Entanglement bound for multipartite pure states based on local measurements

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Phys. Rev. A **84**, 042308 — Published 5 October 2011

DOI: 10.1103/PhysRevA.84.042308

Entanglement bound for multipartite pure states based on local measurements

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Abstract

An entanglement bound based on local measurements is introduced for multipartite pure states. It is the upper bound of the geometric measure and the relative entropy of entanglement. It is the lower bound of minimal measurement entropy. For pure bipartite states, the bound is equal to the entanglement entropy. The bound is applied to pure tripartite qubit states and the exact tripartite relative entropy of entanglement is obtained for a wide class of states.

1 Introduction

One of the open problem in quantum information theory is to quantify the entanglement of a multipartite quantum state. Many entanglement measures for pure or mixed multipartite states have been proposed [1] [2], among them are the tangle [3] [4], the Schmidt measure [5] [6] which is the logarithmic of the minimal number of product terms that comprise the state vector, the geometric measure [7] [8] [9] which is defined in terms of the maximal fidelity of the state vector and the set of pure product states, the relative entropy of entanglement [10] [11], and the robustness [12] [13]. The last three are related with each other [14] [15] [16] [17] [18] [19] and they are equal for some of the states such as stabilizer states, symmetric basis and anti-symmetric basis states. All these entanglement measures are not operationally defined. In bipartite system, however, the entanglement cost and the distillable entanglement are operational entanglement measures. If bipartite entanglement measures satisfy some properties, it turns out that their regularizations are bounded by distillable entanglement from one side and by entanglement cost from the other side[2]. For a pure bipartite state, the two bounds are equal and the entanglement is simply the entropy of the reduced density matrix thus has a clear information theoretical meaning. We will investigate the possibility of extending these entropic and operational definitions of entanglement to multipartite pure states in this paper, we will propose an entanglement measurement bound (EMB) which is an entanglement measure for pure tripartite qubit states. The bound is based on the results of local measurements. Local measurement or local discrimination had been used as upper bound of certain entanglement measures. For a graph state, "Pauli persistency" has been used as an upper bound of Schmidt measure [6], a quantity based on LOCC measurements has been used as an upper bound [20] of geometric measure.

2 The justification of the definition of entanglement measurement bound

What is the usefulness of a bipartite entangled state? One answer should be that we can use it for cryptography. If Alice's part is measured in spin up, Bob's part should definitely in spin up for a bipartite spin entangled Bell state $\Phi = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle_A \left|\uparrow\right\rangle_B + \left|\downarrow\right\rangle_A \left|\downarrow\right\rangle_B)$. Thus using n pairs of Bell state, we can get a shared string of n bits for Alice and Bob through measurements. If Alice measures her part in the basis of $\left|\phi\right\rangle = \cos\phi\left|\uparrow\right\rangle + \sin\phi\left|\downarrow\right\rangle$, $\left|\phi^{\perp}\right\rangle = -\sin\phi\left|\uparrow\right\rangle + \cos\phi\left|\downarrow\right\rangle$, then the measurement results should be that Alice and Bob are simultaneously in state $\left|\phi\right\rangle$ or they are simultaneously in state $\left|\phi^{\perp}\right\rangle$, each with probability $\frac{1}{2}$. Thus if we have n pairs of Bell state, we can get a shared string of n bits regardless of the measurement basis Alice chosen. When we have a less entangled state $\cos\theta\left|\uparrow\right\rangle_A\left|\uparrow\right\rangle_B + \sin\theta\left|\downarrow\right\rangle_A\left|\downarrow\right\rangle_B$, what can we do for the purpose of cryptography? Certainly, we can measure one of the part, say Alice in the spin up and down basis. The result turns out to be a shared string of length n, with the probability $\cos^2\theta$ for spin up, and the probability $\sin^2\theta$ for spin down. The information contained in such a string can be calculated to be $nH_2(\cos^2\theta)$, where $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. However, if Alice measures her part in a rather arbitrary basis $\left|\phi\right\rangle$ and $\left|\phi^{\perp}\right\rangle$, Bob will get his state in $\left|\phi_B\right\rangle \sim (\cos\phi\cos\theta\left|\uparrow\right\rangle_B + \sin\phi\sin\theta\left|\downarrow\right\rangle_B)$ and $\left|\phi_B^{\perp}\right\rangle \sim (-\sin\phi\cos\theta\left|\uparrow\right\rangle_B + \cos\phi\sin\theta\left|\downarrow\right\rangle_B)$ states with probabilities $p(\theta,\phi) = \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi$ and $1-p(\theta,\phi)$, respectively. We can get a shared string of length n with total information content $nH_2(p(\theta,\phi))$. Notice that $p(\theta,\phi) = \cos 2\theta\cos^2\phi + \sin^2\theta$, the minimal information content occurs at the case of $\cos^2\phi = 1$ or 0. So at least we can get a shared string with information content $nH_2(\cos^2\theta)$, which is the entanglement of the n pairs of the state $\cos\theta\left|\uparrow\right\rangle_A\left|\uparrow\right\rangle_B + \sin\theta\left|\downarrow\right\rangle_A\left|\downarrow\right\rangle_B$. Hence, the entanglement of a pure bipartite entangled state is the minimal shared information content obtained by measurement. This point had been proved in Ref. [21] in the context of measurement entropy. For completeness, we will give an alternative proof in the following.

For a bipartite state $|\psi\rangle = \sum_{i,j=1}^{d} A_{ij} |i\rangle |j\rangle$, where $|i\rangle$ and $|j\rangle$ are the orthogonal basis, it is well known that the entanglement entropy is the entropy of the reduced density matrix when one of the partite is traced out. We have $\rho_A = Tr_B(|\psi\rangle \langle\psi|) = \sum_{i,j,k} A_{ij} A_{kj}^* |i\rangle \langle k|$. Thus in the basis $|i\rangle$, the reduced state is $\rho_A = AA^\dagger$,where A is the matrix with entries A_{ij}. A unitary transformation U diagonalizes ρ_A to $\Lambda = diag\{\lambda_1, \ldots, \lambda_d\}$, thus $AA^\dagger = U\Lambda U^\dagger$. So that the singular value decomposition of matrix A is $A = U\sqrt{\Lambda}V^T$, where V is some other unitary transformation. We have the Schmidt decomposition

$$|\psi\rangle = \sum_k \sqrt{\lambda_k} \left|\varphi_k^{(1)}\right\rangle \left|\varphi_k^{(2)}\right\rangle, \tag{1}$$

with orthogonal basis $\left|\varphi_k^{(1)}\right\rangle = \sum_i U_{ik} |i\rangle$, $\left|\varphi_k^{(2)}\right\rangle = \sum_j V_{jk} |j\rangle$. The entanglement of the state is

$$E(|\psi\rangle) = -\sum_{i=1}^{d} \lambda_k \log_2 \lambda_k. \tag{2}$$

Now, we use the measurement to obtain shared digital information from the state vector $|\psi\rangle = \sum_{i,j=1}^{d} A_{ij} |i\rangle |j\rangle$. Suppose the state vector is projected to Alice's measurement base vector $|c\rangle = \sum_i^d c_i |i\rangle$, then Bob's state will be proportional to $\langle c |\psi\rangle = \sum_{i,j} c_i^* A_{ij} |j\rangle$, the probability of which is

$$p = \sum_j \left| \sum_i c_i^* A_{ij} \right|^2 . \tag{3}$$

Let's consider which measurement basis yields the optimal probability p. This is an optimal of p with respect to $\{c_i\}$ subjected to $\sum_i |c_i|^2 = 1$. With the Lagrange multiplier λ, we can write the optimal equation as $\frac{\partial L}{\partial c_i^*} = 0$, where $L = p - \lambda(\sum_i |c_i|^2 - 1)$. The optimal equation then reads

$$\sum_{i,j} A_{ij} A_{kj}^* c_k - \lambda c_k = 0, \tag{4}$$

or

$$(AA^\dagger - \lambda)\mathbf{c} = 0,$$

where $\mathbf{c} = (c_1, \ldots, c_d)^T$. The optimal probability should be $p = \sum_{i,j} c_i^* A_{ij} A_{kj}^* c_k = \mathbf{c}^\dagger AA^\dagger \mathbf{c} = \mathbf{c}^\dagger \lambda \mathbf{c} = \lambda$. So the optimal probability is the eigenvalue of the reduced density matrix $\rho_A = AA^\dagger$. Hence if we use eigenvectors of ρ_A as the measurement basis, the average information of each shared digit is the entanglement of the state. Denote the eigensystem of ρ_A as $\{\lambda_k, \mathbf{c}^k\}$, let's see if the unitary transformed basis $\{U\mathbf{c}^k\}$ decrease the entropy $H(\mathbf{p}) = -\sum_{i=1}^{d} p_k \log_2 p_k$ or not, where $p_k = \mathbf{c}^{k\dagger} U^\dagger AA^\dagger U \mathbf{c}^k$. Denote the elements of U in the basis of $\mathbf{c}^k$, we have $U_{ij} = \mathbf{c}^{i\dagger} U \mathbf{c}^j$. Using the spectrum decomposition of $AA^\dagger$, then

$$\begin{aligned} p_k &= \sum_i \lambda_i \mathbf{c}^{k\dagger} U^\dagger \mathbf{c}^i \mathbf{c}^{i\dagger} U \mathbf{c}^k \\ &= \sum_i \lambda_i |U_{ik}|^2 . \end{aligned} \tag{5}$$

Notice that function $f(x) = -x \log_2 x$ is concave, that is, for $\alpha \in [0, 1]$, one has $f(\alpha x_1 + (1 - \alpha)x_2) \geq \alpha f(x_1) + (1 - \alpha)f(x_2)$. Then $f(p_k) \geq \sum_i |U_{ik}|^2 f(\lambda_i)$, where the unitarity of U is used. Hence

$$\begin{aligned} H(\mathbf{p}) &= \sum_k f(p_k) \geq \sum_{i,k} |U_{ik}|^2 f(\lambda_i) \\ &= \sum_i f(\lambda_i) = E(|\psi\rangle). \end{aligned} \tag{6}$$

We get the desired result.

2.1 Definition

Definition 1 *For an $n-$partite state $|\psi\rangle$, let $\mathbf{p}$ be the probability vector with multiple subscripts, the components of $\mathbf{p}$ are $p_{i_1,i_2,\ldots,i_N} = \left|\left\langle \phi_{i_1}^{(1)}\right| \otimes \left\langle \phi_{i_2|i_1}^{(2)}\right| \otimes \cdots \left\langle \phi_{i_N|i_1 i_2 \ldots i_N}^{(N)}\right| \cdot |\psi\rangle\right|^2$. Here $\left|\phi_{i_j|i_1 i_2 \ldots i_{j-1}}^{(j)}\right\rangle$ (denoted simply as $\left|\phi_{i_j}^{(j)}\right\rangle$ hereafter) $(i_j = 0, 1, \ldots, d_j - 1)$ are the orthonormal basis of $j-th$ partite when the measurement results for the former parties are $i_1, i_2, \ldots, i_{j-1}$, respectively. The EMB of $|\psi\rangle$ is defined as the minimal entropy of the measurement probability vector, that is,*

$$E_{MB}(|\psi\rangle) = \min_{\mathbf{p}} H(\mathbf{p}). \tag{7}$$

The minimization is over all possible local orthogonal measurements.

Notice that the local measurements can be carried out step by step, for each result of the first partite measurement, one can choose an orthogonal basis to measure the second partite residue state. So one may have d_1 different projection measurements for the second partite. When measuring the third partite, one can have $d_1 d_2$ projection measurements, and so on. The choice of $j - th$ partite basis can rely on all the former measurement results. The total number of measurements is $1 + d_1 + d_1 d_2 + \ldots + d_1 d_2 \cdots d_{N-1}$. Meanwhile, the minimization in (7) is also with respect to all permutation of the parties.

There is an definition of entanglement measure based on measurement[21]. In the definition of [21], each partite has only one kind of (complete) measurement, the total number of the measurements is $N - 1$. The E_{Hmin} in [21] is no less than our entanglement measurement bound $E_{MB}(|\psi\rangle)$ by definition.

3 As upper bounds of entanglement measures

3.1 Coarse grain

In the multipartite case it is useful to compare EMB according to different partitions, where the components

$1,...,N$ are grouped into disjoint sets. Any sequence $(A_1,...,A_N)$ of disjoint subsets $A \in V$ with $\bigcup_{i=1}^{N} A_i = \{1,...,N\}$ will be called a partition of V . We will write

$$(A_1,...A_N) \leq (B_1,...,B_M), \tag{8}$$

if $(A_1,...A_N)$ is a finer partition than $(B_1,...,B_M)$. EMB is non-increasing under a coarser grain of the partition. If two components are merged to form a new component, then EMB can only decrease. This is because that the minimization in the definition of EMB Eq. (7) can also be seen as with respect to all possible local measurement hierarchies. A local measurement hierarchy of a finer partition $(A_1,...A_N)$ is definitely a local measurement hierarchy of the coarser grain partition $(B_1,...,B_M)$, while the inverse may not be true. So from $(A_1,...A_N)$ to $(B_1,...,B_M)$, the set of local measurement hierarchies is enlarged, the minimization may reach further lower value. We have

$$E_{MB}^{(A_1,...A_N)}(|\psi\rangle) \geq E_{MB}^{(B_1,...B_M)}(|\psi\rangle), \tag{9}$$

where we specify the partition as the superscript of EMB. So any coarser partition is a lower bound of the finer partition for EMB. Especially, lower bound of EMB for a tripartite pure state is the bipartite pure state entanglement, which is easily obtained. There are three bipartitions of a tripartite state, the tighter lower bound is the partition with largest entanglement.

3.2 Upper bound of geometric measure

Suppose $E_{MB}(|\psi\rangle)$ is achieved by the probability vector $\mathbf{p}$ with components $p_{i_1,i_2,...,i_n}$. For all $p_{i_1,i_2,...,i_n} = \left|\left\langle \phi_{i_1}^{(1)}\right| \otimes \left\langle \phi_{i_2}^{(2)}\right| \otimes \cdots \left\langle \phi_{i_N}^{(N)}\right| \cdot |\psi\rangle\right|^2$, we may denote the largest one as $p_{0,0,...,0} = \left|\left\langle \phi_0^{(1)}\right| \otimes \left\langle \phi_0^{(2)}\right| \otimes \cdots \left\langle \phi_0^{(N)}\right| \cdot |\psi\rangle\right|^2$. So $p_{0,0,...,0} \geq p_{i_1,i_2,...,i_N}$. Then $p_{i_1,i_2,...,i_n} \log_2 p_{i_1,i_2,...,i_N} \leq p_{i_1,i_2,...,i_N} \log_2 p_{0,0,...,0}$

$$\begin{aligned} E_{MB}(|\psi\rangle) &= -\sum_{i_1,i_2,...,i_N} p_{i_1,i_2,...,i_N} \log_2 p_{i_1,i_2,...,i_N} \\ &\geq -\sum_{i_1,i_2,...,i_N} p_{i_1,i_2,...,i_N} \log_2 p_{0,0,...,0} \\ &\geq -\log_2 p_{0,0,...,0} \\ &\geq E_G(|\psi\rangle). \end{aligned} \tag{10}$$

The last inequality comes from the fact that the geometric measure $E_G(|\psi\rangle) = \min -\log_2 F$, where $F = \left|\left\langle \varphi^{(1)}\right| \otimes \left\langle \varphi^{(2)}\right| \otimes \cdots \left\langle \varphi^{(N)}\right| \cdot |\psi\rangle\right|^2$, the minimization is over all possible product state $\left|\varphi^{(1)}\right\rangle \otimes \left|\varphi^{(2)}\right\rangle \otimes \cdots \left|\varphi^{(N)}\right\rangle$. The largest fidelity F should be no less than some special fidelity $p_{0,0,...,0}$.

3.3 Upper bound of the relative entropy of entanglement

Suppose the orthogonal expansion of $|\psi\rangle = \sum_{i_1,i_2,...,i_N} \xi_{i_1,i_2,...,i_N} \left|\phi_{i_1}^{(1)}\right\rangle \otimes \left|\phi_{i_2}^{(2)}\right\rangle \otimes \cdots \left|\phi_{i_N}^{(N)}\right\rangle$ with $\xi_{i_1,i_2,...,i_N} = \left\langle \phi_{i_1}^{(1)}\right| \otimes \left\langle \phi_{i_2}^{(2)}\right| \otimes \cdots \left\langle \phi_{i_N}^{(N)}\right| \cdot |\psi\rangle$ be the optimal expansion that achieves the measure entanglement, that is $E_{MB}(|\psi\rangle) = -\sum_{i_1,i_2,...,i_N} p_{i_1,i_2,...,i_N} \log_2 p_{i_1,i_2,...,i_N}$ with $p_{i_1,i_2,...,i_N} = |\xi_{i_1,i_2,...,i_N}|^2$. Let's construct the separable state

$$\begin{aligned} \omega &= \sum_{i_1,i_2,...,i_N} p_{i_1,i_2,...,i_N} \left|\phi_{i_1}^{(1)}\right\rangle \left|\phi_{i_2}^{(2)}\right\rangle \cdots \left|\phi_{i_N}^{(N)}\right\rangle \\ &\quad \times \left\langle \phi_{i_1}^{(1)}\right| \left\langle \phi_{i_2}^{(2)}\right| \cdots \left\langle \phi_{i_N}^{(N)}\right|. \end{aligned} \tag{11}$$

The relative entropy of $|\psi\rangle$ with respect to ω is $-Tr\,|\psi\rangle\langle\psi| \log_2 \omega = -\langle\psi| \log_2 \omega |\psi\rangle = -\langle\psi| \sum_{i_1,i_2,...,i_N} \left|\phi_{i_1}^{(1)}\right\rangle \left|\phi_{i_2}^{(2)}\right\rangle \cdots \left|\phi_{i_N}^{(N)}\right\rangle \log_2 p_{i_1,i_2,...,i_N} \left\langle \phi_{i_1}^{(1)}\right| \left\langle \phi_{i_2}^{(2)}\right| \cdots \left\langle \phi_{i_N}^{(N)}\right| \psi\rangle = E_{MB}(|\psi\rangle)$. It is larger than or equal to the relative entropy of entanglement $E_R(|\psi\rangle)$. Since the separable state ω is just one of the full separable states, it may not be the full separable state that achieves the minimal relative entropy for state $|\psi\rangle$. So we have

$$E_{MB}(|\psi\rangle) \geq E_R(|\psi\rangle). \tag{12}$$

More concretely, we will consider the pure tripartite qubit state in the next section.

4 Pure tripartite qubit state

It is well known that GHZ state and W state are two different kinds of pure tripartite states that are not convertible with each other under stochastic local operation and classical communication (SLOCC). We may write the states in computational basis as $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. The three parties are called Alice, Bob and Charlie. When they share GHZ state, if Alice measures her part with result 0, then the states of Bob and Charlie are in 0 without further measurement. When Alice measures 1, the other two parts are also in 1. A common string of bits among the three parts can be established when one of them measures in computational basis. However, when Alice measures in $|\phi\rangle = \cos\phi\,|0\rangle + \sin\phi\,|1\rangle$, $|\phi^\perp\rangle = -\sin\phi\,|0\rangle + \cos\phi\,|1\rangle$ basis, the joint state of Bob and Charlie will be left to the entangled state $\cos\phi\,|00\rangle + \sin\phi\,|11\rangle$ or $-\sin\phi\,|00\rangle + \cos\phi\,|11\rangle$. Further measurement should be performed to determine the state of Bob as well as Charlie. So GHZ state measured in computational basis is a rather special case when only one step of measurement is required to transform the tripartite entanglement to shared bits. In general, we need two steps of measurements to convert the tripartite quantum correlation to classical correlation.

In computational basis, a pure tripartite qubit state can be written as

$$|\psi\rangle = \sum_{i,j,k=0}^{1} A_{ijk}\,|i\rangle\,|j\rangle\,|k\rangle\,, \tag{13}$$

the normalization takes $\sum_{i,j,k=0}^{1} |A_{ijk}|^2 = 1$. Let the measurement basis of Alice are $|\phi_a\rangle = a_0^*\,|0\rangle + a_1^*\,|1\rangle$,

$\left|\phi_a^{\perp}\right\rangle = -a_1\left|0\right\rangle + a_0\left|1\right\rangle$, the basis of Bob are $\left|\phi_b\right\rangle = b_0^*\left|0\right\rangle + b_1^*\left|1\right\rangle$, $\left|\phi_b^{\perp}\right\rangle = -b_1\left|0\right\rangle + b_0\left|1\right\rangle$ when Alice is projected to $\left|\phi_a\right\rangle$, the basis of Bob are $\left|\phi_b'\right\rangle = b_0'^*\left|0\right\rangle + b_1'^*\left|1\right\rangle$, $\left|\phi_b'^{\perp}\right\rangle = -b_1'\left|0\right\rangle + b_0'\left|1\right\rangle$ when Alice is projected to $\left|\phi_a^{\perp}\right\rangle$. Suppose the state $\left|\psi\right\rangle$ be projected to $\left|\phi_a\right\rangle\left|\phi_b\right\rangle$ for Alice and Bob's parts, then Charlie should be left in $\left\langle\phi_a\phi_b\right|\left.\psi\right\rangle$ $=\sum_{k=0}^{1}(\sum_{i,j=0}^{1}A_{ijk}a_ib_j)\left|k\right\rangle$, the probability of measurement is $p_{ab} = \sum_{k=0}^{1}\left|\sum_{i,j=0}^{1}A_{ijk}a_ib_j\right|^2$. We may write $p_{ab} = p_a p_{b|a}$, where p_a is the probability of projecting $\left|\psi\right\rangle$ to $\left|\phi_a\right\rangle$, and $p_{b|a}$ is the probability of projecting further to $\left|\phi_b\right\rangle$. For all possible local measurements, we consider the minimal entropy of the probability distribution $\{p_{ab}, p_{ab^{\perp}}, p_{a^{\perp}b'}, p_{a^{\perp}b'^{\perp}}\}$, alternatively, we may write it as $\{p_{00},\ p_{01}, p_{10}, p_{11}\}$. The entropy of the measurement should be

$$\begin{aligned} E_{MB}(\left|\psi\right\rangle) &= \min(-\sum_{i,j=0}^{1}p_{ij}\log p_{ij}) \\ &= \min[-\sum_{i}^{1}(p_i\log p_i + p_i\sum_j p_{j|i}\log p_{j|i}]. \end{aligned}$$

So we may solve the problem by minimizing the entropy of conditional distribution $p_{j|i}$ by first fixing p_i, that is, after Alice's part is measured in some basis that is not known to Bob and Charlie, Bob choose some basis to minimize the entropy of conditional distribution $p_{j|i}$. Since the joint state of Bob and Charlie is left to (unnormalized) $\left\langle\phi_a\right|\left.\psi\right\rangle = \sum_{i,j,k=0}^{1}A_{ijk}a_i\left|j\right\rangle\left|k\right\rangle$ for some quite general measurement base $\left|\phi_a\right\rangle$ of Alice. From the result of bipartite case, we have

$$\min -\sum_{j=0}^{1}p_{j|i}\log p_{j|i} = E(\left|\psi_i\right\rangle). \tag{14}$$

Thus the minimization problem turns out to be

$$E_{MB}(\left|\psi\right\rangle) = \min_{\{a_0,a_1\}}[-\sum_{i=0}^{1}(p_i\log p_i + p_i E(\left|\psi_i\right\rangle)]. \tag{15}$$

Where

$$\left|\psi_0\right\rangle = p_0^{-1/2}\sum_{i,j,k=0}^{1}A_{ijk}a_i\left|j\right\rangle\left|k\right\rangle, \tag{16}$$

$$\left|\psi_1\right\rangle = p_1^{-1/2}\sum_{j,k=0}^{1}(-A_{0jk}a_1^* + A_{1jk}a_0^*)\left|j\right\rangle\left|k\right\rangle; \tag{17}$$

with

$$p_0 = \sum_{j,k=0}^{1}\left|A_{0jk}a_0 + A_{1jk}a_1\right|^2, \tag{18}$$

$$p_1 = \sum_{j,k=0}^{1}\left|-A_{0jk}a_1^* + A_{1jk}a_0^*\right|^2. \tag{19}$$

In practical calculation, we can choose $a_0 = \cos\theta$, $a_1 = \sin\theta e^{i\varphi}$, thus $E_{MB}(\left|\psi\right\rangle)$ is given by the minimization over $\{\theta,\varphi\}$. The bipartite entanglement at RHS of (15) can easily be evaluated with concurrence. Alternatively, we may write EMB as

$$E_{MB}(\left|\psi\right\rangle) = \min_{\{a_0,a_1\}}[\sum_{i=0}^{1}S(B_iB_i^{\dagger})], \tag{20}$$

where S is the von Neumann entropy of a matrix, $S(\varrho) = -Tr(\varrho\log_2\varrho)$,

$$B_0 = a_0A_0 + a_1A_1, \tag{21}$$

$$B_1 = -a_1^*A_0 + a_0^*A_1, \tag{22}$$

with A_0 and A_1 are the matrices of elements $(A_0)_{jk} = A_{0jk}$, $(A_1)_{jk} = A_{1jk}$.

For E_{Hmin} in [21], Bob's measurement is independent of Alice's measurements. Suppose the measurement basis of Alice be $\left|\phi_a\right\rangle = a_0^*\left|0\right\rangle + a_1^*\left|1\right\rangle$, $\left|\phi_a^{\perp}\right\rangle = -a_1\left|0\right\rangle + a_0\left|1\right\rangle$, the basis of Bob be $\left|\phi_b\right\rangle = b_0^*\left|0\right\rangle + b_1^*\left|1\right\rangle$, $\left|\phi_b^{\perp}\right\rangle = -b_1\left|0\right\rangle + b_0\left|1\right\rangle$, respectively. Then

$$E_{Hmin}(\left|\psi\right\rangle) = \min(-\sum_{i,j=0}^{1}p'_{ij}\log p'_{ij}). \tag{23}$$

where $p'_{lm} = \sum_{k=0}^{1}\left|\sum_{ij}A_{ijk}a_{li}b_{mj}\right|^2$, with $(a_{00}, a_{01}, a_{10}, a_{11}) = (a_0, a_1, -a_1^*, a_0^*)$ and $(b_{00}, b_{01}, b_{10}, b_{11}) = (b_0, b_1, -b_1^*, b_0^*)$. The minimization in (23) is more difficult than that of (20). The calculation of the geometric measure involves minimization over the product state of three qubits and thus is more difficult than the calculation of EMB in (20). Only for symmetric tripartite state, the calculation of the geometric measure can be reduced as shown later.

4.1 Superposition of GHZ and W' states

It has been known [22] [23] that any pure tripartite qubit state can be local unitarily transformed to the standard form

$$\begin{aligned} \left|\psi\right\rangle &= q_0\left|000\right\rangle + q_1\left|011\right\rangle + q_2\left|101\right\rangle \\ &\quad + q_3\left|110\right\rangle + q_4e^{i\gamma}\left|111\right\rangle. \end{aligned} \tag{24}$$

Where q_i are positive, $\gamma \in [-\pi/2, \pi/2]$. The concurrences of $\left|\psi_0\right\rangle$ and $\left|\psi_1\right\rangle$ are $C_0 = 2p_0^{-1}\left|q_0q_1\cos^2\theta + q_0q_4\sin\theta\cos\theta e^{i(\gamma+\varphi)} - q_2q_3\sin^2\theta e^{2i\varphi}\right|$, with probability $p_0 = (q_0^2 + q_1^2)\cos^2\theta + (q_2^2 + q_3^2 + q_4^2)\sin^2\theta + 2q_1q_4\sin\theta\cos\theta\cos(\gamma + \varphi)$, and $C_1 = 2p_1^{-1}\left|q_0q_1\sin^2\theta - q_0q_4\sin\theta\cos\theta e^{i(\gamma+\varphi)} - q_2q_3\cos^2\theta e^{2i\varphi}\right|$, with probability $p_1 = (q_0^2+q_1^2)\sin^2\theta + (q_2^2+q_3^2+q_4^2)\cos^2\theta - 2q_1q_4\sin\theta\cos\theta\cos(\gamma+\varphi)$, respectively.

A special case is the superposition of GHZ and W' state, $\left|GHZ - W'\right\rangle = \cos\alpha\left|GHZ\right\rangle + \sin\alpha\left|W'\right\rangle$, which is a standard tripartite state with $q_0 = q_4 = \frac{1}{\sqrt{2}}\cos\alpha$, $q_1 = q_2 = q_3 = \frac{1}{\sqrt{3}}\sin\alpha$, $\gamma = 0$. The state is widely used in evaluating the tangle of symmetric tripartite mixed state. For

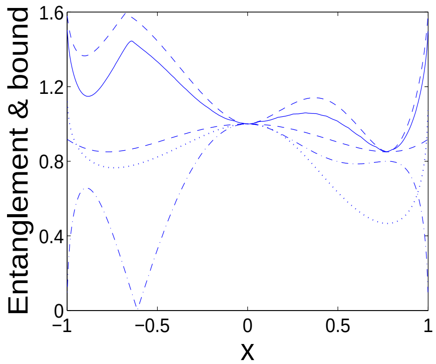

Figure 1: The Entanglement with respect to x, the portion of W' state in the superposition of W' and GHZ state. The solid line is the entanglement measurement bound, the dotted line is the geometric measure, the up dashed line is E_{Hmin} of Ref. [21], the down dashed line is the bipartite entanglement, the dash-dot line is the tangle.

this superposition state, we have calculated EMB for parameter $x = \sin\alpha$, $\alpha \in [-\pi/2, \pi/2]$. The results are shown in Fig.1. Also shown in Fig.1 are the tangle, the geometric measure and the bipartition entanglement of the state. The tangle of the state is [24]

$$\tau(|GHZ - W'\rangle) = \left|\cos^4\alpha + \frac{8}{9}\sqrt{6}\sin^3\alpha\cos\alpha\right|.$$

According to the permutation symmetry of the state, the geometric measure for this state is [25] [26] [27]

$$E_G = \min_{\phi} -\log_2\left|\langle GHZ - W'|\,(|\phi\rangle)^{\otimes 3}\right|^2, \qquad (25)$$

where $|\phi\rangle$ is a qubit state.

4.2 Bipartite lower bound

For a general state $|\psi\rangle = \sum_{i,j,k=0}^{1} A_{ijk}|i\rangle|j\rangle|k\rangle$, we may project it to state $|\phi_{ab}\rangle = \sum_{i,j=0}^{1} c_{ij}^{*}|i\rangle|j\rangle$ with joint measurement of Alice and Bob, then Charlie should be left in $\langle\phi_{ab}|\,\psi\rangle = \sum_{k=0}^{1}(\sum_{i,j=0}^{1} A_{ijk}c_{ij})|k\rangle$. The bipartition is a coarser grain of a tripartition, so

$$E_{bi}(|\psi\rangle) \le E_{MB}(|\psi\rangle). \qquad (26)$$

The bipartite lower bound is $E_{bi}(|\psi\rangle) = \min\{H_2(x_1), H_2(x_2), H_2(x_3)\}$, with $x_m = \frac{1}{2}(1 + \sqrt{1 - C_m^2})$. The concurrence $C_m = 2\sqrt{\left|d_{00}^{(m)}d_{11}^{(m)} - d_{01}^{(m)}d_{10}^{(m)}\right|}$, with $d_{ii'}^{(1)} = \sum_{j,k=0}^{1} A_{ijk}A_{i'jk}^{*}$, $d_{jj'}^{(2)} = \sum_{i,k=0}^{1} A_{ijk}A_{ij'k}^{*}$, $d_{kk'}^{(3)} = \sum_{i,j=0}^{1} A_{ijk}A_{ijk'}^{*}$. For a $|GHZ - W'\rangle$ state, the three concurrences are equal to

$$C = \sqrt{\cos^4\alpha + \frac{4}{3}\sin^2\alpha\cos^2\alpha + \frac{8}{9}\sin^4\alpha}. \qquad (27)$$

So the lower bound of EMB of $|GHZ - W'\rangle$ state is $H_2(\frac{1}{2}(1 + \sqrt{1 - C^2}))$.

4.3 A special superposition state with equal tripartite EMB and bipartite entanglement

It can be seen from figure 1 that there is a superposition of GHZ and W' state whose tripartite EMB and bipartite entanglement are equal. The state is a $|GHZ - W'\rangle$ state with $x = \sin\alpha = \sqrt{\frac{3}{5}}$, we will denote it as $|\Omega\rangle$ in the following. Then

$$|\Omega\rangle = \frac{1}{\sqrt{5}}(|000\rangle + |011\rangle + |101\rangle + |110\rangle + |111\rangle). \qquad (28)$$

The bipartite entanglement is $E_{bi}(|\Omega\rangle) = S(\rho_C)$, where $\rho_C = Tr_{AB}(|\Omega\rangle\langle\Omega|) = \frac{2}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{3}{5}|1\rangle\langle 1|$. Then

$$E_{bi}(|\Omega\rangle) = H_2[\frac{1}{2}(1 + \frac{1}{\sqrt{5}})] \approx 0.8505. \qquad (29)$$

For the tripartite EMB, the eigenvalues of $B_0B_0^\dagger$ and $B_1B_1^\dagger$ in Eq.(20) are

$$\lambda_{0\pm} = \frac{1}{10}(2 + K \pm \sqrt{5}K), \qquad (30)$$

$$\lambda_{1\pm} = \frac{1}{10}[3 - K \pm \sqrt{5}(1 - K)], \qquad (31)$$

respectively, where $K = |a_1|^2 + a_0a_1^* + a_1a_0^* = \sin^2\theta + \sin 2\theta\cos\varphi$. Notice that $\lambda_{0+} + \lambda_{1-} = \frac{1}{10}(5 + \sqrt{5})$, $\lambda_{0-} + \lambda_{1+} = \frac{1}{10}(5 - \sqrt{5})$, the minimal entropy summation in Eq.(20) should be achieved by maximal K or minimal K. The maximal and minimal values of K are $\frac{1}{2}(1 \pm \sqrt{5})$, respectively. Either of them leads to the same eigenvalues $\{0, 0, \frac{1}{2}(1 + \frac{1}{\sqrt{5}}), \frac{1}{2}(1 - \frac{1}{\sqrt{5}})\}$. The tripartite EMB then is

$$E_{MB}(|\Omega\rangle) = H_2[\frac{1}{2}(1 + \frac{1}{\sqrt{5}})] = E_{bi}(|\Omega\rangle). \qquad (32)$$

For any bipartition of $|\Omega\rangle$, the bipartition relative entropy of entanglement $E_{bi}^{R}(|\Omega\rangle)$ is just the entropy of the reduced density matrix, so $E_{bi}^{R}(|\Omega\rangle) = E_{bi}(|\Omega\rangle)$. However, the tripartite relative entropy of entanglement $E_R(|\Omega\rangle)$ should be no less than the bipartite one, as can be seen from the definition of the relative entropy of entanglement. So we have

$$E_{bi}(|\Omega\rangle) = E_{MB}(|\Omega\rangle) \ge E_R(|\Omega\rangle) \ge E_{bi}^{R}(|\Omega\rangle). \qquad (33)$$

So that all of them are equal for state $|\Omega\rangle$. We thus obtain the exact value of $E_{MB}(|\Omega\rangle)$ and the tripartite relative entropy of entanglement $E_R(|\Omega\rangle)$ for state $|\Omega\rangle$.

We may consider the minimal measurement entropy E_{Hmin} defined in [21] for $|\Omega\rangle$ state. The measurement basis can be $|\phi_a\rangle|\phi_b\rangle, |\phi_a\rangle\,|\phi_b^{\perp}\rangle, |\phi_a^{\perp}\rangle|\phi_b\rangle, |\phi_a^{\perp}\rangle\,|\phi_b^{\perp}\rangle$. the probabilities of the measurements are $p_{00}' = \frac{1}{5}[1 + xy]$, $p_{01}' = \frac{1}{5}[1 + x(1 - y)]$, $p_{10}' = \frac{1}{5}[1 + (1 - x)y]$, $p_{11}' = \frac{1}{5}[1 + (1 - x)(1 - y)]$. Where $x = (|a_0 + a_1|^2 - |a_0|^2) \in [\frac{1}{2}(1 - \sqrt{5}),$

$\frac{1}{2}(1+\sqrt{5})]$, $y = (|b_0 + b_1|^2 - |b_0|^2) \in [\frac{1}{2}(1-\sqrt{5}), \frac{1}{2}(1+\sqrt{5})]$. Then the minimal entropy of the measurement is

$$E_{Hmin}(|\Omega\rangle) = \min - \sum_{i,j=0}^{1} p'_{ij} \log_2 p'_{ij} = H_2[\frac{1}{2}(1 + \frac{1}{\sqrt{5}})]. \quad (34)$$

We can see that $E_{Hmin}(|\Omega\rangle) = E_{MB}(|\Omega\rangle)$.

4.4 Conditions for equal of EMB and minimal measurement entropy

For a general tripartite state $|\psi\rangle$, we have $E_{Hmin}(|\psi\rangle) \geq E_{MB}(|\psi\rangle)$, with the equality holds only when the basis $|\phi_b\rangle, |\phi_b^\perp\rangle$ coincides with the basis $|\phi'_b\rangle, |\phi_b'^\perp\rangle$. The basis $|\phi_b\rangle, |\phi_b^\perp\rangle$ are the eigenvectors of $B_0 B_0^\dagger$, while the basis $|\phi'_b\rangle, |\phi_b'^\perp\rangle$ are the eigenvectors of $B_1 B_1^\dagger$. Hence only when matrix $B_0 B_0^\dagger$ commutes with $B_1 B_1^\dagger$ can we have $E_{Hmin}(|\psi\rangle) = E_{MB}(|\psi\rangle)$.

The A_0 and A_1 matrices for the standard form of tripartite state (24) are

$$A_0 = \begin{bmatrix} q_0 & 0 \\ 0 & q_1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & q_2 \\ q_3 & q_4 e^{i\gamma} \end{bmatrix}. \quad (35)$$

Notice that $B_0 B_0^\dagger + B_1 B_1^\dagger = A_0^2 + A_1 A_1^\dagger \equiv \mathcal{A}$, so the condition for the equality should be

$$[B_0 B_0^\dagger, \mathcal{A}] = 0. \quad (36)$$

If we require that $B_0 B_0^\dagger$ commutes $B_1 B_1^\dagger$ for all measurements of the Alice's qubit, then condition (36) reduces to $[A_0, A_1] = 0, [A_0, A_1^\dagger] = 0$ and $[A_1, A_1^\dagger] = 0$. These are equivalent to $(q_0 - q_1)q_2 = 0, (q_0 - q_1)q_3 = 0, q_2 = q_3, q_2 e^{i\gamma} = q_3 e^{-i\gamma}, q_2 e^{i\gamma} = q_3 e^{-i\gamma}$. The solutions should be either

$$q_0 = q_1, q_2 = q_3, \gamma = 0, \quad (37)$$

or

$$q_2 = q_3 = 0. \quad (38)$$

The corresponding states are

$$\begin{aligned} |\Omega_1\rangle &= q_0(|000\rangle + |011\rangle) + q_2(|101\rangle + |110\rangle) + q_4|111\rangle, \\ |\Omega_2\rangle &= q_0|000\rangle + q_1|011\rangle + q_4 e^{i\gamma}|111\rangle. \end{aligned}$$

For $|\Omega_1\rangle$ state, we choose $B_0 = \cos\theta A_0 + \sin\theta A_1$. Let $\det(B_0) = 0$ to determine θ, then the eigenvalues of $B_0 B_0^\dagger$ are 0 and $(TrB_0)^2$. We have

$$E_{MB}(|\Omega_1\rangle) = E_{bi}(|\Omega_1\rangle) = H_2[\frac{1}{2}(1 + \sqrt{1 - C^2})], \quad (39)$$

with $C^2 = 4q_0^2[2(1 - 2q_0^2) - q_4^2]$. The tripartite relative entropy of entanglement $E_R(|\Omega_1\rangle)$ is obtained to be equal to $E_{bi}(|\Omega_1\rangle)$ since it is in between $E_{MB}(|\Omega_1\rangle)$ and $E_{bi}(|\Omega_1\rangle)$. Similar results can be obtained for states $|\Omega'_1\rangle = q_0(|000\rangle + |101\rangle) + q_3(|011\rangle + |110\rangle) + q_4|111\rangle$ and $|\Omega''_1\rangle = q_0(|000\rangle + |110\rangle) + q_1(|101\rangle + |011\rangle) + q_4|111\rangle$.

For $|\Omega_2\rangle$ state, the equality of EMB and the bipartite entanglement do not hold in general, however, when $q_1 = 0$, we do have the equality. But the situation seems rather trivial.

4.5 LOCC monotone for completely measurement of a pure tripartite state

A fundamental property of an entanglement measure is that it should not increase under LOCC. Local measurement will not increase the entanglement of a state on average. To illustrate the detail meanings of EMB under LOCC, let's consider the tripartite qubit state first. Given a pure tripartite qubit state (13) with coefficients A_{ijk}, we can calculate the bound with formula (15) where the default first step measurement is on Alice's qubit. We may first measure Bob's qubit or Charlie's qubit. The results may differ. The bound should be the minimum of the three by definition. We denote it as

$$E_{MB}(|\psi\rangle) = \min\{E_{MB}^A(|\psi\rangle), E_{MB}^B(|\psi\rangle), E_{MB}^C(|\psi\rangle)\},$$

where $E_{MB}^i(|\psi\rangle)$ is calculated with formula (15) when *ith* partite is measured first. One the other hand, after a measurement on Alice's partite, the state left should be (16) with probability (18) or (17) with probability (19). The maximal average entanglement after local measurement on Alice's partite can be denoted as

$$E_{LOCC}^A(|\psi\rangle) = \max_{a_0, a_1}[-\sum_{i=0}^{1} p_i E(|\psi_i\rangle)]. \quad (40)$$

We may measure Bob's or Charlie's qubit first, the maximal average entanglement after a local measurement then is

$$\begin{aligned} E_{LOCC}(|\psi\rangle) &= \max\{E_{LOCC}^A(|\psi\rangle), \\ &\quad E_{LOCC}^B(|\psi\rangle), E_{LOCC}^C(|\psi\rangle)\}. \end{aligned} \quad (41)$$

If we have

$$E_{MB}(|\psi\rangle) \geq E_{LOCC}(|\psi\rangle), \quad (42)$$

then the EMB is an LOCC monotone, we may call it measurement entanglement and denoted as $E_M(|\psi\rangle)$. In the following we will prove that (42) is true for a pure tripartite state in the sense of completely measurement of the first partite.

Theorem 1 *Entanglement measurement bound for a pure tripartite qubit state is an LOCC monotone.*

Proof: Suppose that EMB of a tripartite pure state $|\psi\rangle$ is achieved by measuring Alice's partite first, then we have

$$E_{MB}(|\psi\rangle) = E_{MB}^{(A)}(|\psi\rangle) \geq E^{(AB,C)}(|\psi\rangle) \quad (43)$$

by (9), where $E^{(AB,C)}(|\psi\rangle)$ is the bipartite entanglement. When we measure on Alice or Bob of AB part, the average entanglement of the remained part will not exceed $E^{(AB,C)}(|\psi\rangle)$ according to the monotonicity of bipartite entanglement [28], namely,

$$\begin{aligned} E^{(AB,C)}(|\psi\rangle) &\geq E_{LOCC}^A(|\psi\rangle), \\ E^{(AB,C)}(|\psi\rangle) &\geq E_{LOCC}^B(|\psi\rangle). \end{aligned}$$

Similarly, we also have $E_{MB}^{(A)}(|\psi\rangle) \geq E^{(B,AC)}(|\psi\rangle)$ and the monotonicity of bipartite entanglement shows that $E^{(B,AC)}(|\psi\rangle) \geq E_{LOCC}^{C}(|\psi\rangle)$. Thus (42) is proved, and the theorem follows.

For a $d_1 \times d_2 \times d_3$ tripartite state with completely measurement of the each partite, we have

$$E_M(|\psi\rangle) = E_{MB}(|\psi\rangle) \geq E_{LOCC}(|\psi\rangle).$$

The completely measurement means that the state of N parties is projected to $N-1$ parties after the measurement.

5 Conclusion

The entanglement bound based on local measurements is introduced for multipartite pure states. The measurement sequence is a dependent one, for each step of measurement, the basis rely on the former measurement results. The entanglement measurement bound defined in this paper is a lower bound of a multipartite entanglement measure called *minimal measurement entropy* which is based on independent measurements of the parties. The entanglement measurement bound is also the upper bound of *geometric measure* and *the relative entropy of entanglement.* The property of coarser grain for the bound is derived. Based on the coarser grain of the bound and the fact that in bipartite case the bound is equal to the relative entropy of entanglement, we obtain the lower and upper bounds for the relative entropy of entanglement of a tripartite state. For a tripartite qubit state we derive the condition when the lower and upper bound coincide. The exact relative entropy of entanglement follows for a class of tripartite qubit states in the form of $|\Omega_1\rangle = q_0(|000\rangle + |011\rangle) + q_2(|101\rangle + |110\rangle) + q_4 |111\rangle$ or their qubit permutation states. It is an interesting phenomenon that the tripartite relative entropy of entanglement is equal to the bipartite relative entropy of entanglement while the tangle is nonzero for these states. For tripartite qubit states, the bound itself is an entanglement monotone. Further works can be done on whether the bound is an LOCC monotone or not in general.

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Acknowledgment

Funding by the National Natural Science Foundation of China (Grant No. 60972071), Natural Science Foundation of Zhejiang Province (Grant No. Y6100421), Zhejiang Province Science and Technology Project (Grant No. 2009C31060), Zhejiang Province Higher Education Bureau Program (Grant No. Y200906669) are gratefully acknowledged.

References

[1] M. B. Plenio and S. Virmani, Quant. Inf. Comp. **7**, 1 (2007).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[3] V. Coffman , J. Kunduand W. K. Wootters Phys. Rev. A **61** 052306 (2000).

[4] A. Miyake, Phys. Rev. A **67,** 012108 (2003).

[5] J. Eisert and H.J. Briegel, Phys. Rev. A **64**, 022306 (2001).

[6] M. Hein, J. Eisert and H.J. Briegel, Phys. Rev. A **69**, 062311 (2004).

[7] A. Shimony, Ann. N. Y. Acad. Sci. **755**, 675 (1995).

[8] T.-C. Wei and P. M. Goldbart, Phys. Rev. A **68**, 042307 (2003).

[9] D. C. Brody and L. P. Hughston, J. Geom. and Phys.**38**, 19 (2001).

[10] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[11] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).

[12] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).

[13] A. W. Harrow and M. A. Nielsen, Phys. Rev. A **68**, 012308 (2003).

[14] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Phys. Rev. Lett. **96**, 040501 (2006).

[15] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Phys. Rev. A **77**, 012104 (2008).

[16] T.-C. Wei, M. Ericsson, P. M. Goldbart, and W. J. Munro, Quantum Inf. Comput. **4**, 252 (2004).

[17] D. Cavalcanti, Phys. Rev. A **73**, 044302 (2006).

[18] T.-C. Wei, Phys. Rev. A **78**, 012327 (2008).

[19] H. Zhu, L. Chen, and M. Hayashi, New J. Phys. **12**, 083002 (2010).

[20] D. Markham, A. Miyake and S. Virmani New J. Phys. **9** 194 (2007).

[21] Chernyavskiy, arXiv: 0905.0201[quant-ph].

[22] H. A. Carteret, A. Higuchi, and A. Sudbery, J. Math. Phys. **41**, 7932 (2000).

[23] S. Tamaryan, T.-C. Wei, and D.K. Park, Phys. Rev. A **80**, 052315 (2009).

[24] C Eltschka, A Osterloh, J Siewert, and A Uhlmann, New J. Phys. **10**, 043014 (2008).

[25] R. Hübener, M. Kleinmann, T.-C. Wei, C. G. Guillén, and O. Gühne, Phys. Rev. A **80**, 032324 (2009).

[26] M. Hayashi, D. Markham, M. Murao, M. Owari, and S.Virmani, J. Math. Phys. **50**, 122104 (2009).

[27] T.-C. Wei and S. Severini, J. Math. Phys. **51**, 092203 (2010).

[28] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).