Entanglement classes of symmetric Werner states

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The symmetric Werner states for n qubits, important in the study of quantum nonlocality and useful for applications in quantum information, have a surprisingly simple and elegant structure in terms of tensor products of Pauli matrices. Further, each of these states forms a unique local unitary equivalence class, that is, no two of these states are interconvertible by local unitary operations.

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Quantum information, motivated by practical applications in computation and cryptography, has been instrumental in shedding light on fundamental theoretical questions in physics and computer science. These include violation of Bell inequalities and local hidden variable theories [1–3], new proofs of classical information theorems [4], and new physical principles such as information causality [5]. Basic questions about states have been a driving theme. For example, when is a given state separable? When does it exhibit correlations that are nonclassical? Considering quantum states as resources raises the question of interconvertibility. When can one given state be transformed by local operations into a second given state? In general, these questions are difficult. In this article, we consider a class of states that has been demonstrated to have interesting and useful properties for which we give a structure theorem and answer the interconvertibility question. We introduce a novel analysis based on the identification of symmetric density matrices with real polynomials in three variables and apply the representation theory of $SO(3)$ on this space of polynomials.

The symmetric Werner states lie in the intersection of two important classes of composite states of subsystems of equal dimension. Symmetric states, that is, states that are invariant under permutation of subsystems, are the subject of recent work including: geometric measure of entanglement [6–10], efficient tomography [11], classification of states equivalent under stochastic local operations and classical communication (SLOCC) [12–14], and our own work on classification of states equivalent under local unitary (LU) transformations [15, 16]. Werner states, defined to be those states invariant under the action of any particular single qubit unitary operator acting on all n qubits, have found a multitude of uses in quantum information science. Originally introduced in 1989 for two particles [2] to distinguish between classical correlation and Bell inequality satisfaction, Werner states have found use in the description of noisy quantum channels [17], as examples in nonadditivity claims [18], and in the study of deterministic purification [19]. In what may

prove to be a practical application to computing in noisy environments, Werner states comprise decoherence-free subspaces for collective decoherence [20–22].

An understanding of structure and entanglement properties of mixed Werner states is known for bipartite and tripartite systems of arbitrary dimension [2, 23], but remains an open problem for higher numbers of component subsystems. It is natural to restrict the problem to subclasses that might be more tractable. In previous work [24] we have fully classified local unitary equivalence classes of *pure* Werner states. In this article, we consider the case of *symmetric* Werner states (pure and mixed), which include the singlet and the uniform mixture of symmetric Dicke states. The symmetric Werner states for n qubits have a surprisingly simple and elegant structure: their density matrices consist of linear combinations of symmetrized products of $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$ and the identity. Further, each of these states forms a unique local unitary equivalence class, that is, no two of these states are interconvertible by local unitary operations.

We show how symmetric mixed states of n qubits can be represented as real polynomials in three variables. We then show how to use representation theory of $SO(3)$ on polynomials to obtain the main result, Theorem 1 below.

I. PRELIMINARIES

Let $\mathcal{W}$ denote the 4-dimensional real vector space of 2×2 Hermitian matrices. A convenient basis for $\mathcal{W}$ is $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where σ_0 is the 2×2 identity matrix, and $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$ are the Pauli matrices.

The set of n-qubit density matrices is a proper subset of the vector space $\mathcal{W}^{\otimes n}$, where every element ρ (whether or not ρ is positive or has trace 1) can be uniquely written in the form $\rho = \sum_I s_I \sigma_I$, where $I = i_1 i_2 \ldots i_n$ is a multiindex with $i_k = 0, 1, 2, 3$ for $1 \leq k \leq n$, and σ_I denotes

$$\sigma_I = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n},$$

with real coefficients s_I.

Given a permutation π of $\{1, 2, \ldots, n\}$, let P_π denote the operator on Hilbert space that carries out the corre-

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sponding permutation of qubits. Here are two examples.

$$P_{(23)} \left| 11000 \right\rangle = \left| 10100 \right\rangle$$

$$P_{(23)} \left(\sigma_j \otimes \sigma_k \otimes \sigma_l \right) P_{(23)}^{-1} = \sigma_j \otimes \sigma_l \otimes \sigma_k$$

Define a symmetrization operator on density matrices as

$$\mathrm{Sym}(\rho) = \frac{1}{n!} \sum_\pi P_\pi \rho P_\pi^{-1}.$$

An n-qubit symmetric density matrix ρ is one for which $P_\pi \rho P_\pi^{-1} = \rho$ for all permutations π, or equivalently, one for which $\mathrm{Sym}(\rho) = \rho$. We denote by $\mathrm{Sym}^n \mathcal{W}$ the n-fold symmetric power of $\mathcal{W}$. It is the subspace of elements of $\mathcal{W}^{\otimes n}$ that are invariant under qubit permutation. Every n-qubit symmetric density matrix ρ is an element of $\mathrm{Sym}^n \mathcal{W}$, and can be written

$$\rho = \frac{1}{2^n} \sum c_{n_1 n_2 n_3} \mathrm{Sym} \left(\sigma_0^{\otimes n_0} \otimes \sigma_1^{\otimes n_1} \otimes \sigma_2^{\otimes n_2} \otimes \sigma_3^{\otimes n_3} \right), \tag{1}$$

where the sum is over non-negative integers n_0, n_1, n_2, and n_3 such that $n_0 + n_1 + n_2 + n_3 = n$. The coefficients $c_{n_1 n_2 n_3}$ are real. The collection of n-qubit symmetric density matrices is a proper subset of $\mathrm{Sym}^n \mathcal{W}$, since the latter contains Hermitian matrices that are not positive semi-definite, and Hermitian matrices for which the trace is not 1.

Let $\mathbb{R}_n[x, y, z]$ be the set of polynomials of degree at most n in three variables x, y, and z with real coefficients. For each n, there is a linear map $F_n : \mathrm{Sym}^n \mathcal{W} \to \mathbb{R}_n[x, y, z]$ defined by

$$\frac{1}{2^n} \mathrm{Sym} \left(\sigma_0^{\otimes n_0} \otimes \sigma_1^{\otimes n_1} \otimes \sigma_2^{\otimes n_2} \otimes \sigma_3^{\otimes n_3} \right) \mapsto x^{n_1} y^{n_2} z^{n_3}.$$

In this way, we may associate a polynomial of degree at most n with each n-qubit symmetric mixed state. The polynomial associated with (1) is

$$F_n(\rho) = \sum_{\substack{n_1, n_2, n_3 \\ n_1 + n_2 + n_3 \le n}} c_{n_1 n_2 n_3} x^{n_1} y^{n_2} z^{n_3}. \tag{2}$$

For each n, the map F_n is an invertible linear map. Table I lists these polynomials for some example symmetric mixed states.

Since F_n is a linear map, the polynomial for a mixture of symmetric mixed states is the mixture of the polynomials.

$$F_n(p_1 \rho_1 + p_2 \rho_2) = p_1 F_n(\rho_1) + p_2 F_n(\rho_2)$$

A product of polynomials $\mathbb{R}_n[x, y, z] \times \mathbb{R}_m[x, y, z] \to \mathbb{R}_{n+m}[x, y, z]$ represents the symmetrized tensor product of states.

$$F_n(\rho_1) F_m(\rho_2) = F_{n+m}(\mathrm{Sym}(\rho_1 \otimes \rho_2))$$

Let $g \in SU(2)$. Define $T_g : \mathrm{Sym}^n \mathcal{W} \to \mathrm{Sym}^n \mathcal{W}$ to be the symmetric transformation of each qubit by g.

$$T_g(\rho) = g^{\otimes n} \rho (g^\dagger)^{\otimes n}$$

Symmetric Mixed State ρ	n	$F_n(\rho)$
$\left\| 0 \right\rangle \left\langle 0 \right\|$	1	$1 + z$
$\left\| 1 \right\rangle \left\langle 1 \right\|$	1	$1 - z$
$\left\| + \right\rangle \left\langle + \right\| = \frac{1}{2}(\left\| 0 \right\rangle + \left\| 1 \right\rangle)(\left\langle 0 \right\| + \left\langle 1 \right\|)$	1	$1 + x$
Totally mixed, $\frac{1}{2} \left\| 0 \right\rangle \left\langle 0 \right\| + \frac{1}{2} \left\| 1 \right\rangle \left\langle 1 \right\|$	1	1
$\left\| 00 \right\rangle \left\langle 00 \right\|$	2	$(1 + z)^2$
Totally mixed	2	1
$\frac{1}{2} \left\| 00 \right\rangle \left\langle 00 \right\| + \frac{1}{2} \left\| 11 \right\rangle \left\langle 11 \right\|$	2	$1 + z^2$
$\frac{1}{2} \left\| 0 \right\rangle \left\langle 0 \right\| \otimes \frac{I}{2} + \frac{1}{2} \frac{I}{2} \otimes \left\| 0 \right\rangle \left\langle 0 \right\|$	2	$1 + z$
Singlet, $\frac{1}{2}(\left\| 01 \right\rangle - \left\| 10 \right\rangle)(\left\langle 01 \right\| - \left\langle 10 \right\|)$	2	$1 - x^2 - y^2 - z^2$
$\frac{1}{2}(\left\| 00 \right\rangle - \left\| 11 \right\rangle)(\left\langle 00 \right\| - \left\langle 11 \right\|)$	2	$1 - x^2 + y^2 + z^2$
$\frac{1}{2}(\left\| 00 \right\rangle + \left\| 11 \right\rangle)(\left\langle 00 \right\| + \left\langle 11 \right\|)$	2	$1 + x^2 - y^2 + z^2$
$\frac{1}{2}(\left\| 01 \right\rangle + \left\| 10 \right\rangle)(\left\langle 01 \right\| + \left\langle 10 \right\|)$	2	$1 + x^2 + y^2 - z^2$
Uniform Dicke mixture	2	$1 + \frac{1}{3}(x^2 + y^2 + z^2)$
$\left\| 000 \right\rangle \left\langle 000 \right\|$	3	$(1 + z)^3$
GHZ, $\frac{1}{2}(\left\| 000 \right\rangle + \left\| 111 \right\rangle)(\left\langle 000 \right\| + \left\langle 111 \right\|)$	3	$1 + 3z^2 + x^3 - 3xy^2$
W	3	$(1 + z) \cdot$ $(1 + 2x^2 + 2y^2 - z^2)$
Uniform Dicke mixture	3	$1 + x^2 + y^2 + z^2$
Uniform Dicke mixture	4	$1 + 2(x^2 + y^2 + z^2)$ $+ \frac{1}{5}(x^2 + y^2 + z^2)^2$

TABLE I: Polynomials for some symmetric mixed states. The density matrix for the W state is $\rho_W = \frac{1}{3}(\left\| 100 \right\rangle + \left\| 010 \right\rangle + \left\| 001 \right\rangle)(\left\langle 100 \right\| + \left\langle 010 \right\| + \left\langle 001 \right\|)$. The totally mixed 1-qubit state is $\frac{I}{2} = \frac{1}{2} \left\| 0 \right\rangle \left\langle 0 \right\| + \frac{1}{2} \left\| 1 \right\rangle \left\langle 1 \right\|$.

If we define $R_g : \mathbb{R}_n[x, y, z] \to \mathbb{R}_n[x, y, z]$ to be the transformation on polynomials defined by

$$R_g(f)(x, y, z) = f((x, y, z)\Phi(g)), \tag{3}$$

where f is a polynomial, $\Phi : SU(2) \to SO(3)$ is the homomorphism [27] that associates a rotation in $\mathbb{R}^3$ with each 2×2 unitary, and $(x, y, z)\Phi(g)$ denotes a row vector multiplied by a 3×3 orthogonal matrix, then the following diagram commutes.

$$
\begin{array}{ccc}
\mathbb{R}_n[x, y, z] & \xrightarrow{R_g} & \mathbb{R}_n[x, y, z] \\
\Big\uparrow{\scriptstyle F_n} & & \Big\uparrow{\scriptstyle F_n} \\
\mathrm{Sym}^n \mathcal{W} & \xrightarrow{T_g} & \mathrm{Sym}^n \mathcal{W}
\end{array}
$$

It is curious in Table I that the pure W state appears as the product of other polynomials, and hence as the symmetrized tensor product of Hermitian matrices. Since a symmetrization is a mixture, it seems contradictory that the pure W state could be a mixture. The resolution is that the factor $1 + 2x^2 + 2y^2 - z^2$ corresponds to a Hermitian matrix with a negative eigenvalue, and hence does not represent a density matrix. It is an interesting question to ask whether there is physical significance in the factor-ability of the W state.

II. SYMMETRIC WERNER STATES

Theorem 1. *Let ρ be an n-qubit symmetric mixed state. We claim that $g^{\otimes n} \rho (g^\dagger)^{\otimes n} = \rho$ for all $g \in SU(2)$ if and*

only if $F_n(\rho)$ is a linear combination of terms of the form $(x^2 + y^2 + z^2)^m$, for $0 \le m \le n/2$. Further, any two such states are local unitarily inequivalent. That is, there is no product $g = g_1 \otimes g_2 \otimes \cdots \otimes g_n$ of local unitaries g_i such that $g\rho g^\dagger = \rho'$ unless $\rho = \rho'$.

It is clear that polynomials of the form $F_n(\rho) = \sum_{m=0}^{\lfloor n/2 \rfloor} b_m (x^2 + y^2 + z^2)^m$ are invariant under the $SU(2)$ action (3) since $x^2 + y^2 + z^2$ is invariant under any rotation in $SO(3)$.

Conversely, suppose that $g^{\otimes n} \rho (g^\dagger)^{\otimes n} = \rho$ for all $g \in SU(2)$. Then $R_g(F_n(\rho)) = F_n(\rho)$ for all $g \in SU(2)$. We seek to find the subspace of $\mathbb{R}_n[x, y, z]$ that transforms as the trivial irreducible representation of $SO(3)$. Homogeneous polynomials in three variables x, y, and z are known to be reducible representations of $SO(3)$ [25][28]. If V_l is the irreducible representation of $SO(3)$ with dimension $2l + 1$, then the homogeneous polynomials of degree p in three variables decompose into irreducible representations as

$$\bigoplus_{j=0}^{\lfloor p/2 \rfloor} V_{p-2j}. \tag{4}$$

When expressed as a sum of irreducible representations, the homogeneous polynomials of degree p in three variables contain one dimension of the trivial representation (V_0) if p is even, and zero dimensions if p is odd [25]. The vector space $\mathbb{R}_n[x, y, z]$ of polynomials of degree at most n is a direct sum of vector spaces of homogeneous polynomials of degree p, for $0 \le p \le n$. Therefore, the space $\mathbb{R}_n[x, y, z]$ contains $\lfloor n/2 \rfloor + 1$ dimensions of the trivial representation. Since we have identified all of these dimensions, $F_n(\rho)$ must be a linear combination of the polynomials given.

Finally, let ρ, ρ' be symmetric Werner states, and suppose there is a local unitary operation $g = g_1 \otimes g_2 \otimes \cdots \otimes g_n$ such that $g\rho g^\dagger = \rho'$. We show in [16] that for $n \ge 3$ there exists an element $h \in SU(2)$ such that $h^{\otimes n} \rho (h^\dagger)^{\otimes n} = g\rho g^\dagger$. Because ρ is a Werner state, we have $\rho = \rho'$. For $n = 2$, it is straightforward, albeit tedious, to check that the coefficients of λ, λ^2 in the characteristic polynomial $\det(\rho(a) - \lambda\mathrm{Id})$ for the state $\rho(a) := \mathrm{Id}/4 + \frac{a}{4}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$ are $\frac{1-3a^2-2a^3}{16}$ and $\frac{3-3a^2}{8}$, respectively. This implies that if $\rho(a)$ is local unitary equivalent to $\rho(a')$, then $a = a'$.

III. SUMMARY AND OUTLOOK

We have shown how symmetric Werner states have a simple structure expressed in a basis of tensors of Pauli matrices, and that decompositions in this basis are local unitary invariants. Natural next questions are

- Can we give bounds on coefficients in the polynomials $F_n(\rho)$?

- Can we find conditions for separability using these coefficients?

- Can we say what level of reduced density matrices determines symmetric Werner states? (That is, if we are given a collection of k-qubit reduced density matrices for some k, can we determine whether there is a unique ρ that has these reduced density matrices?)

Evidence that the last question is not just wishful thinking is that the partial trace operation is particularly nice in Pauli tensor coordinates. We have

$$\mathrm{tr}_k(\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}) = \begin{cases} 2\sigma_{i_1} \otimes \cdots \widehat{\sigma_{i_k}} \cdots \otimes \sigma_{i_n} & \text{if } i_k = 0 \\ 0 & \text{otherwise} \end{cases}$$

where the wide hat symbol means omit the kth factor. Progress on any such aspect of the N-representability problem would be of interest.

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[27] Φ is given in a natural way by the adjoint action $SU(2) \to SO(su(2))$, so that $\Phi(g)(M) = gMg^{\dagger}$, and we identify $su(2)$ with $\mathbb{R}^3$ by $A \leftrightarrow (1,0,0)$, $B \leftrightarrow (0,1,0)$, $C \leftrightarrow (0,0,1)$. See [26].

[28] To be precise, the cited work considers the representation $\mathbb{C}[x,y,z]$. In the case of $SO(3)$, the irreducible submodules of this complex representation are in one-to-one correspondence with the *real* irreducible submodules of $\mathbb{R}[x,y,z]$ via complexification. See [26, Ch.2 Sec.6].