Noise correlations in the expansion of an interacting one-dimensional Bose gas from a regular array

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Phys. Rev. A **84**, 043632 — Published 20 October 2011

DOI: 10.1103/PhysRevA.84.043632

Noise correlations in the expansion of an interacting 1D Bose gas from a regular array

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(Dated: September 27, 2011)

We consider the one dimensional expansion of a system of interacting bosons, starting from a regular array. Without interactions the familiar Hanbury Brown and Twiss effect for bosons gives rise to a series of peaks in the density-density correlations of the expanded system. Infinitely repulsive particles likewise give a series of dips, a signature of the underlying description in terms of free fermions. In the intermediate case of finite interaction the noise correlations consist of a set of Fano resonance lineshapes, with an asymmetry parameter determined by the scattering phase shift of a pair of particles, and a width depending on the initial momentum spread of the particles.

The Hanbury Brown and Twiss (HBT) effect [1] is a fundamental signature of quantum statistics appearing in quantum optics, atomic and mesoscopic physics, and nuclear collisions [2–6]. It is most dramatically manifested as an interference effect in the intensity correlations due to two or more incoherent sources, with a sign depending on the statistics of particles: positive correlations for bosons; negative for fermions. In Ref. [7] it was pointed out that the time-of-flight images ubiquitous in ultracold atomic physics contain such noise signals. After this realization noise correlations following expansion were used to characterize many-body states in a number of groundbreaking experiments [8–10].

In most known instances of the HBT effect interactions between particles do not play a significant role once expansion begins (or after particles leave the sources), either because these effects are weak or due to the spatial separation of the sources. In this paper we consider the one-dimensional expansion of a system of particles, where strong interaction effects are unavoidable. Indeed, in 1D the paths giving rise to the HBT effect must cross. Note that many works assume interactions are 'switched off' before expansion – see e.g. Ref. [11] – a technically difficult procedure involving rapidly changing scattering lengths near a Feshbach resonance.

The situation that we will consider is illustrated in Fig. 1. Particles are initially confined to a regular 1D lattice of spacing Δ, with one particle per site. At time $t = 0$ the lattice potential is removed, though the potential restricting the particles' motion to one dimension remains. We are concerned with the density correlations present after some time t, when the system has expanded to many times its original size (analogous to the 'far field' limit in optics). Thus we have in mind a 1D version of the experiment of Ref. [9], in which noise correlations were measured in the expansion of a 3D atomic Mott insulating state from an optical lattice. A recent experiment demonstrated the preparation of such a 1D state in a slightly different context [12].

If we restrict ourselves to the (physically realistic) case of δ-function interactions, the result in the impenetrable limit of infinite repulsion can be obtained immediately. In this case the system can be mapped (as far as any observable involving density is concerned) to a system of noninteracting *fermions* [13]. Thus the HBT effect in this limit will be that of free fermions. The aim of this paper is to describe the crossover from the bosonic to the fermionic HBT effect as interactions increase and identify signatures not present in either limit.

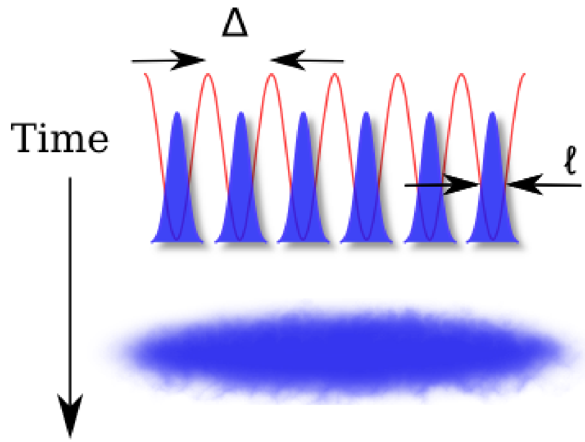

FIG. 1. (Color online) 1D expansion of atoms from an optical lattice of spacing Δ. The spread of the initial wavepackets is ℓ. Noise correlations will be present in an absorption image of the expanded cloud.

To introduce some ideas and notation we briefly describe the familiar HBT effect in this setting. We assume Gaussian initial wavefunctions corresponding to harmonic oscillator length $\ell = \sqrt{\hbar/m\omega}$, $\varphi_\alpha(y) = \frac{1}{(\pi\ell^2)^{1/4}} \exp\left[-\frac{(y-\alpha\Delta)^2}{2\ell^2}\right]$. The overlap $e^{-\Delta^2/4\ell^2}$ between neighboring sites is assumed to be negligible. After a period t of free evolution these wavefunctions have the form

$$\varphi_\alpha(x; t \gg \ell^2) \to \sqrt{\frac{\ell}{i\sqrt{\pi}t}} \exp\left[\left(\frac{i}{2t} - \frac{\ell^2}{2t^2}\right)(x-\alpha\Delta)^2\right]. \quad (1)$$

(Where we have set $\hbar = m = 1$) If we consider a pair of identical particles on sites α and $\alpha + 1$, the two-particle wavefunction is $\Psi_2(x_1, x_2; t) =$

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$\frac{1}{\sqrt{2}}\left[\varphi_\alpha(x_1;t)\varphi_{\alpha+1}(x_2;t) \pm \varphi_\alpha(x_2;t)\varphi_{\alpha+1}(x_1;t)\right]$, with $\pm$ for bosons and fermions respectively. The corresponding probability density is then

$$|\Psi_2(x_1,x_2;t)|^2 \to \frac{\ell^2}{\pi t^2} e^{-\ell^2(\xi_1^2+\xi_2^2)}\left[1 \pm \cos\left(\left[\xi_1-\xi_2\right]\Delta\right)\right], \tag{2}$$

where the variables $\xi_{1,2} = x_{1,2}/t$ correspond to the velocities of the two particles. The oscillatory second term describes the HBT effect, with a sign dependent on the statistics of the particles. For an array of N particles the density-density correlation function develops peaks due to the contributions of higher harmonics arising from pairs of particles separated by multiples of Δ

$$\mathcal{C}(x_1,x_2;t) \equiv \int dx_3\cdots dx_N |\Psi_N(x_1,x_2,\ldots,x_N;t)|^2.$$
$$\to \frac{\ell^2}{\pi t^2} e^{-\ell^2(\xi_1^2+\xi_2^2)}\left[1 \pm \frac{2\pi}{N}\sum_{n=-\infty}^{\infty}\delta(\Delta\left[\xi_1-\xi_2\right]-2\pi n)\right] \tag{3}$$

In a trajectory picture the HBT effect arises as a cross-term between trajectories that do and do not exchange pairs of particles (see Fig. 3, bottom)

We turn now to the central subject of this paper: the HBT effect in the presence of interactions between the particles. We assume that the evolution of the system for $t > 0$ is governed by the N-particle Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{i<j}\delta(x_i - x_j). \tag{4}$$

The $c \to 0$ and $c \to \infty$ limits can be described in terms of free bosons and free fermions, respectively. The density-density correlations reflect this, corresponding to the plus sign in Eq. (3) in the former case and the minus sign in the latter. Our main result is Eq. (16) below, valid when $e^{-2c\Delta} \ll 1$. We find that in the crossover regime the density-density correlations consist not of a series of symmetric peaks or dips but rather of *Fano lineshapes* (see Fig. 2)

$$\frac{\left[q_n\Gamma_n/2 + (\varepsilon-\eta_n)\right]^2}{\Gamma_n^2/4 + (\varepsilon-\eta_n)^2} \tag{5}$$

where $\varepsilon = \Delta(\xi_1 - \xi_2) - 2\pi n$ represents the deviation from the n^{th} peak. Γ_n and η_n are defined below Eq. (16), while the asymmetry parameter q_n is expressed in terms of the two particle scattering matrix

$$S(k) = -\frac{c - ik}{c + ik}, \tag{6}$$

by the relation

$$\arg S(2\pi n/\Delta) = 2q_n/(q_n^2 - 1). \tag{7}$$

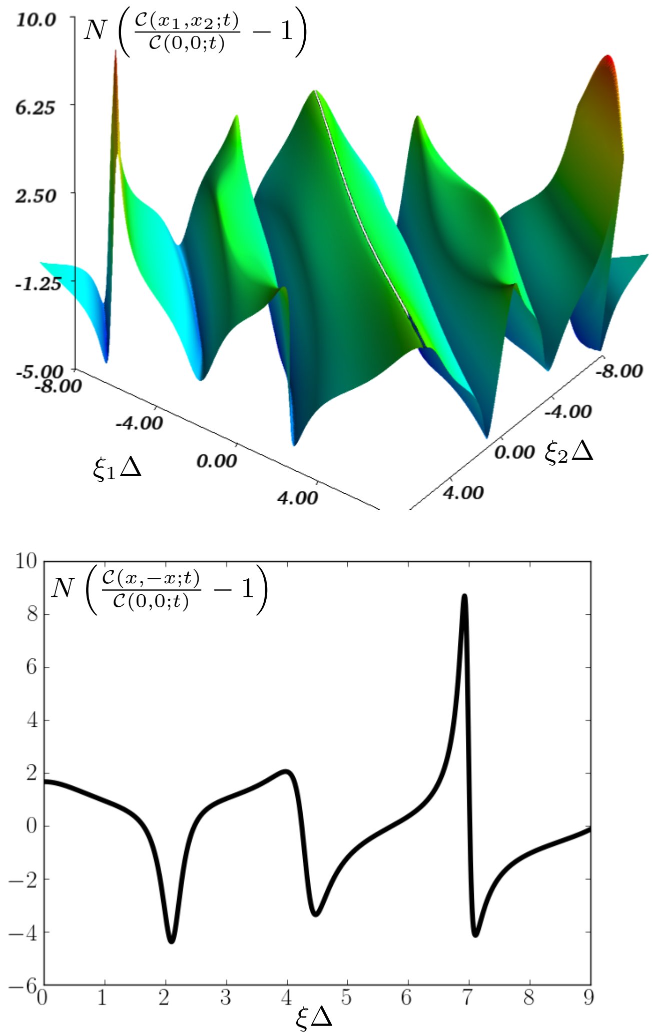

FIG. 2. (Color online) (Top) Normalized correlation function $N\left(\frac{\mathcal{C}(x_1,x_2;t)}{\mathcal{C}(0,0;t)}-1\right)$ from Eq. (16) for $c\Delta = 2$, $\ell/\Delta = 0.2$. (Bottom) A slice with $x_1 = -x_2$ for the same parameters, showing the evolution of the Fano asymmetry between successive peaks.

This illustrates the evolution from $q_n = \infty$ for free bosons (resonance lineshape) to $q_n \to 0$ as $c \to \infty$ (antiresonance). The asymmetry of the lineshape is the first qualitative feature of the crossover regime. The second is the finite width Γ_n, which vanishes in the two limits.

The surprising simplicity of our result is a consequence of the integrability of the Hamiltonian Eq. (4) [14]. The N-particle scattering it describes is *nondiffractive*, consisting of pairwise scattering that either preserves or exchanges the momenta of the scattering particles. It was realized only recently that this allows the time dependence of the N-particle propagator describing the amplitude for particles at $y_1,\ldots,y_N$ to arrive at $x_1,\ldots x_N$ after time t to be written explicitly for $c > 0$ as [15]

$$\mathcal{G}_N(x_1, x_2, \ldots, x_N|y_1, y_2, \ldots, y_N; t) = \sum_{\sigma\in\mathcal{S}_N} \int \cdots \int A_\sigma \prod_{j=1}^{N} e^{ik_{\sigma(j)}(x_j - y_{\sigma(j)})} e^{-\frac{it}{2}\sum_j k_j^2} \frac{dk_1}{2\pi} \cdots \frac{dk_N}{2\pi} \tag{8}$$

where $\mathcal{S}_N$ denotes the symmetric group of degree N, and

$$A_\sigma = \prod \left\{ S(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) : x_\alpha < x_\beta \text{ but } y_{\sigma(\alpha)} > y_{\sigma(\beta)} \right\}. \tag{9}$$

We note parenthetically that the propagator in the attractive ($c < 0$) case is considerably more complicated due to the presence of bound states and was found in a recent preprint [16].

To verify the remarkable formula Eq. (8) one should first observe that it satisfies the boundary condition $\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \mathcal{G}_N|_{x_i=x_j} = c\,\mathcal{G}_N|_{x_i=x_j}$ imposed by the interaction. Next we must check that the initial condition $\mathcal{G}_N(\mathbf{x}|\mathbf{y}; 0) = \sum_\sigma \prod_i \delta(x_i - y_{\sigma(j)})$ is obeyed. This follows from the fact that the integral

$$\int \cdots \int A_\sigma \prod_{j=1}^{N} e^{ik_{\sigma(j)}(x_j - y_{\sigma(j)})} \frac{dk_1}{2\pi} \cdots \frac{dk_N}{2\pi} \tag{10}$$

is nonzero only for $A_\sigma = 1$ i.e. when the $\{x_i\}$ are in the same order as the $\{y_i\}$. This in turn is a consequence of the following Golden Rule that we will use repeatedly for integrals of this type : *a particle moving to the left (right) must be overtaken by another particle moving to the left (right)*. In the present case the Golden Rule restricts us to $A_\sigma = 1$, from which the product of δ-functions follows.

To understand the origin of the Golden Rule, consider the integral over $k_{\sigma(j)}$ in Eq. (10). The result can be viewed as the Fourier transform of the product of factors in A_σ involving $k_{\sigma(j)}$, evaluated at $x_j - y_{\sigma(j)}$, which is thus the convolution of the Fourier transforms of these factors. Because $S(k)$ is holomorphic in the lower half plane for $c > 0$, its Fourier transform is supported in $[0, \infty)$. Thus for $x_j - y_{\sigma(j)} > 0$ we must have at least one factor $S(k_{\sigma(j)} - k_{\sigma(i)})$ where $k_{\sigma(j)}$ appears first (particle is overtaken moving to the right). Likewise for $x_j - y_{\sigma(j)} < 0$ we must have at least one factor $S(k_{\sigma(k)} - k_{\sigma(j)})$ where $k_{\sigma(j)}$ appears second (particle is overtaken moving to the left).

The time evolution of our array can be found by convolving the propagator with the Gaussian initial wavepackets

$$\Psi_N(\mathbf{x}; t) = \frac{1}{\sqrt{N!}} \int \mathcal{G}_N(\mathbf{x}, \mathbf{y}; t) \prod_j \varphi_j(y_j) d\mathbf{y} \tag{11}$$

The utility of this expression would seem to be hampered by the momentum integrals and the sum over permutations in Eq. (8). However, the former may be evaluated in the stationary phase approximation at long times

$$\mathcal{G}_N(\mathbf{x}|\mathbf{y}; t) \to \left(\frac{1}{2\pi i t}\right)^{N/2} \sum_{\sigma\in\mathcal{S}_N} A'_\sigma \prod_{j=1}^{N} e^{i\left(\frac{t}{2}\xi_j^2 - \xi_j y_{\sigma(j)}\right)}, \tag{12}$$

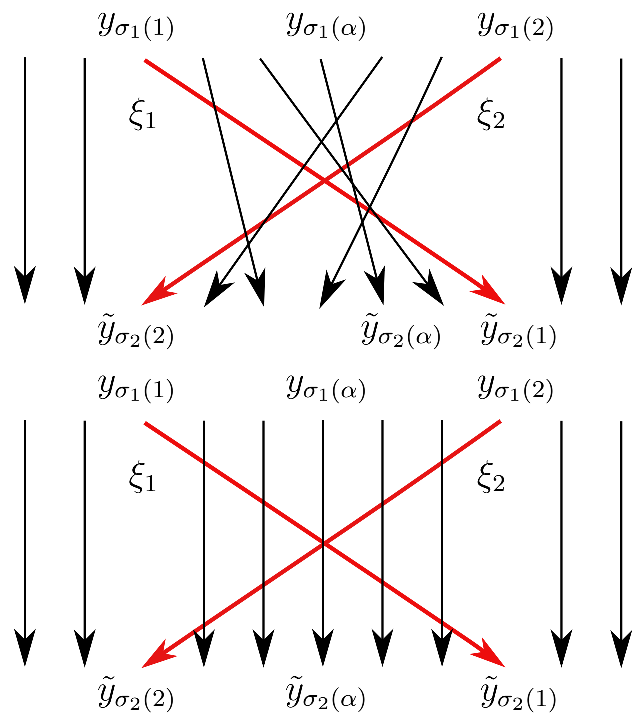

FIG. 3. (Color online) (Top) Diagram representing a term that survives integration over x_j, $j \neq 1, 2$. At the top are the positions $\{y_{\sigma_1(i)}\}$, and at the bottom are $\{\tilde{y}_{\sigma_2(i)}\}$. The thick red lines correspond to the two coordinates x_1 and x_2 that are not integrated over in the two-body density matrix Eq. (3). (Bottom) Usual contribution to the noninteracting HBT effect.

where again we have used the variables $\xi_j = x_j/t$, and the stationary phase integral assumes that these are order one in the long time limit. In the above A'_σ denotes

$$A'_\sigma = \prod \left\{ S(\xi_\alpha - \xi_\beta) : x_\alpha < x_\beta \text{ but } y_{\sigma(\alpha)} > y_{\sigma(\beta)} \right\}. \tag{13}$$

To evaluate the probability distribution we require the 'forward and back' propagator

$$\mathcal{G}_N(\mathbf{x}|\mathbf{y}; t)\mathcal{G}_N^*(\mathbf{x}|\tilde{\mathbf{y}}; t) \to \left(\frac{1}{2\pi t}\right)^N \sum_{\sigma_1,\sigma_2\in\mathcal{S}_N} A'_{\sigma_1} A'^*_{\sigma_2} \prod_j e^{-i\xi_j\left(y_{\sigma_1(j)} - \tilde{y}_{\sigma_2(j)}\right)}. \tag{14}$$

In this expression the scattering phases have the explicit form

$$A'_{\sigma_1} A'^*_{\sigma_2} = \prod \left\{ S(\xi_\alpha - \xi_\beta) : \sigma_1(\alpha) > \sigma_1(\beta), \sigma_2(\alpha) < \sigma_2(\beta) \right\}, \tag{15}$$

Unlike the individual A'_σ, we see that the form of the product does not depend upon the ordering of the $\{x_j\}$. Since we need to integrate over all but two of the $\{x_j\}$ to find the density correlation function (see Eq. (3)), this fact is tremendously useful, as it tells us that the integrals have the same form as Eq. (10), and allows us to apply the Golden Rule. The only non-trivial terms (i.e. without $\sigma_1(\alpha) = \sigma_2(\alpha)$ for all α) are of the form illustrated in Fig. 3 (top). x_1 and x_2 are exempted from the Golden Rule and correspond to the only particles not overtaken. Thus we must have $\sigma_1(1) = \sigma_2(2)$ and $\sigma_2(1) = \sigma_1(2)$

Despite this simplification there would still seem to be a great many terms to sum in Eq. (14). We will now show that the remaining terms can be grouped according to the order of their contribution in the parameter $e^{-2c\Delta}$, with the lower powers amenable to explicit evaluation. Since the parameter $c\Delta$ is the same as the usual Lieb–Liniger parameter $\gamma \equiv c/n$, with the density $n = \Delta^{-1}$, the use of $e^{-2c\Delta}$ as a small parameter is not too restrictive.

Let us first consider the terms that give rise to the usual HBT effect in the case of noninteracting particles (Fig. 3, bottom). Each of the x_α with $\sigma_1(\alpha) = \sigma_2(\alpha)$ lying between $\sigma_1(1) = \sigma_2(2)$ and $\sigma_2(1) = \sigma_1(2)$ brings a factor $S(\xi_2 - \xi_\alpha)S(\xi_\alpha - \xi_1)$ if $\sigma_1(1) < \sigma_1(2)$ and $S(\xi_1 - \xi_\alpha)S(\xi_\alpha - \xi_2)$ if $\sigma_1(2) < \sigma_1(1)$. After integrating over the $\{x_\alpha : \alpha \neq 1, 2\}$ and convolving with the Gaussian wavepackets Eq. (11), we can sum all such contributions in a geometric series to give

$$\mathcal{C}(x_1, x_2 : t) \to \frac{\ell^2}{\pi t^2} e^{-\ell^2(\xi_1^2+\xi_2^2)} \times \left[1 + \frac{2}{N}\mathrm{Re}\left(\frac{S(\xi_2-\xi_1)e^{i\Delta(\xi_1-\xi_2)}}{1-e^{i\Delta(\xi_1-\xi_2)}\zeta(\xi_1,\xi_2)}\right)\right], \quad (16)$$

which generalizes Eq. (3) to the interacting case. In Eq. (16) we have defined the function

$$\zeta(\xi_1, \xi_2) = 1 - 2\sqrt{\pi}c\ell S(\xi_2 - \xi_1 - ic) \times\left[e^{\ell^2(c-i\xi_1)^2}\mathrm{erfc}(\ell\,[c - i\xi_1]) + e^{\ell^2(c+i\xi_2)^2}\mathrm{erfc}(\ell\,[c + i\xi_2])\right],$$

where $\mathrm{erfc}(x)$ is the complementary error function $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$. Eq. (16) is shown in Fig. 2

In the limit that $\zeta(\xi_1, \xi_2)$ is close to unity, Eq. (16) can be interpreted as a series of Fano lineshapes Eq. (5) with $\eta_n = -\mathrm{Im}\,\zeta$, $\Gamma_n = 2(1 - \mathrm{Re}\,\zeta) > 0$, and q_n as given in Eq. (7). Fig. 2 (bottom) illustrates the evolution of q_n between successive peaks from smaller (close to antiresonance) to larger values.

The physical origin of the asymmetry q_n lies in the scattering phase of particles 1 and 2 with each other, while the width Γ_n arises from the collisions of these particles with those that they pass, whose momentum has a Gaussian distribution and gives rise to a distribution of scattering phases. Γ_n vanishes in the limits $c \to 0$ and $c \to \infty$, but also when $\ell \to 0$. In the last case this is a consequence of the typical momenta of the particles

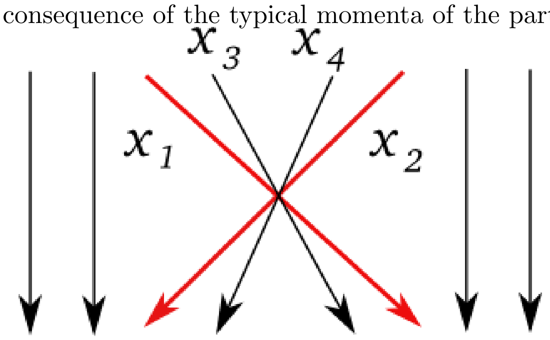

FIG. 4. (Color online) Simplest 'new' contribution, smaller by $e^{-2c\Delta}$.

becoming large (except for particles 1 and 2 whose momenta are fixed by x_1 and x_2) and the scattering phase for their collisions approaching zero (see Eq. (6)).

Let us now show that the remaining contributions are small in the parameter $e^{-2c\Delta}$. Consider the first 'non HBT' diagram shown in Fig. 4. Evaluating this diagram gives the contribution

$$\frac{16c^2\ell^4}{t^2}e^{-2c\Delta}e^{4i\Delta(\xi_1-\xi_2)}e^{-\ell^2(\xi_1^2+\xi_2^2)} \times \exp\left[\ell^2((c-i\xi_1)^2 + (c+i\xi_2)^2)\right] \times S(\xi_2-\xi_1)S(\xi_2-\xi_1-ic)^2 S(\xi_2-\xi_1-2ic) \quad (17)$$

where, as always, we ignore the overlap $e^{-\Delta^2/4\ell^2}$ between neighboring sites. The two exponential factors $e^{-c\Delta}$ arise from the pole in the upper half plane of x_3 coming from the $S(\xi_3 - \xi_1)$ factor, and from the pole in the lower half plane of x_4 coming from the $S(\xi_2-\xi_4)$ factor. In the same way, one can show that the power of $e^{-c\Delta}$ appearing in a contribution is at least twice the total number of moves to the right (or to the left).

In conclusion, we have shown that the HBT effect of interacting particles in one dimension has a number of interesting features that distinguish it from the noninteracting problem, most notably an asymmetry and finite width in the peaks of the density-density correlation function of the expanded system. The calculation hinges upon the integrability of the 1D Bose gas, and indeed appears to depend essentially upon $c > 0$ for the form of the propagator Eq. (8) to be valid.

The author would like to acknowledge the support of the NSF through grant DMR-0846788 and Research Corporation through a Cottrell Scholar award, and thanks Tom Jackson for a useful conversation.

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