Detecting non-Abelian geometric phases with three-level Λ systems

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Detecting Non-Abelian Geometric Phases with Three-Level Lambda Atoms

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We show that a non-Abelian gauge potential in two nearly degenerated dressed states may be induced by two laser beams interacting with a three-level Λ atomic system. We demonstrate that the populations of the atomic states at the end of a composed path formed by two closed loops are dependent on the order of those two loops, showing an unambiguous signature of the non-Abelian geometric phase. Through numerical calculations, we show that non-Abelian feature of the geometric phases can be tested under realistic conditions.

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It was discovered by Berry that a geometric phase, in addition to the usual dynamical phase, is accumulated on the wave function of a quantum system, provided that the Hamiltonian is cyclic and adiabatic[1]. Since then the concept of geometric phase has been generalized to much looser conditions, such as non-adiabatic, non-cyclic conditions or even an open system[2–5]. In addition, it has found many applications in various fields of physics[2]. Particulary, since the geometric phase has global feature and is independent on the details of evolution path, it has been used to achieve quantum gates, and the quantum gates realized with pure geometric phases have intrinsic fault-tolerant features[6–9].

According to the commutation characters of the gauge potential related to the geometric phases, the geometric phase factor in a quantum system can be divided into two types: Abelian geometric phase related to usually a non-degenerated state and non-Abelian geometric phase corresponding to the degenerated states [1, 10]. Applying both the Abelian and non-Abelian geometric phase, a pure geometric quantum computer can be formed. As the first step of questing for a pure geometric quantum computer, the Abelian geometric phase has been demonstrated in many systems[2, 11, 12]. However, the key character of non-Abelian structure, also known as the noncommutative effect, has not been detected yet[13–18]. It is believed that the detecting of the non-Abelian geometric phase will be more complicated[8]. The nuclear magnetic resonance(NMR) experiment detected the geometric phase of the two-fold degenerate states[19]. However, only the Abelian part of the gauge structures has been experimentally observed in this experiment due to the limitation of the NMR implementation[14]. The observation of the non-Abelian geometric phase is important in the field of quantum theory and it is also a crucial step toward fault-tolerant geometric quantum computation.

In this paper, we demonstrate that a geometric phase associated with the non-Abelian gauge potential will also be induced by two laser beams interacting with a three-level Λ system. The scheme will be performed with the cold atomic system, which is well known as a perfect platform for the quantum information processes. Actually the three-level Λ atoms are seldom to be used to explore the non-Abelian geometric phases since the eigenstates are non-degenerate. However, we demonstrate that the gauge potentials associated with the nearly degenerated two states in the large detuning case are non-Abelian, and thus the non-Abelian geometric phases can also be unambiguously detected through comparing the population of the atoms in the end of two composed pathes (C_1C_2 and C_2C_1 specified below) formed by two closed evolution loops, where the dynamic phase of each subloop can be removed by multi-loop method. Comparing with the manipulating of the four level tripod atomic system[13, 15, 17], the three level Λ system proposed here is more simple and stable for the Alkali atoms and thus we provide a promising method to detect the non-Abelian geometric phases.

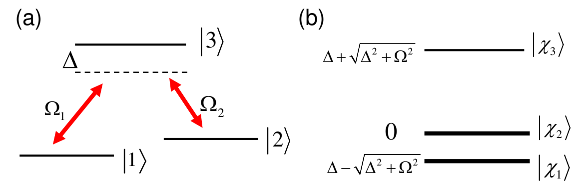

FIG. 1: (color online). (a)Three-level Λ atomic system coupled with two laser beams characterized by the Rabi frequencies Ω_1 and Ω_2 with a large single-photon detuning Δ. (b)The dressed states and the corresponding energy levels.

The system we consider is a cold atomic gas with each has a three-level Λ-type configuration as shown in Fig. 1(a). The ground states $|1\rangle$ and $|2\rangle$ are coupled to an excited state $|3\rangle$ by two laser beams with the corresponding Rabi frequencies Ω_1 and Ω_2, respectively, but with the same large detuning Δ. The atom can be Rb^{87} and we choose $5S^{1/2}$ $F=1,2$ and $5P^{3/2}$ $F'=3$ to implement such configuration. In the interaction picture,the laser-atom interaction Hamiltonian H reads

$$H=-\hbar(\Omega_1|1\rangle\langle 3|+\Omega_2|2\rangle\langle 3|+2\Delta|3\rangle\langle 3|)+h.c., \quad (1)$$

where the Rabi frequencies $\Omega_1=\Omega\sin\theta e^{i\varphi}$ and $\Omega_2=$

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$\Omega\cos\theta$ with $\Omega = \sqrt{|\Omega_1|^2+|\Omega_2|^2}$($\theta$, φ are the variable parameters). In this paper, we assume that the parameters θ, φ and Ω may be time-dependent but position-independent. Usually it is more convenient to study the dress state representation, while the dressed states are the eigen-states of the Hamiltonian (1). The dressed states $|\chi\rangle = (|\chi_1\rangle,\ |\chi_2\rangle,\ |\chi_3\rangle)^{Tr}$ (as shown in Fig.1(b)) of the Hamiltonian H are specified by $|\chi\rangle = \Gamma(|1\rangle,\ |2\rangle,\ |3\rangle)^{Tr}$($Tr$ denotes the transposition), where

$$\Gamma = \begin{pmatrix} \cos\theta & -\sin\theta e^{-i\varphi} & 0 \\ \sin\theta\cos\gamma e^{i\varphi} & \cos\theta\cos\gamma & -\sin\gamma \\ \sin\theta\sin\gamma e^{i\varphi} & \cos\theta\sin\gamma & \cos\gamma \end{pmatrix}, \qquad (2)$$

and γ is given by $\tan\gamma = (\sqrt{\Delta^2+\Omega^2}-\Delta)/\Omega$, with the corresponding eigenvalues $\lambda = (\ 0, \Delta-\sqrt{\Delta^2+\Omega^2},\ \Delta+\sqrt{\Delta^2+\Omega^2})$ [20]. Under the large detuning case $\Delta \gg \Omega$, we could get $\gamma = 0$ for $\tan\gamma \to 0$. Then the two dressed states $\{|\chi_1\rangle,\ |\chi_2\rangle\}$ have negligible contribution from the excited state $|\chi_3\rangle$.Thus a subspace is spanned by the two lower dressed states$\{|\chi_1\rangle,\ |\chi_2\rangle\}$, where

$$\begin{array}{l} |\chi_1\rangle = \cos\theta|1\rangle - \sin\theta e^{-i\varphi}|2\rangle, \\ |\chi_2\rangle = \sin\theta e^{i\varphi}|1\rangle + \cos\theta|2\rangle. \end{array} \qquad (3)$$

In this subspace, the spontaneous emission can be neglected for $(\lambda_2-\lambda_1)/(\lambda_3-\lambda_2) \approx 0$. Then the jumping from $|\chi_1\rangle,\ |\chi_2\rangle$ to $|\chi_3\rangle$ can also be neglected. Thus the adiabatic condition can be fulfilled. With the two degenerate states ($\Omega \gg \Delta - \sqrt{\Delta^2+\Omega^2} \approx 0$ when $\Delta \gg \Omega$) shown in Eq.3, the non-Abelian adiabatic geometric phases can be induced in the subspace spanned by $|\chi_1\rangle$ and $|\chi_2\rangle$.

We briefly recall the general formalism of non-Abelian geometric phases[10, 13]. Considering a Hamiltonian whose time-dependence is described by a close curve $\mathbf{C}$ in a parameter space, i.e.: $H(t) = H(R_\mu(t))$ ($\mu = 1, 2, \cdots, n$) with $R(t) \in \mathbf{C}$, which has a set of N degenerate levels. According to the adiabatic theorem, an instantaneous state $|\Psi_a\rangle$ can always be expanded as a superposition of $|\eta_a\rangle$: $|\Psi_a\rangle = \sum_b |\eta_b\rangle U_{ba}$, where $|\eta_a\rangle$ is an N-fold degenerate set of orthogonal instantaneous eigenstates of the Hamiltonian $H(t)$. Substituting the wave function $|\Psi_a\rangle$ into the Schrödinger equation $i\frac{d}{dt}|\Psi_a\rangle = H(t)|\Psi_a\rangle$, we get:

$$\dot{U}_{ba} = -\sum_c \langle\eta_b|\dot{\eta_c}\rangle U_{ca}. \qquad (4)$$

We can use the relationship $\langle\eta_b|\dot{\eta_c}\rangle = \sum_\mu A_{ab\mu}(dR^\mu/dt)$ to define a gauge potential as follows

$$A_{ab\mu} = \langle\eta_a|\frac{\partial}{\partial R^\mu}|\eta_b\rangle. \qquad (5)$$

Then a matrix of formal solution to Eq.(4) can be obtained by direct integration,

$$U_{ab} = [\mathcal{P}\exp(-\int A_\mu dR^\mu)]_{ab}, \qquad (6)$$

where $\mathcal{P}$ denotes the time-ordering operator. The quantity $A = \sum_\mu A_\mu dR^\mu$ is defined in Eq.(5). Substituting Eq.(3) into (5), we have

$$\begin{aligned} A_\theta &= -i\sigma_y\cos\varphi + i\sigma_x\sin\varphi, \\ A_\varphi &= -i\sin^2\theta\sigma_z + i\sin\theta\cos\theta\cos\varphi\sigma_x \\ &\quad -i\sin\theta\cos\theta\sin\varphi\sigma_y. \end{aligned} \qquad (7)$$

Clearly the gauge potential $\mathbf{A}$ has non-Abelian feature. Substituting Eq.(7) into (6), we can obtain

$$U = \exp(-\int A_\theta d\theta - \int A_\varphi d\varphi). \qquad (8)$$

Note that

$$\begin{array}{l} \vartheta_x = -\int\sin\varphi d\theta - \int\sin\theta\cos\theta\cos\varphi d\varphi, \\ \vartheta_y = \int\cos\varphi d\theta + \int\sin\theta\cos\theta\sin\varphi d\varphi, \\ \vartheta_z = \int\sin^2\theta d\varphi, \end{array} \qquad (9)$$

then the evolution operator is given by

$$U = e^{i\overrightarrow{\sigma}\overrightarrow{\vartheta}(t)}, \qquad (10)$$

where $\overrightarrow{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\overrightarrow{\vartheta}(t) = (\vartheta_x, \vartheta_y, \vartheta_z)$. Therefore the entire phase factor which contains geometric phase as well as dynamic phase and the evolution operator in the three level Λ system is obtained.

We now turn to construct two closed pathes that can be used to detect non-Abelian features of the entire phases. The detection of these features of the pure geometric phases will come after we have applied additional processes to cancel the dynamic phases. From now on, we choose two specific closed paths in the parameter space(both parameters θ and φ vary with time) to analyze the non-Abelian features numerically. We choose the Rabi frequencies in the closed loop C_1 as

$$\Omega_1 = \Omega_0 f(t),\ \Omega_2 = \Omega_0 e^{-t^2/\tau^2}, \qquad (11)$$

where $\varphi = \frac{2\pi}{\tau}t$ and $f(t)$ is

$$f(t) = \begin{cases} \sin(\frac{\pi t}{\tau}), & 0 \le t \le \tau \\ 0. & \text{others} \end{cases} \qquad (12)$$

Thus the parameter $\theta(t)$ is determined by

$$\tan\theta(t) = \frac{|\Omega_1|}{|\Omega_2|} = \sin(\frac{\pi t}{\tau})e^{t^2/\tau^2}. \qquad (13)$$

From 0 to τ, the parameters (θ, φ) vary from $(0, 0)$ to $(0, 2\pi)$, so a closed path in the parameter space is formed by modulating the laser beams. Substituting Eq.(11) into (10), we obtain the gauge potential A_1, and the corresponding evolution operator is

$$U_1 = \mathcal{P}\exp(-\oint_{C_1} A_1^\mu d\chi^\mu). \qquad (14)$$

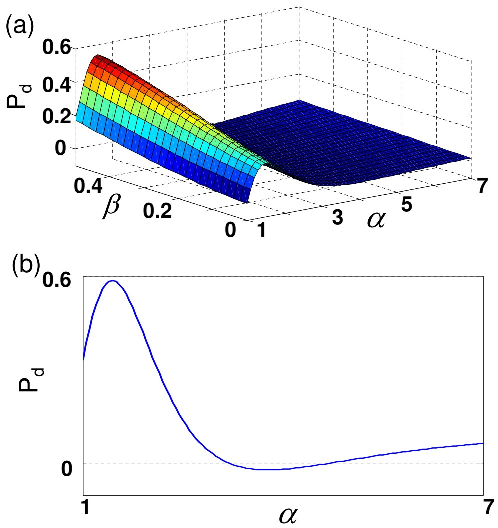

FIG. 2: (color online). The population difference P_d versus different parameters of α and β. The population difference is induced by the entire phase which contains geometric phase as well as dynamic phase. (b)The population difference P_d versus parameter α when $\beta = 0.8$.

Similarly, we set the Rabi frequencies of the closed loop C_2 as follows

$$\Omega_1 = \alpha\Omega_0 f(t),\ \Omega_2 = \Omega_0 e^{-(t-\beta)^2/\tau^2}. \tag{15}$$

Here we have introduced the variables α, β with α being an amplitude attenuation and β being a time delay to differentiate the two closed paths. Obviously, the starting and ending point of C_2 are set to be the same as C_1. The corresponding evolution operator of C_2 is given by

$$U_2 = \mathcal{P}\exp\left(-\oint_{C_2} A_2^\mu d\chi^\mu\right), \tag{16}$$

where A_2 is the corresponding gauge potential along the path C_2. Supposed that the initial eigenstates $|\chi_{1,2}\rangle_i$ and the final eigenstates $|\chi_{1,2}\rangle_f$ after the evolution of C_1, C_2 are given by

$$|\chi_1\rangle_i = |\chi_1\rangle_f = |1\rangle,\ |\chi_2\rangle_i = |\chi_2\rangle_f = |2\rangle. \tag{17}$$

If the system is prepared to the state$|\Psi\rangle_i = |\chi_1\rangle_i$ initially, after undergoing a composed path(first C_1, and then C_2), the total evolution operator should be $U = U_2U_1$. On the other hand, if the system undergoes the counter-order composed path, the evolution operator is $U' = U_1U_2$. Now, we will focus on the elements U_{11} and U'_{11} in the matrix U and U', respectively. We find that $P = |U_{11}|^2$ and $P' = |U'_{11}|^2$ are the probabilities of the final state in the state $|1\rangle$ corresponding to the end of the evolution U and U', respectively. Thus the population difference between the two composite pathes is given by

$$P_d = P - P' = |U_{11}|^2 - |U'_{11}|^2. \tag{18}$$

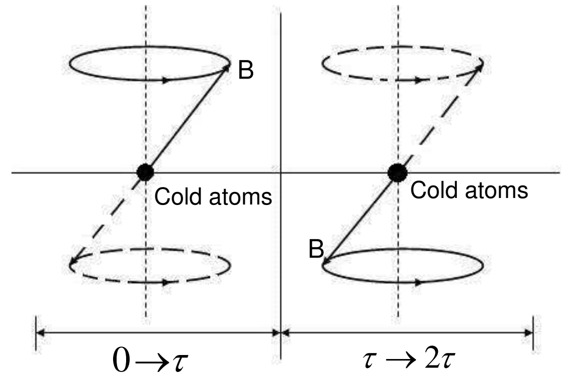

FIG. 3: Schematic representation of the multi-loop method to cancel the effects of dynamic phases. The cold atoms are interacting with lasers, its Hamiltonian evolves in the parameter space as that a spin particle subject to an effective magnetic field (denoted as **B**).

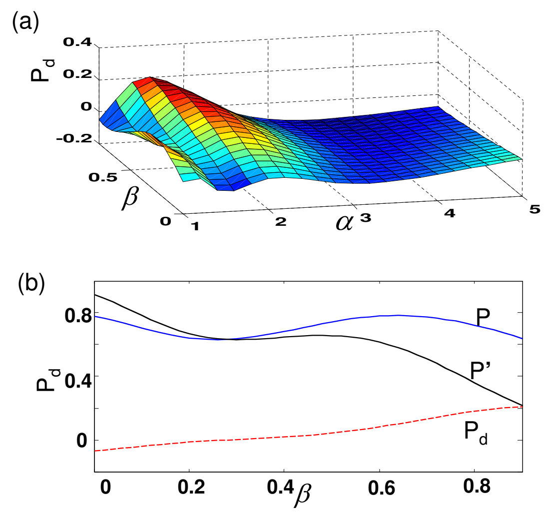

FIG. 4: (color online). The population difference P_d versus the parameters of α and β. The population difference is induced by the pure geometric phase after canceling the dynamic phase. (b)The population difference P_d versus parameter β when $\alpha = 0.8$.

Therefore,we can detect the noncommutative feature through measuring P_d. If $P_d \neq 0$, the non-Abelian gauge structure is confirmed.

The quantitative results of P_d versus different α and β are plotted in Fig.2(a). Since the operation time of the laser beams is τ, let $U_1(U_2)$ implement during time $0 \to \tau$, while $U_2(U_1)$ implement from τ to 2τ for the evolution $U(U')$, the non-Abelian gauge structure of the entire phase can be demonstrated from the difference P_d. Fig.2(b) is a further detailed diagram with parameters $\beta = 0.8$ and α varies from 1 to 7.

The approximately degenerated subspace spanned by the two lower dressed states $|\chi_1\rangle, |\chi_2\rangle$ stands for an effective spin$-1/2$ system. Similar to this spin$-1/2$ system, the effective magnetic field will introduce an external dynamic phase, which will also contribute to the population difference. In order to detect the effects induced from pure geometric phase, we should develop a method to remove the dynamic phase. A general approach is the multi-loop method to cancel the dynamical phases: let the state $|\Psi\rangle$ evolve along the time-reversal path of the first-period loop during the second period[21]. Although the dynamic phase of each single loop doesn't vanish, the total dynamic phase will be canceled by the overlapping of the two loops. In our system, this procedure can be achieved by reversing the effective magnetic field $\mathbf{B}(2\tau - t) = -\mathbf{B}(t)$ on the same loop of the first-period $[0, \tau)$ (as shown in Fig.3). The effective magnetic field $\mathbf{B}$ is parameterized by θ, φ, and it naturally satisfies the relationship: $\mathbf{B}(\theta, \varphi) = -\mathbf{B}(-\theta, -\varphi)$ in the rotating frame. It is straightforward to find that, θ, φ in equation(10) under the time-reversal transform $t \to -t$ become

$$\begin{aligned} &\theta(t) \to -\theta(-t),\ \varphi(t) \to -\varphi(-t), \\ &d\theta(t) \to d\theta(-t),\ d\varphi(t) \to -d\varphi(-t). \end{aligned} \tag{19}$$

Substituting Eq.(19) into (9), we can get $\vartheta(-t)$. Hence, we have the evolution of the time-reversal loop $U_- = e^{i\overrightarrow{\sigma}\overrightarrow{\vartheta}(-t)}$, so the pure geometric evolution operator can be given by

$$U_g = \frac{1}{2} U_- U. \tag{20}$$

According to Eq.(20), the population difference P_d with evolution U_g is shown in Fig.4(a). The noncommutative property is still very clear for the non-vanishing difference P_d. As shown in Fig.4(b), $P_d = 21.01\%$ when $\beta = 0.89$ and $\alpha = 1.42$, which is high enough to be detected in the experiments.

In addition, we briefly discuss the experimental feasibility of the proposed scheme. Firstly, we need $\Delta \gg \Omega_0$(adiabatic condition); Secondly, $\Omega_0 \gg 1/\Gamma_c$ (Γ_c is the coherence time of the dark state, stimulated Raman adiabatic passage condition). When setting the Rabi frequency $\Omega_0 = 2\pi \times 0.1MHz$ and the detuning $\Delta \geq 2\pi \times 100MHz$, which are feasible in experiment[22]. Thus, all the conditions can be fulfilled.

In conclusion, we have proposed an experimentally feasible scheme to detect the pure non-Abelian geometric phases with three level Λ atoms. By designing two specific composite cyclic evolutions formed by C_1C_2 and C_2C_1, we have shown in detail that the non-Abelian geometric phases can be observed through detecting the population of the internal states. The observation of the truly non-Abelian geometric phase is important in the field of quantum theory as well as a crucial step toward fault-tolerant geometric quantum computation.

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