\cdots k}^2$. Certainly, the parties can transform the states $|\psi_{ij\cdots k}\rangle_{2N}$ $(i, j, k = 0, 1,)$ into the states $|\phi_{ij\cdots k}\rangle_{2N}$ with a bit-flip operation $\sigma_x = |H\rangle\langle V| + |V\rangle\langle H|$ on each of the N photons A_2, B_2, $\cdots$, and Z_2.

By measuring the photons A_2, B_2, $\cdots$, and Z_2 with the basis $X = \{|\pm\rangle\}$ and performing some unitary operations or not, the parties can obtain a new N-photon system which is in the states $|\Phi_0\rangle_N = \frac{1}{\sqrt{2}}(|HH\cdots H\rangle + |VV\cdots V\rangle)_{A_1,B_1,\cdots,Z_1}$, $\cdots$, $|\Phi_{ij\cdots k}\rangle_N = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle + |\overline{i}\overline{j}\cdots\bar{k}\rangle)_{A_1,B_1,\cdots,Z_1}$, $\cdots$, and $|\Phi_{2^{N-1}-1}\rangle_N = \frac{1}{\sqrt{2}}(|HV\cdots V\rangle + |VH\cdots H\rangle)_{A_1,B_1,\cdots,Z_1,A_2,B_2,\cdots,Z_2}$ with the probabilities f_0^2, $\cdots$, $f_{ij\cdots k}^2$, $\cdots$, and $f_{2^{N-1}-1}^2$, respectively, similar to the conventional entanglement purification for three-photon systems shown in Sec.II B. In this way, the parties can obtain a new ensemble of N-photon systems ρ_N' with the fidelity $f_0' =$

$\frac{f_0^2}{f_0^2+\cdots+f_{ij\cdots k}^2+\cdots+f_{2^{N-1}-1}^2}$ from the original ensemble in the state ρ_N.

In the conventional N-photon entanglement purification for phase-flip errors, the parties can only obtain a new assemble, similar to the case for three-photon systems. In detail, with a phase-flip error on the state $|\Phi_0^+\rangle_N$, it becomes $|\Phi_0^-\rangle_N$. Here

$$|\Phi_0^-\rangle_N = \frac{1}{\sqrt{2}}(|HH\cdots H\rangle - |VV\cdots V\rangle)_{A,B,\cdots,Z}. \quad \text{(A6)}$$

With a Hadamard operation on each photon, the states $|\Phi_0^+\rangle_N$ and $|\Phi_0^-\rangle_N$ are transformed into the states $|\Psi_0^+\rangle_N$ and $|\Psi_0^-\rangle_N$, respectively. Here

$$\begin{aligned}
|\Psi_0^+\rangle_N &= \frac{1}{2^{\frac{N+1}{2}}}[(|H\rangle+|V\rangle)_A(|H\rangle+|V\rangle)_B\cdots(|H\rangle+|V\rangle)_Z \\
&+ (|H\rangle-|V\rangle)_A(|H\rangle-|V\rangle)_B\cdots(|H\rangle-|V\rangle)_Z], \\
|\Psi_0^-\rangle_N &= \frac{1}{2^{\frac{N+1}{2}}}[(|H\rangle+|V\rangle)_A(|H\rangle+|V\rangle)_B\cdots(|H\rangle+|V\rangle)_Z \\
&- (|H\rangle-|V\rangle)_A(|H\rangle-|V\rangle)_B\cdots(|H\rangle-|V\rangle)_Z].
\end{aligned}$$

The number of $|V\rangle$ is even in each item of the state $|\Psi_0^+\rangle_N$, while it is odd in the state $|\Psi_0^-\rangle_N$. With this feature, the relative probability of the state $|\Psi_0^-\rangle_N$ will be depressed.

The density matrix of an ensemble over noisy channels with only phase-flip errors can be written as

$$\rho'_N = P_0|\Psi_0^+\rangle_N\langle\Psi_0^+| + P_1|\Psi_0^-\rangle_N\langle\Psi_0^-|. \quad \text{(A7)}$$

Here P_0 and P_1 represent the probabilities of the states $|\Psi_0^+\rangle_N$ and $|\Psi_0^-\rangle_N$ in the ensemble ρ'_N, respectively, and $P_0 + P_1 = 1$. For each pair of the entangled N-photon systems picked out from this ensemble, say $A_1B_1\cdots Z_1$ and $A_2B_2\cdots Z_2$, their state can be viewed as the mixture of four pure states, i.e., $|\Psi_0^+\rangle_N\otimes|\Psi_0^+\rangle_N$, $|\Psi_0^+\rangle_N\otimes|\Psi_0^-\rangle_N$, $|\Psi_0^-\rangle_N\otimes|\Psi_0^+\rangle_N$, and $|\Psi_0^-\rangle_N\otimes|\Psi_0^-\rangle_N$ with the probabilities P_0^2, P_0P_1, P_1P_0, and P_1^2, respectively.

The parties let their photons pass through their QNDs, as the same as those shown in Fig.2. That is, Alice lets her photons A_1 and A_2 pass through her QND. So do the other parties. The parties only keep the instances in which the number of odd parities is even when they perform a parity-check measurement on their two photons independently. In these instances, the photons come from the state $|\Psi_0^+\rangle_N\otimes|\Psi_0^+\rangle_N$ or $|\Psi_0^-\rangle_N\otimes|\Psi_0^-\rangle_N$. It is not difficult to prove that the instances come from the cross-combinations $|\Psi_0^-\rangle_N\otimes|\Psi_0^+\rangle_N$ and $|\Psi_0^+\rangle_N\otimes|\Psi_0^-\rangle_N$ lead the number of odd parities to be odd. By performing the conventional N-photon entanglement purification for phase-flip errors, similar to the case for three-photon systems, the N parties can obtain a new ensemble in the states $|\Phi_0^+\rangle_N$ and $|\Phi_0^-\rangle_N$ with the probabilities $\frac{P_0^2}{P_0^2+P_1^2}$ and $\frac{P_1^2}{P_0^2+P_1^2}$, respectively. The fidelity of the state $|\Phi_0^+\rangle_N$ is $P'_0 = \frac{P_0^2}{P_0^2+P_1^2} = \frac{P_0^2}{P_0^2+(1-P_0)^2}$. When $P_0 > 1/2$, $P'_0 > P_0$.

2. High-efficiency multipartite entanglement purification for bit-flip errors with entanglement link

In the conventional multipartite entanglement purification for bit-flip errors, the parties do not take the cross-combinations $|\Phi_{lr\cdots q}^+\rangle_N\otimes|\Phi_{ij\cdots k}^+\rangle_N$ and $|\Phi_{ij\cdots k}^+\rangle_N\otimes|\Phi_{lr\cdots q}^+\rangle_N$ ($l, r, \cdots, q \in \{0,1\}$ and $l \neq i$, $r \neq j$, $\cdots$, or $q \neq k$) into account for obtaining some high-fidelity N-photon entangled systems. However, the parties can first obtain some high-fidelity N'-photon entangled systems ($2 \leq N' < N$) from the cross-combinations and then obtain some high-fidelity N-photon entangled systems with entanglement link, similar to the entanglement purification for three-photon entangled systems (as discussed in Sec.II C 1 and Sec. II C 2). In this time, the probability that the $2N$-photon system is in the cross-combination $|\Phi_{lr\cdots q}^+\rangle_N\otimes|\Phi_{ij\cdots k}^+\rangle_N$ or $|\Phi_{ij\cdots k}^+\rangle_N\otimes|\Phi_{lr\cdots q}^+\rangle_N$ is $f_{lr\cdots q}f_{ij\cdots k}$. The high-fidelity entangled subsystem which can be obtained by the N parties is composed of N'_s photons. Here N'_s is a integer number and it is larger than $\frac{N-1}{2}$ but not larger than $\frac{N+1}{2}$. Certainly, the more the number of the photons in each system, the more the kinds of the entanglement purification with entanglement link. Let us use four-photon systems as an example to describe the principle of high-efficiency multipartite entanglement purification with entanglement link. It is more complicated than that in the case with three-photon systems.

For four-photon systems, the sixteen GHZ states can be written as

$$\begin{aligned}
|\Phi_0^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HHHH\rangle \pm |VVVV\rangle)_{ABCD}, \\
|\Phi_1^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HHHV\rangle \pm |VVVH\rangle)_{ABCD}, \\
|\Phi_2^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HHVH\rangle \pm |VVHV\rangle)_{ABCD}, \\
|\Phi_3^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HHVV\rangle \pm |VVHH\rangle)_{ABCD}, \\
|\Phi_4^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HVHH\rangle \pm |VHVV\rangle)_{ABCD}, \\
|\Phi_5^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HVHV\rangle \pm |VHVH\rangle)_{ABCD}, \\
|\Phi_6^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HVVH\rangle \pm |VHHV\rangle)_{ABCD}, \\
|\Phi_7^\pm\rangle_{ABCD} &= \frac{1}{\sqrt{2}}(|HVVV\rangle \pm |VHHH\rangle)_{ABCD}. \quad \text{(A8)}
\end{aligned}$$

Here the subscripts A, B, C, and D represent the photons kept by the parties in quantum communication Alice, Bob, Charlie, and Dean, respectively. When we only consider the entanglement purification for bit-flip errors, we assume that the ensemble of four-photon systems after the transmission over a noisy channel is in the state

$$\rho_4 = \sum_{m=0}^{7} f''_m|\Phi_m^+\rangle_{ABCD}\langle\Phi_m^+|. \quad \text{(A9)}$$

Except for the 8 identity-combinations $|\Phi_m^+\rangle_{A_1B_1C_1D_1} \otimes |\Phi_m^+\rangle_{A_2B_2C_2D_2}$ kept for obtaining some four-photon systems with a high fidelity, the 56 cross-combinations $|\Phi^+_{lr\cdots q}\rangle_4 \otimes |\Phi^+_{ij\cdots k}\rangle_4$ are discarded in the conventional four-photon entanglement purification. In fact, the parties can first distil some three-photon entangled systems and two-photon entangled systems from the cross-combinations, and then obtain some four-photon entangled systems with entanglement link.

When the parities of the four parties with QNDs are even, even, even, and odd (we abbreviate them as (even, even, even, odd) below), the 8-photon system $A_1B_1C_1D_1A_2B_2C_2D_2$ is in the states

$$\begin{aligned}
|\zeta_1\rangle &= \frac{1}{\sqrt{2}}(|HHHH\rangle_{A_1B_1C_1D_1}|HHHV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVVV\rangle_{A_1B_1C_1D_1}|VVVH\rangle_{A_2B_2C_2D_2}), \\
|\zeta_2\rangle &= \frac{1}{\sqrt{2}}(|HHHV\rangle_{A_1B_1C_1D_1}|HHHH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVVH\rangle_{A_1B_1C_1D_1}|VVVV\rangle_{A_2B_2C_2D_2}), \\
|\zeta_3\rangle &= \frac{1}{\sqrt{2}}(|HHVH\rangle_{A_1B_1C_1D_1}|HHVV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVHV\rangle_{A_1B_1C_1D_1}|VVHH\rangle_{A_2B_2C_2D_2}), \\
|\zeta_4\rangle &= \frac{1}{\sqrt{2}}(|HHVV\rangle_{A_1B_1C_1D_1}|HHVH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVHH\rangle_{A_1B_1C_1D_1}|VVHV\rangle_{A_2B_2C_2D_2}), \\
|\zeta_5\rangle &= \frac{1}{\sqrt{2}}(|HVHH\rangle_{A_1B_1C_1D_1}|HVHV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHVV\rangle_{A_1B_1C_1D_1}|VHVH\rangle_{A_2B_2C_2D_2}), \\
|\zeta_6\rangle &= \frac{1}{\sqrt{2}}(|HVHV\rangle_{A_1B_1C_1D_1}|HVHH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHVH\rangle_{A_1B_1C_1D_1}|VHVV\rangle_{A_2B_2C_2D_2}), \\
|\zeta_7\rangle &= \frac{1}{\sqrt{2}}(|HVVH\rangle_{A_1B_1C_1D_1}|HVVV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHHV\rangle_{A_1B_1C_1D_1}|VHHH\rangle_{A_2B_2C_2D_2}),
\end{aligned}$$

and

$$\begin{aligned}
|\zeta_8\rangle &= \frac{1}{\sqrt{2}}(|HVVV\rangle_{A_1B_1C_1D_1}|HVVH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHHH\rangle_{A_1B_1C_1D_1}|VHHV\rangle_{A_2B_2C_2D_2})
\end{aligned}$$

with the probabilities of $\frac{1}{2}f_0''f_1''$, $\frac{1}{2}f_0''f_1''$, $\frac{1}{2}f_2''f_3''$, $\frac{1}{2}f_2''f_3''$, $\frac{1}{2}f_4''f_5''$, $\frac{1}{2}f_4''f_5''$, $\frac{1}{2}f_6''f_7''$, and $\frac{1}{2}f_6''f_7''$, respectively.

The four parties in quantum communication can obtain a three-photon system by performing a Hadamard operation on each of the five photons $D_1A_2B_2C_2D_2$ and then measuring them with the basis Z. When the number of the outcomes $|V\rangle$ in the measurements is even, the three-photon system $A_1B_1C_1$ is in the states $|\Phi_0^+\rangle_{A_1B_1C_1}$, $|\Phi_1^+\rangle_{A_1B_1C_1}$, $|\Phi_2^+\rangle_{A_1B_1C_1}$, and $|\Phi_3^+\rangle_{A_1B_1C_1}$ with the probabilities $\frac{1}{2}f_0''f_1''$, $\frac{1}{2}f_2''f_3''$, $\frac{1}{2}f_4''f_5''$, and $\frac{1}{2}f_6''f_7''$, respectively. When the number of the outcomes $|V\rangle$ is odd, the system $A_1B_1C_1$ is in the states $|\Phi_0^-\rangle_{A_1B_1C_1}$, $|\Phi_1^-\rangle_{A_1B_1C_1}$, $|\Phi_2^-\rangle_{A_1B_1C_1}$, and $|\Phi_3^-\rangle_{A_1B_1C_1}$ with the probabilities $\frac{1}{2}f_0''f_1''$, $\frac{1}{2}f_2''f_3''$, $\frac{1}{2}f_4''f_5''$, and $\frac{1}{2}f_6''f_7''$, respectively. With a phase-flip operation σ_z on one of the three photons $A_1B_1C_1$, the parties can obtain the state $|\Phi_m^+\rangle_{A_1B_1C_1}$ from the state $|\Phi_m^-\rangle_{A_1B_1C_1}$ $(m = 0, 1, 2, 3)$. That is, the three parties Alice, Bob, and Charlie obtain a three-photon ensemble in the state

$$\rho''_{ABC} = \sum_{m=0}^{3} f_m'''|\Phi_m^+\rangle_{ABC}\langle\Phi_m^+|, \tag{A10}$$

where

$$\begin{aligned}
f_0''' &= \frac{f_0''f_1''}{f_0''f_1''+f_2''f_3''+f_4''f_5''+f_6''f_7''}, \\
f_1''' &= \frac{f_2''f_3''}{f_0''f_1''+f_2''f_3''+f_4''f_5''+f_6''f_7''}, \\
f_2''' &= \frac{f_4''f_5''}{f_0''f_1''+f_2''f_3''+f_4''f_5''+f_6''f_7''}, \\
f_3''' &= \frac{f_6''f_7''}{f_0''f_1''+f_2''f_3''+f_4''f_5''+f_6''f_7''}.
\end{aligned} \tag{A11}$$

When the parities are (even, even, odd, even), the three parties Alice, Bob, and Dean obtain a three-photon ensemble ρ''_{ABD}, similar to the case with the outcomes (even, even, even, odd). So do the ensembles ρ''_{ACD} and ρ''_{BCD}.

When two of the four parties obtain the odd parity with their QNDs, they can obtain a two-photon ensemble ρ''_{AB}, ρ''_{AC}, ρ''_{AD}, ρ''_{BC}, ρ''_{BD} or ρ''_{CD}, by performing some Hadamard operations and measurements with the basis Z. Let us use the outcomes (even, even, odd, odd) as an example to describe the principle. In this time, the 8-photon system is in the states

$$\begin{aligned}
|\xi_1\rangle &= \frac{1}{\sqrt{2}}(|HHHH\rangle_{A_1B_1C_1D_1}|HHVV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVVV\rangle_{A_1B_1C_1D_1}|VVHH\rangle_{A_2B_2C_2D_2}), \\
|\xi_2\rangle &= \frac{1}{\sqrt{2}}(|HHVV\rangle_{A_1B_1C_1D_1}|HHHH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVHH\rangle_{A_1B_1C_1D_1}|VVVV\rangle_{A_2B_2C_2D_2}), \\
|\xi_3\rangle &= \frac{1}{\sqrt{2}}(|HHHV\rangle_{A_1B_1C_1D_1}|HHVH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVVH\rangle_{A_1B_1C_1D_1}|VVHV\rangle_{A_2B_2C_2D_2}), \\
|\xi_4\rangle &= \frac{1}{\sqrt{2}}(|HHVH\rangle_{A_1B_1C_1D_1}|HHHV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VVHV\rangle_{A_1B_1C_1D_1}|VVVH\rangle_{A_2B_2C_2D_2}), \\
|\xi_5\rangle &= \frac{1}{\sqrt{2}}(|HVHH\rangle_{A_1B_1C_1D_1}|HVVV\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHVV\rangle_{A_1B_1C_1D_1}|VHHH\rangle_{A_2B_2C_2D_2}), \\
|\xi_6\rangle &= \frac{1}{\sqrt{2}}(|HVVV\rangle_{A_1B_1C_1D_1}|HVHH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHHH\rangle_{A_1B_1C_1D_1}|VHVV\rangle_{A_2B_2C_2D_2}), \\
|\xi_7\rangle &= \frac{1}{\sqrt{2}}(|HVHV\rangle_{A_1B_1C_1D_1}|HVVH\rangle_{A_2B_2C_2D_2} \\
&\quad + |VHVH\rangle_{A_1B_1C_1D_1}|VHHV\rangle_{A_2B_2C_2D_2}),
\end{aligned}$$

and

$$|\xi_8\rangle = \frac{1}{\sqrt{2}}(|HVVH\rangle_{A_1B_1C_1D_1}|HVHV\rangle_{A_2B_2C_2D_2} + |VHHV\rangle_{A_1B_1C_1D_1}|VHVH\rangle_{A_2B_2C_2D_2})$$

with the probabilities of $\frac{1}{2}f_0''f_3''$, $\frac{1}{2}f_0''f_3''$, $\frac{1}{2}f_1''f_2''$, $\frac{1}{2}f_1''f_2''$, $\frac{1}{2}f_4''f_7''$, $\frac{1}{2}f_4''f_7''$, $\frac{1}{2}f_5''f_6''$, and $\frac{1}{2}f_5''f_6''$, respectively. For obtaining two-photon entangled systems, Alice, Bob, Charlie, and Dean first perform a Hadamard operation on each of the six photons $C_1D_1A_2B_2C_2D_2$ and then measure them with the basis Z. When the number of the outcomes $|V\rangle$ is even, the two-photon system A_1B_1 is in the states $|\phi^+\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{A_1B_1}$ and $|\psi^+\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|HV\rangle+|VH\rangle)_{A_1B_1}$ with the probabilities $\frac{1}{2}(f_0''f_3''+f_1''f_2'')$ and $\frac{1}{2}(f_4''f_7''+f_5''f_6'')$, respectively. When the number of the outcomes $|V\rangle$ is odd, the two-photon system A_1B_1 is in the states $|\phi^-\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle)_{A_1B_1}$ and $|\psi^-\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)_{A_1B_1}$ with the probabilities $\frac{1}{2}(f_0''f_3'' + f_1''f_2'')$ and $\frac{1}{2}(f_4''f_7'' + f_5''f_6'')$, respectively. With a phase-flip operation σ_z on the photon A_1, the states $|\phi^-\rangle_{A_1B_1}$ and $|\psi^-\rangle_{A_1B_1}$ will be transformed into the states $|\phi^+\rangle_{A_1B_1}$ and $|\psi^+\rangle_{A_1B_1}$, respectively. That is, Alice and Bob obtain a two-photon ensemble in the state

$$\rho''_{AB} = f^0_{AB}|\phi^+\rangle_{AB}\langle\phi^+| + f^1_{AB}|\psi^+\rangle_{AB}\langle\psi^+|, \quad \text{(A12)}$$

where

$$f^0_{AB} = \frac{f_0''f_3''+f_1''f_2''}{f_0''f_3''+f_1''f_2''+f_4''f_7''+f_5''f_6''},$$
$$f^1_{AB} = \frac{f_4''f_7''+f_5''f_6''}{f_0''f_3''+f_1''f_2''+f_4''f_7''+f_5''f_6''}. \quad \text{(A13)}$$

With three-photon ensembles ρ''_{ABC}, ρ''_{ABD}, ρ''_{ACD}, and ρ''_{BCD}, and two-photon ensembles ρ''_{AB}, ρ''_{AC}, ρ''_{AD}, ρ''_{BC}, ρ''_{BD}, and ρ''_{CD}, Alice, Bob, Charlie, and Dean can obtain some four-photon systems with entanglement link. In detail, for a system composed of a three-photon entangled subsystem ABC and a two-photon entangled subsystem A_1D, Alice, Bob, Charlie, and Dean can obtain a four-photon entangled system $ABCD$ by performing a QND measurement on the photons AA_1 and performing a single-photon measurement on the photon A_1 with the basis Z after Alice takes a Hadamard operation on the photon A_1, as shown in Fig. 7. Certainly, they can obtain a four-photon entangled system $ABCD$ from the complicated system composed of two three-photon subsystems ABC and A_1C_2D by performing a QND measurement on the photons A and A_1, and another on the photons C and C_1, shown in Fig. 8. By performing a single-photon measurement on the photon A_1 and another on the photon C_1 after a Hadamard operation on each of these two photons, Alice, Bob, Charlie, and Dean can obtain a four-photon entangled system. Also, the four parties can obtain a four-photon entangled systems from three two-photon entangled subsystems, shown in Fig. 9.

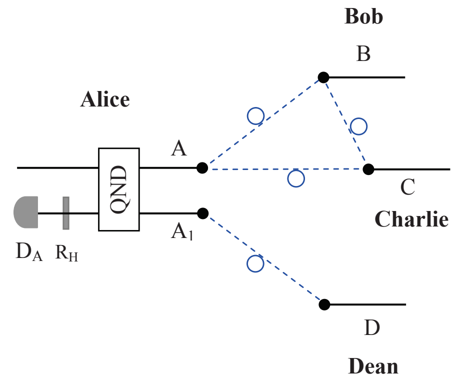

FIG. 7: (Color online) The principle of the entanglement link for producing a four-photon entangled system from a three-photon entangled subsystem and a two-photon entangled subsystem with a QND.

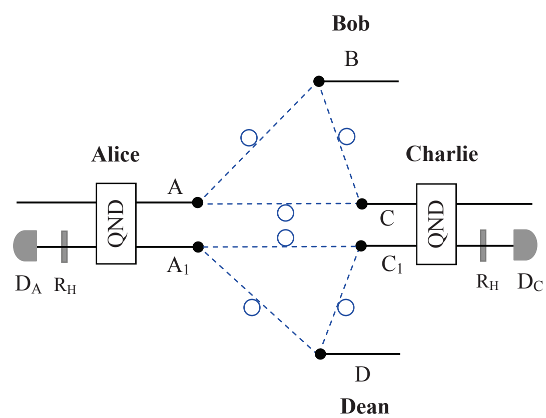

FIG. 8: (Color online) The principle of the entanglement link for producing a four-photon entangled system from two three-photon entangled subsystems with two QNDs.

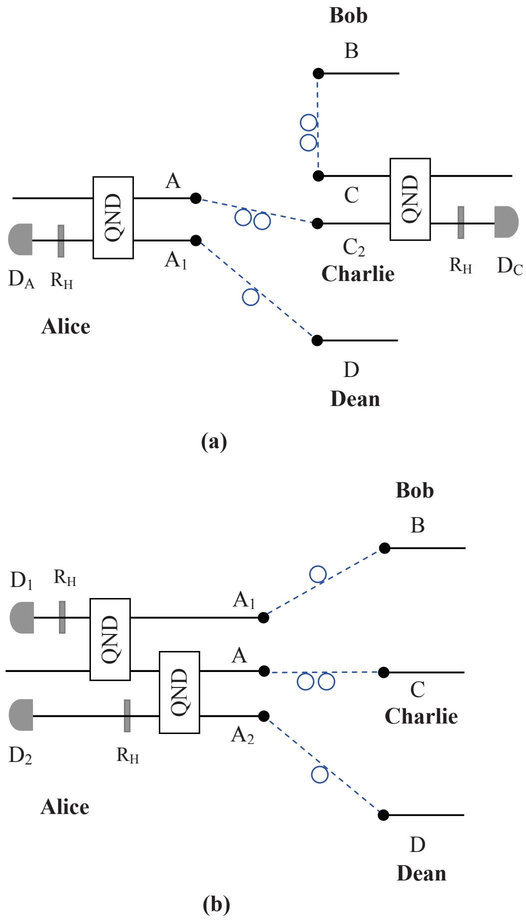

FIG. 9: (Color online) The principle of the entanglement link for producing a four-photon entangled subsystem from three two-photon entangled subsystems with two QNDs. There are two topological structures: (a) a symmetrical structure in which Alice shares a two-photon entangled system with Charlie and another with Dean, and Charlie shares another two-photon entangled system with Bob; (b) an unsymmetrical structure in which Alice shares a two-photon entangled system with each of the other three parties.

[1] M. A. Nielsen and I. L. Chuang *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[4] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. **68**, 557 (1992).
[5] G. L. Long and X. S. Liu, Phys. Rev. A **65**, 032302 (2002).
[6] X. H. Li, F. G. Deng, and H. Y. Zhou, Phys. Rev. A **78**, 022321 (2008).
[7] S. Pironio, A. Acín, N. Brunner, N. Gisin, S. Massar, and V. Scarani, New J. Phys. **11**, 045021 (2009).
[8] C. H. Bennett and G. Brassard, *in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984), pp. 175-179.
[9] F. G. Deng and G. L. Long, Phys. Rev. A **69**, 052319 (2004).
[10] W. Y. Hwang, Phys. Rev. Lett. **91**, 057901 (2003).
[11] F. G. Deng and G. L. Long, Phys. Rev. A **70**, 012311 (2004).
[12] X. B. Wang, Phys. Rev. Lett. **94**, 230503 (2005).
[13] H. K. Lo, X. F. Ma, and K. Chen, Phys. Rev. Lett. **94**, 230504 (2005).
[14] T. Schmitt-Manderbach, H. Weier, M. Furst, R. Ursin, F. Tiefenbacher, T. Scheidl, J. Perdigues, Z. Sodnik, C. Kurtsiefer, J. G. Rarity, A. Zeilinger, and H. Weinfurter, Phys. Rev. Lett. **98**, 010504 (2007).
[15] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[16] X. S. Liu, G. L. Long, D. M. Tong, and L. Feng, Phys. Rev. A **65**, 022304 (2002).
[17] A. Grudka and A. Wójcik, Phys. Rev. A **66**, 014301 (2002).
[18] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).
[19] A. Karlsson, M. Koashi, and N. Imoto, Phys. Rev. A **59** 162 (1999).
[20] L. Xiao, G. L. Long, F. G. Deng, and J. W. Pan, Phys. Rev. A **69**, 052307 (2004).
[21] R. Cleve, D. Gottesman, and H. K. Lo, Phys. Rev. Lett. **83**, 648 (1999).
[22] A. M. Lance, T. Symul, W. P. Bowen, B. C. Sanders, and P. K. Lam, Phys. Rev. Lett. **92**, 177903 (2004).
[23] F. G. Deng, X. H. Li, C. Y. Li, P. Zhou, and H. Y. Zhou, Phys. Rev. A **72**, 044301 (2005); *ibid*, Europ. Phys. J. D **39**, 459 (2006).
[24] X. H. Li, P. Zhou, C. Y. Li, H. Y. Zhou, and F. G. Deng, J. Phys. B **39**, 1975 (2006).
[25] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[26] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Phys. Rev. Lett. **77**, 2818 (1996).
[27] J. W. Pan, C. Simon, and A. Zellinger, Nature (London) **410**, 1067 (2001).
[28] C. Simon and J. W. Pan, Phys. Rev. Lett. **89**, 257901 (2002).
[29] Y. B. Sheng, F. G. Deng, and H. Y. Zhou, Phys. Rev. A **77**, 042308 (2008).
[30] Y. B. Sheng and F. G. Deng, Phys. Rev. A **81**, 032307 (2010).
[31] Y. B. Sheng and F. G. Deng, Phys. Rev. A **82**, 044305 (2010).
[32] X. H. Li, Phys. Rev. A **82**, 044304 (2010).
[33] F. G. Deng, Phys. Rev. A **83**, 062316 (2011).
[34] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[35] M. Murao, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight, Phys. Rev. A **57**, R4075 (1998).
[36] Y. B. Sheng, F. G. Deng, and H. Y. Zhou, Eur. Phys. J. D **55**, 235 (2009).
[37] Y. B. Sheng, F. G. Deng, and G. L. Long, Phys. Lett. A **375**, 396 (2010).
[38] K. Nemoto and W. J. Munro, Phys. Rev. Lett. **93**, 250502 (2004).
[39] S. D. Barrett, P. Kok, K. Nemoto, R. G. Beausoleil, W. J. Munro, and T. P. Spiller, Phys. Rev. A **71**, 060302(R) (2005).
[40] Y. B. Sheng, F. G. Deng, and G. L. Long, Phys. Rev. A. **82**, 032318 (2010).
[41] C. Wang, Y. Zhang, and G. S. Jin, Quant. Inform. Comput. **11**, 988 (2011).
[42] Y. B. Sheng, F. G. Deng, and H. Y. Zhou, Phys. Rev. A. **77**, 062325 (2008); *ibid*, Quant. Inform. Comput. **10**, 272 (2010).
[43] S. D. Barrett, P. Kok, K. Nemoto, R. G. Beausoleil, W. J. Munro, and T. P. Spiller, Phys. Rev. A **71**, 060302 (2005).
[44] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowing, and G. J. Milburn, Rev. Mod. Phys. **79**, 135 (2007).
[45] P. Kok, Phys. Rev. A **77**, 013808 (2008).
[46] H. F. Hofmann, K. Kojima, S. Takeuchi, and K. Sasaki, J. Opt. B: Quantum Semiclassical Opt. **5**, 218 (2003).
[47] C. Wittmann, U. L. Andersen, M. Takeoka, and G. Leuchs, e-print arXiv: 1002.0232.
[48] J. Gea-Banacloche, Phys. Rev. A **81**, 043823 (2010).
[49] J. H. Shapiro, Phys. Rev. A **73**, 062305 (2006).
[50] J. H. Shapiro and M. Razavi, New J. Phys. **9**, 16 (2007).
[51] I. Friedler, D. Petrosyan, M. Fleischhauer, and G. Kurizki, Phys. Rev. A **72**, 043803 (2005).
[52] I. Friedler, G. Kurizki, and D. Petrosyan, Phys. Rev. A **71**, 023803 (2005).
[53] I. Friedler, G. Kurizki, and D. Petrosyan, Europhys. Lett. **68**, 625 (2004).
[54] G. J. Pryde, J. L. OBrien, A. G. White, S. D. Bartlett, and T. C. Ralph, Phys. Rev. Lett. **92**, 190402 (2004).
[55] T. C. Ralph, S. D. Bartlett, J. L. OBrien, G. J. Pryde, and H. M. Wiseman, Phys. Rev. A **73**, 012113 (2006).
[56] G. J. Pryde, J. L. OBrien, A. G. White, T. C. Ralph, and H. M. Wiseman, Phys. Rev. Lett. **94**, 220405 (2005).
[57] C. Bonato, F. Haupt, S. S. R. Oemrawsingh, J. Gudat, D. P. Ding, M. P. van Exter, and D. Bouwmeester, Phys. Rev. Lett., **104**, 160503 (2010).
[58] C. Wang, Y. Zhang, and G. S. Jin, Phys. Rev. A **84**, 032307 (2011).