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# Two-dimensional Noncommutative atom Gas with Anandan interaction

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Landau like quantization of the Anandan system in a special electromagnetic field is studied. Unlike the cases of the AC system and the HMW system, the torques of the system on the magnetic dipole and the electric dipole don't vanish. By constructing Heisenberg algebra, the Landau analog levels and eigenstates on commutative space, NC space and NC phase space are obtained respectively. By using the coherent state method, some statistical properties of such free atom gas are studied and the expressions of some thermodynamic quantities related to revolution direction are obtained. Two particular cases of temperature are discussed and the more simple expressions of the free energy on the three spaces are obtained. We give the relation between the value of  $\sigma$  and revolution direction clearly, and find Landau like levels of the Anandan system are invariant and the levels between the AC system and the HMW system are interchanged each other under Maxwell dual transformations on the three spaces. The two sets of eigenstates labelled by  $\sigma$  can be related by a supersymmetry transformation on commutative space, but the phenomenon don't occur on NC situation. We emphasize that some results relevant to Anandan interaction are suitable for the cases of AC interaction and HMW interaction under special conditions.

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## 1. INTRODUCTION

There are many papers to study the topological and geometrical effects of charged and neutral particles in the presence of electromagnetic fields in their quantum dynamics. One well known topological effect is the Aharonov-Bohm (AB) effect which shows that a quantum charge circulating a magnetic flux line accumulates a quantum topological phase and demonstrates the physical significance of magnetic vector potential.[1] The effect can be observed by using matter-wave interferometry. Another well known quantum phase is the Aharonov-Casher (AC) phase acquired by a neutral particle with a non-zero magnetic dipole moment, circulating a straight line of charge [2], which is a non-dispersive quantum geometrical phase [3] and was observed experimentally in a neutron interferometer [4] and in a neutral atomic Ramsey interferometer [5]. Its Maxwell dual phase is the He-Mckellar-Wilkens (HMW) phase which implies that a neutral particle with a non-zero electric dipole moment moving around a line of magnetic monopoles would accumulate a quantum phase.[6] Wei, Han and Wei proposed a practical experimental configuration to test this effect [7], and Dowling et al proposed two other experimental schemes for it [8]. Recently, another non-dispersive quantum geometrical phase was proposed by Anandan. It describes that a neutral particle with permanent electric dipole moment and non-vanishing magnetic dipole moment in the presence of external magnetic and electric fields in the relativistic and non-relativistic case accumulates a quantum phase.[9]-[11] This phase is called the Anandan phase.

The interaction of the magnetic field with a charged particle in two dimensions plays an important role in Landau quantization and quantum *Hall effect*. Inspired by the work of Paredes *et al.*, proving the existence of anionic excitation in rotating Bose-Einstein condensates [12], Ericsson and Sjöqvist used the AC interaction and developed an analog of Landau quantization which provide the possibility of an atomic quantum *Hall effect* [13]. Following these steps, Ribeiro *et al.* made use of the HMW interaction to generate another analog of Landau quantization.[14]

Recently, the study of physics effects on noncommutative (NC) space and noncommutative phase space has attracted much interest.[15]-[26] There are many researches in NC quantum mechanics such as the AB effect [27]-[31], the AC effect [32, 33], the HMW effect [14, 34], Landau levels [35]-[37], quantum *Hall effect* [38]-[40], and so on. Unlike the case in usual quantum mechanics, Passos *et al.* have demonstrated that the AC phase, the HMW phase and the Anandan phase are quantum geometric dispersive phases on NC space and NC phase space.[41] They

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have also obtained noncommutative Landau like quantization with the AC interaction and the HMW interaction respectively.[42] In this letter, we will use the Anandan interaction for another analog of Landau quantization in the non-relativistic limit on commutative space (C), NC space and NC phase space. Unlike the Landau like quantization of the AC system and the HMW system, both torques of the Anandan system on the magnetic dipole and the electric dipole don't vanish. We use the coherent state method to give out the expressions of some thermodynamic quantities related to revolution direction for the free atom gas with the Anandan interaction and discuss the free energy of the system on two particular cases of temperature on the three spaces. By the description of atom orbits, we give the relation between the value of  $\sigma$  and revolution direction. We find that Landau like levels of the Anandan system are invariant and the levels between the AC system and the HMW system are interchanged each other under Maxwell dual transformations on the three spaces. We think the levels of the AC system and the HMW system are similar under the same revolution direction which is different from the point of Ribeiro *et al.*[14] and analyze their mistake from the angles of revolution direction and Maxwell duality. Some difference between commutative space and NC situation is also discussed from the supersymmetry transformation. Some results from the Anandan system can be restricted to the ones of the AC system and the HMW system.

This paper is organized as follows: In section 2 and 3 the Landau like quantization of a neutral atom with permanent electric and magnetic dipole moments in a special electromagnetic field on commutative space and NC situation is studied. In section 4 some thermodynamic properties of such free atom gas are studied and two particular cases of temperature are discussed on commutative space, NC space and NC phase space. Some results are restricted to the AC system and the HMW system. In section 5 we study the relation between the value of  $\sigma$  and revolution direction by the coherent state, discuss the levels of the Anandan system, the AC system and the HMW system by Maxwell duality, and analyze the mistake of Ribeiro *et al.* by the angles of revolution direction and Maxwell duality. In section 6, by super-symmetric study, we give out the difference of the system between commutative space and NC situation. Finally, in section 7 we present our conclusions.

## 2. LANDAU LEVELS ANALOG FOR NEUTRAL ATOMS ON COMMUTATIVE SPACE

In the non-relativistic limit, the Hamiltonian for a neutral spin-half atom possessing a non-zero electric dipole moment  $\mathbf{d}$  and a non-zero magnetic dipole moment  $\mathbf{u}$ , moving in an external electromagnetic fields, can be described by the Anandan Hamiltonian [41],

$$H = \frac{1}{2m}[\mathbf{P} - c^{-2}(\mathbf{u} \times \mathbf{E}) + c^{-2}(\mathbf{d} \times \mathbf{B})]^2 - \frac{u\hbar}{2mc^2}\nabla \cdot \mathbf{E} + \frac{d\hbar}{2mc^2}\nabla \cdot \mathbf{B} \quad (1)$$

where the terms of  $O(E^2)$  and  $O(B^2)$  are neglected. Considering the quantum dynamics of a particle, the Hamiltonian in fact contains two other physical situations, the AC effect ( $u \neq 0$  and  $d = 0$ ) and the HMW effect ( $u = 0$  and  $d \neq 0$ ). In the present problem, the kinematic momentum is  $\Pi = -i\hbar\nabla - c^{-2}(\mathbf{u} \times \mathbf{E}) + c^{-2}(\mathbf{d} \times \mathbf{B})$  and the Anandan vector potential is defined as  $\mathbf{A} = (\mathbf{u} \times \mathbf{E}) - (\mathbf{d} \times \mathbf{B})$ . Thus the associated field strength is  $\mathbf{B}_{eff} = \nabla \times \mathbf{A}$ . Here we consider the atom moves in the  $x - y$  plane, and its magnetic dipole moment as well as electric dipole moment are in the  $z$ -direction. At the same time, the electric field configuration  $E = \frac{\rho_e}{2}r\hat{e}_\phi$  and the magnetic field configuration  $B = \frac{\rho_m}{2}r\hat{e}_\phi$  mentioned in Ref.[14] are still used. Obviously,  $\mathbf{B}_{eff}$  is uniform and the conditions for electrostatics and magnetostatics are satisfied. But unlike the cases of Landau analogous quantization with the AC interaction and the HMW interaction, both torques on the magnetic dipole and the electric dipole don't vanish, because the atom with momentum  $\langle \Pi \rangle$  sees an effective magnetic field  $\mathbf{B}' \cong \mathbf{B} + \mathbf{v} \times \mathbf{E}/c$  and an effective electric field  $\mathbf{E}' \cong \mathbf{E} - \mathbf{v} \times \mathbf{B}/c$  in its own reference frame. The Hamiltonian (1) in such dipole-field configuration can be written as

$$H = \frac{1}{2m}[(p_x + \frac{u\rho_e - d\rho_m}{2c^2}y)\hat{e}_x + (p_y - \frac{u\rho_e - d\rho_m}{2c^2}x)\hat{e}_y]^2 - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2} \quad (2)$$

Further, the Hamiltonian can be expressed as

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{8}m\omega^2(x^2 + y^2) + \frac{\omega}{2}(p_x y - p_y x) - \frac{\omega\hbar}{2} \quad (3)$$

with the cyclotron frequency  $\omega = \sigma \frac{|u\rho_e - d\rho_m|}{mc^2} = \frac{u\rho_e - d\rho_m}{mc^2}$  where  $\sigma = \pm 1$ . Thus, the cyclotron frequencies of the AC system and the HMW system are  $\omega_{AC} = \sigma \frac{|u\rho_e|}{mc^2} = \frac{u\rho_e}{mc^2}$  and  $\omega_{HMW} = \sigma \frac{|d\rho_m|}{mc^2} = \frac{-d\rho_m}{mc^2}$ , respectively. If we introduce

the operators,

$$\begin{aligned} a_x &= \sqrt{\frac{m|\omega|}{8\hbar}}\left(x + \frac{2ip_x}{m|\omega|}\right) - i\sqrt{\frac{m|\omega|}{8\hbar}}\left(y + \frac{2ip_y}{m|\omega|}\right), & a_x^+ &= \sqrt{\frac{m|\omega|}{8\hbar}}\left(x - \frac{2ip_x}{m|\omega|}\right) + i\sqrt{\frac{m|\omega|}{8\hbar}}\left(y - \frac{2ip_y}{m|\omega|}\right), \\ a_y &= i\sqrt{\frac{m|\omega|}{8\hbar}}\left(x + \frac{2ip_x}{m|\omega|}\right) - \sqrt{\frac{m|\omega|}{8\hbar}}\left(y + \frac{2ip_y}{m|\omega|}\right), & a_y^+ &= -i\sqrt{\frac{m|\omega|}{8\hbar}}\left(x - \frac{2ip_x}{m|\omega|}\right) - \sqrt{\frac{m|\omega|}{8\hbar}}\left(y - \frac{2ip_y}{m|\omega|}\right), \end{aligned} \quad (4)$$

where they satisfy Heisenberg algebraic relations

$$[a_i, a_j] = 0, [a_i^+, a_j^+] = 0, [a_i, a_j^+] = \delta_{ij}, \quad i, j = x, y, \quad (5)$$

the Hamiltonian becomes

$$H = \frac{\hbar|\omega|}{2}(a_x^+ a_x + a_y^+ a_y + 1) + \frac{\hbar\sigma|\omega|}{2}(a_y^+ a_y - a_x^+ a_x) - \frac{\sigma|\omega|\hbar}{2}. \quad (6)$$

Therefore, the eigenvalues of  $H$  are

$$E_{n_x, n_y} = \frac{\hbar|\omega|}{2}(n_x + n_y + 1) + \frac{\hbar\sigma|\omega|}{2}(n_y - n_x) - \frac{\sigma|\omega|\hbar}{2}, \quad (7)$$

where non-negative integers  $n_x, n_y$  are the eigenvalues of the number operators  $a_x^+ a_x, a_y^+ a_y$ , respectively. The corresponding eigenstates are

$$|n_x, n_y\rangle = \frac{1}{\sqrt{n_x! n_y!}} (a_x^+)^{n_x} (a_y^+)^{n_y} |0, 0\rangle, \quad (8)$$

where  $|0, 0\rangle$  is the vacuum of  $H$ . Thus, the levels for the AC system and the HMW system are

$$E_{n_x, n_y}^{AC} = \frac{\hbar|\omega_{AC}|}{2}(n_x + n_y + 1) + \frac{\hbar\sigma|\omega_{AC}|}{2}(n_y - n_x) - \frac{\sigma|\omega_{AC}|\hbar}{2} \quad (9)$$

and

$$E_{n_x, n_y}^{HMW} = \frac{\hbar|\omega_{HMW}|}{2}(n_x + n_y + 1) + \frac{\hbar\sigma|\omega_{HMW}|}{2}(n_y - n_x) - \frac{\sigma|\omega_{HMW}|\hbar}{2} \quad (10)$$

respectively.

### 3. LANDAU LEVELS ANALOG FOR NEUTRAL ATOMS ON NC PHASE SPACE AND NC SPACE

In order to maintain the Bose-Einstein statistics in NC quantum mechanics, non-commutativity of both space-space and momentum-momentum may be necessary.[26] We need to consider physical problems on this space, NC phase space, where the coordinates  $\hat{x}_i$  and momentums  $\hat{p}_i$  satisfy the following relations:

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\bar{\Theta}_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (11)$$

where the elements of the antisymmetric matrices  $\{\Theta_{ij}\}$  and  $\{\bar{\Theta}_{ij}\}$  are very small and represent space-space non-commutativity and momentum-momentum non-commutativity. The static Schrödinger equation on NC phase space is usually expressed as

$$H(x, p) * \psi = E\psi, \quad (12)$$

where  $H(x, p)$  is the Hamiltonian of the usual quantum system and a star product (Moyal-Weyl product) is defined by

$$(f * g)(x, p) = e^{\frac{i}{2\alpha^2}\Theta_{ij}\partial_i^x\partial_j^x + \frac{i}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p\partial_j^p} f(x, p)g(x, p) = f(x, p)g(x, p) + \frac{i}{2\alpha^2}\Theta_{ij}\partial_i^x f \partial_j^x g|_{x_i=x_j} + \frac{i}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p f \partial_j^p g|_{p_i=p_j}. \quad (13)$$

By replacing the star product by the general ordinary product and shifting coordinates  $x_i$  and momentums  $p_i$  with [18]

$$\begin{aligned} x_i &\longrightarrow \hat{x}_i = \alpha x_i - \frac{1}{2\hbar\alpha}\Theta_{ij}p_j, \\ x_i &\longrightarrow \hat{p}_i = \alpha p_i + \frac{1}{2\hbar\alpha}\bar{\Theta}_{ij}x_j, \end{aligned} \quad (14)$$

the Schrödinger equation can be written as

$$H(\alpha x_i - \frac{1}{2\hbar\alpha}\Theta_{ij}p_j, \alpha p_i + \frac{1}{2\hbar\alpha}\bar{\Theta}_{ij}x_j)\psi = E\psi. \quad (15)$$

Here  $\alpha$  is a scaling constant related to the non-commutativity of phase space and satisfies the relation  $\theta\bar{\theta} = 4\alpha^2(1 - \alpha^2)$ [43]. So the Hamiltonian of the system on NC phase space can be written as

$$H = \frac{1}{2m}[(\alpha p_x + \frac{1}{2\hbar\alpha}\bar{\theta}y) + \frac{u\rho_e - d\rho_m}{2c^2}(\alpha y + \frac{1}{2\hbar\alpha}\theta p_x)]\hat{e}_x + [(\alpha p_y - \frac{1}{2\hbar\alpha}\bar{\theta}x) - \frac{u\rho_e - d\rho_m}{2c^2}(\alpha x - \frac{1}{2\hbar\alpha}\theta p_y)]\hat{e}_y]^2 - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2}, \quad (16)$$

where we set  $\Theta_{xy} = \theta$  and  $\bar{\Theta}_{xy} = \bar{\theta}$ . After organization, the Hamiltonian can be written as

$$H = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{1}{8}M\bar{\omega}^2(x^2 + y^2) + \frac{\bar{\omega}}{2}(p_x y - p_y x) - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2} \quad (17)$$

where the modified mass is

$$M = m(\alpha + \frac{(u\rho_e - d\rho_m)\theta}{4\alpha\hbar c^2})^{-2} \quad (18)$$

and the modified cyclotron frequency is

$$\bar{\omega} = \frac{2\sigma}{m} |(\alpha + \frac{(u\rho_e - d\rho_m)\theta}{4\alpha\hbar c^2})(\frac{\bar{\theta}}{2\alpha\hbar} + \frac{(u\rho_e - d\rho_m)\alpha}{2c^2})| = \frac{2}{m}(\alpha + \frac{(u\rho_e - d\rho_m)\theta}{4\alpha\hbar c^2})(\frac{\bar{\theta}}{2\alpha\hbar} + \frac{(u\rho_e - d\rho_m)\alpha}{2c^2}). \quad (19)$$

For the AC system and the HMW system, the modified masses and cyclotron frequencies are

$$M_{AC} = m(\alpha + \frac{u\rho_e\theta}{4\alpha\hbar c^2})^{-2}, \quad \bar{\omega}_{AC} = \frac{2\sigma}{m} |(\alpha + \frac{u\rho_e\theta}{4\alpha\hbar c^2})(\frac{\bar{\theta}}{2\alpha\hbar} + \frac{u\rho_e\alpha}{2c^2})| = \frac{2}{m}(\alpha + \frac{u\rho_e\theta}{4\alpha\hbar c^2})(\frac{\bar{\theta}}{2\alpha\hbar} + \frac{u\rho_e\alpha}{2c^2}) \quad (20)$$

and

$$M_{HMW} = m(\alpha - \frac{d\rho_m\theta}{4\alpha\hbar c^2})^{-2}, \quad \bar{\omega}_{HMW} = \frac{2\sigma}{m} |(\alpha - \frac{d\rho_m\theta}{4\alpha\hbar c^2})(\frac{\bar{\theta}}{2\alpha\hbar} - \frac{d\rho_m\alpha}{2c^2})| = \frac{2}{m}(\alpha - \frac{d\rho_m\theta}{4\alpha\hbar c^2})(\frac{\bar{\theta}}{2\alpha\hbar} - \frac{d\rho_m\alpha}{2c^2}). \quad (21)$$

Similarly we can define the creation and annihilation operators as

$$\begin{aligned} \bar{a}_x &= \sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(x + \frac{2ip_x}{M|\bar{\omega}|}) - i\sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(y + \frac{2ip_y}{M|\bar{\omega}|}), & \bar{a}_x^+ &= \sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(x - \frac{2ip_x}{M|\bar{\omega}|}) + i\sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(y - \frac{2ip_y}{M|\bar{\omega}|}), \\ \bar{a}_y &= i\sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(x + \frac{2ip_x}{M|\bar{\omega}|}) - \sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(y + \frac{2ip_y}{M|\bar{\omega}|}), & \bar{a}_y^+ &= -i\sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(x - \frac{2ip_x}{M|\bar{\omega}|}) - \sqrt{\frac{M|\bar{\omega}|}{8\hbar}}(y - \frac{2ip_y}{M|\bar{\omega}|}). \end{aligned} \quad (22)$$

Thus, the Hamiltonian on NC phase space becomes the following form

$$H = \frac{\hbar|\bar{\omega}|}{2}(\bar{a}_x^+\bar{a}_x + \bar{a}_y^+\bar{a}_y + 1) + \frac{\hbar\sigma|\bar{\omega}|}{2}(\bar{a}_y^+\bar{a}_y - \bar{a}_x^+\bar{a}_x) - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2}. \quad (23)$$

The energy eigenvalues and the corresponding eigenstates are

$$E_{\bar{n}_x, \bar{n}_y} = \frac{\hbar|\bar{\omega}|}{2}(\bar{n}_x + \bar{n}_y + 1) + \frac{\hbar\sigma|\bar{\omega}|}{2}(\bar{n}_y - \bar{n}_x) - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2} \quad (24)$$

and

$$|\bar{n}_x, \bar{n}_y\rangle = \frac{1}{\sqrt{\bar{n}_x!\bar{n}_y!}}(\bar{a}_x^+)^{\bar{n}_x}(\bar{a}_y^+)^{\bar{n}_y}|0, 0\rangle. \quad (25)$$

The levels for the AC system and the HMW system are

$$E_{\bar{n}_x, \bar{n}_y}^{AC} = \frac{\hbar|\bar{\omega}_{AC}|}{2}(\bar{n}_x + \bar{n}_y + 1) + \frac{\hbar\sigma|\bar{\omega}_{AC}|}{2}(\bar{n}_y - \bar{n}_x) - \frac{u\rho_e\hbar}{2mc^2} \quad (26)$$

and

$$E_{\bar{n}_x, \bar{n}_y}^{HMW} = \frac{\hbar|\bar{\omega}_{HMW}|}{2}(\bar{n}_x + \bar{n}_y + 1) + \frac{\hbar\sigma|\bar{\omega}_{HMW}|}{2}(\bar{n}_y - \bar{n}_x) + \frac{d\rho_m\hbar}{2mc^2}. \quad (27)$$

When  $\bar{\theta} = 0$ , it leads  $\alpha = 1$ [43] and the phase space becomes the NC space where only momentum-momentum is commutative. The energy eigenvalues (24), (26) and (27) become

$$E_{\bar{n}'_x, \bar{n}'_y} = \frac{\hbar|\bar{\omega}'|}{2}(\bar{n}'_x + \bar{n}'_y + 1) + \frac{\hbar\sigma|\bar{\omega}'|}{2}(\bar{n}'_y - \bar{n}'_x) - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2}, \quad (28)$$

$$E_{\bar{n}'_x, \bar{n}'_y}^{AC} = \frac{\hbar|\bar{\omega}'_{AC}|}{2}(\bar{n}'_x + \bar{n}'_y + 1) + \frac{\hbar\sigma|\bar{\omega}'_{AC}|}{2}(\bar{n}'_y - \bar{n}'_x) - \frac{u\rho_e\hbar}{2mc^2}, \quad (29)$$

$$E_{\bar{n}'_x, \bar{n}'_y}^{HMW} = \frac{\hbar|\bar{\omega}'_{HMW}|}{2}(\bar{n}'_x + \bar{n}'_y + 1) + \frac{\hbar\sigma|\bar{\omega}'_{HMW}|}{2}(\bar{n}'_y - \bar{n}'_x) + \frac{d\rho_m\hbar}{2mc^2}, \quad (30)$$

respectively. Here

$$\bar{\omega}' = \frac{\sigma}{mc^2} \left| 1 + \frac{(u\rho_e - d\rho_m)\theta}{4\hbar c^2} \right| (u\rho_e - d\rho_m) = \frac{1}{mc^2} \left( 1 + \frac{(u\rho_e - d\rho_m)\theta}{4\hbar c^2} \right) (u\rho_e - d\rho_m), \quad (31)$$

$$\bar{\omega}'_{AC} = \frac{\sigma}{mc^2} \left| 1 + \frac{u\rho_e\theta}{4\hbar c^2} \right| u\rho_e = \frac{u\rho_e}{mc^2} \left( 1 + \frac{u\rho_e\theta}{4\hbar c^2} \right), \quad (32)$$

$$\bar{\omega}'_{HMW} = \frac{\sigma}{mc^2} \left| 1 - \frac{d\rho_m\theta}{4\hbar c^2} \right| d\rho_m = -\frac{d\rho_m}{mc^2} \left( 1 - \frac{d\rho_m\theta}{4\hbar c^2} \right). \quad (33)$$

When  $\theta = \bar{\theta} = 0$ , the results return to the cases of general quantum mechanics (7), (9) and (10).

#### 4. THERMODYNAMICS ON NC SITUATION

Up to now, we can obtain Landau like levels of the Anandan system for a given  $\sigma$  on commutative space, NC space and NC phase space. Glauber has discussed the coherent states of a harmonic oscillator in detail.[44] Using the coherent state method, we can know the quantum information of the system. Like free electron gas related to Landau problems, we will study some thermodynamical properties of such free atom gas with Anandan interaction. The normalized coherent state of the system on NC phase space is defined as

$$|z_x, z_y\rangle = \text{Exp}\left[-\frac{1}{2}(|z_x|^2 + |z_y|^2)\right] \text{Exp}[z_x \bar{a}_x^\dagger + z_y \bar{a}_y^\dagger] |0, 0\rangle \quad (34)$$

where  $z_x, z_y$  are complex parameters. The mean coordinates of the state  $|z_x, z_y\rangle$  related to  $\sigma$  is given by

$$\begin{aligned} \bar{r}_{\sigma=-1} &= (\text{Re}[Az_x^*(t) + B(-iz_y)], \text{Im}[Az_x^*(t) + B(-iz_y)]), \\ \bar{r}_{\sigma=1} &= (\text{Re}[Az_x^* + B(-iz_y(t))], \text{Im}[Az_x^* + B(-iz_y(t))]), \end{aligned} \quad (35)$$

where  $A = \alpha\sqrt{\frac{2\hbar}{M|\bar{\omega}|}} - \frac{\theta}{2\alpha}\sqrt{\frac{M|\bar{\omega}|}{2\hbar}}$ ,  $B = \alpha\sqrt{\frac{2\hbar}{M|\bar{\omega}|}} + \frac{\theta}{2\alpha}\sqrt{\frac{M|\bar{\omega}|}{2\hbar}}$  and  $z_x^*(t) = z_x^* e^{i|\bar{\omega}|t}$ ,  $z_y(t) = z_y e^{-i|\bar{\omega}|t}$ . Here we consider a two-dimensional system with radius  $R \gg A, B$ . The partition function of the system on NC phase space is  $Z = \text{Tr} e^{-\frac{H}{kT}}$  in the standard way. For  $\sigma = -1$  the partition function can be written as

$$\begin{aligned} Z &= \frac{1}{\pi^2} \int d^2 z_x d^2 z_y \langle z_x, z_y | e^{-\frac{1}{kT} \left[ \frac{\hbar|\bar{\omega}|}{2} (2\bar{a}_x^\dagger \bar{a}_x + 1) - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2} \right]} | z_x, z_y \rangle \\ &= \frac{4}{A^2} e^{-\frac{1}{kT} \left( \frac{\hbar|\bar{\omega}|}{2} - \frac{(u\rho_e - d\rho_m)\hbar}{2mc^2} \right)} \int |z'_x| |z_y| \text{Exp}\left[ \left| \frac{z'_x}{A} \right|^2 \left( e^{-\frac{\hbar|\bar{\omega}|}{kT}} - 1 \right) \right] d|z'_x| d|z_y| \end{aligned} \quad (36)$$

Like Ref.[45], we exclude the coherent states with the mean coordinates outside the size and sum over the other states. Because  $R \gg A, B$  and the exponential falls off rapidly with  $|z'_y|$ , we can integrate  $|z'_y|$  from zero to infinity safely. The result is

$$Z = \frac{R^2}{2\text{Sin}h\left[\frac{\hbar|\bar{\omega}|}{2kT}\right]} \frac{e^{\frac{(u\rho_e - d\rho_m)\hbar}{2mc^2 kT}}}{\left(\alpha\sqrt{\frac{2\hbar}{M|\bar{\omega}|}} + \frac{\theta}{2\alpha}\sqrt{\frac{M|\bar{\omega}|}{2\hbar}}\right)^2}. \quad (37)$$

So for  $\sigma = -1$  the free energy of the system is

$$F = -nkT \ln Z = -nkT \ln \left[ \frac{R^2}{2\text{Sin}h\left[\frac{\hbar|\bar{\omega}|}{2kT}\right]} \frac{e^{\frac{(u\rho_e - d\rho_m)\hbar}{2mc^2 kT}}}{\left(\alpha\sqrt{\frac{2\hbar}{M|\bar{\omega}|}} + \frac{\theta}{2\alpha}\sqrt{\frac{M|\bar{\omega}|}{2\hbar}}\right)^2} \right]. \quad (38)$$

By using the same way, we can obtain the free energy for  $\sigma = 1$

$$F = -nkT \ln \left[ \frac{R^2}{2 \text{Sinh} \left[ \frac{\hbar |\omega|}{2kT} \right]} \frac{e^{\frac{(u\rho_e - d\rho_m)\hbar}{2mc^2 kT}}}{\left( \alpha \sqrt{\frac{2\hbar}{M|\omega|}} - \frac{\theta}{2\alpha} \sqrt{\frac{M|\omega|}{2\hbar}} \right)^2} \right]. \quad (39)$$

If we set  $u \neq 0$  and  $d = 0$ , we can obtain the free energy of the AC system on NC phase space,

$$F = -nkT \ln \left[ \frac{R^2}{2 \text{Sinh} \left[ \frac{\hbar |\omega_{AC}|}{2kT} \right]} \frac{e^{\frac{u\rho_e \hbar}{2mc^2 kT}}}{\left( \alpha \sqrt{\frac{2\hbar}{M_{AC} |\omega_{AC}|}} + \frac{\theta}{2\alpha} \sqrt{\frac{M_{AC} |\omega_{AC}|}{2\hbar}} \right)^2} \right] \quad \text{for } \sigma = -1, \quad (40)$$

and

$$F = -nkT \ln \left[ \frac{R^2}{2 \text{Sinh} \left[ \frac{\hbar |\omega_{AC}|}{2kT} \right]} \frac{e^{\frac{u\rho_e \hbar}{2mc^2 kT}}}{\left( \alpha \sqrt{\frac{2\hbar}{M_{AC} |\omega_{AC}|}} - \frac{\theta}{2\alpha} \sqrt{\frac{M_{AC} |\omega_{AC}|}{2\hbar}} \right)^2} \right] \quad \text{for } \sigma = 1. \quad (41)$$

If we set  $u = 0$  and  $d \neq 0$ , we can obtain the free energy of the HMW system on NC phase space,

$$F = -nkT \ln \left[ \frac{R^2}{2 \text{Sinh} \left[ \frac{\hbar |\omega_{HMW}|}{2kT} \right]} \frac{e^{\frac{-d\rho_m \hbar}{2mc^2 kT}}}{\left( \alpha \sqrt{\frac{2\hbar}{M_{HMW} |\omega_{HMW}|}} + \frac{\theta}{2\alpha} \sqrt{\frac{M_{HMW} |\omega_{HMW}|}{2\hbar}} \right)^2} \right] \quad \text{for } \sigma = -1, \quad (42)$$

and

$$F = -nkT \ln \left[ \frac{R^2}{2 \text{Sinh} \left[ \frac{\hbar |\omega_{HMW}|}{2kT} \right]} \frac{e^{\frac{-d\rho_m \hbar}{2mc^2 kT}}}{\left( \alpha \sqrt{\frac{2\hbar}{M_{HMW} |\omega_{HMW}|}} - \frac{\theta}{2\alpha} \sqrt{\frac{M_{HMW} |\omega_{HMW}|}{2\hbar}} \right)^2} \right] \quad \text{for } \sigma = 1. \quad (43)$$

Further, by restricting  $\theta, \bar{\theta}$ , we can obtain the free energies on NC space and commutative space for the Anandan system, the AC system and the HMW system.

### (1). High Temperature Approximation

For the Anandan system where  $|u\rho_e - d\rho_m|$  is not very small, the free energy for  $\sigma = -1$  on NC phase space can be expressed by

$$F_{-1} = F_{-1}^0 + F_{-1}^\theta + F_{-1}^{\bar{\theta}} \quad (44)$$

where

$$\begin{aligned} F_{-1}^0 &= -nkT \ln \left[ \frac{mR^2 |\omega| e^{-\frac{\hbar |\omega|}{2kT}} \text{Csch} \left[ \frac{\hbar |\omega| \alpha^2}{2kT} \right]}{4\hbar \alpha^2} \right], \\ F_{-1}^\theta &= \frac{kmnT |\omega| \theta}{4\hbar \alpha^2} - \frac{1}{8} \text{Coth} \left[ \frac{\hbar |\omega| \alpha^2}{2kT} \right] mn |\omega|^2 \theta, \\ F_{-1}^{\bar{\theta}} &= \frac{knT \bar{\theta}}{\hbar m |\omega| \alpha^2} - \frac{\text{Coth} \left[ \frac{\hbar |\omega| \alpha^2}{2kT} \right] n \bar{\theta}}{2m}. \end{aligned} \quad (45)$$

In the calculation, we only consider the first order modification from space-space non-commutativity and momentum-momentum non-commutativity and ignore the interaction between them. Similarly, we can obtain for  $\sigma = 1$

$$F_1 = F_1^0 + F_1^\theta + F_1^{\bar{\theta}} \quad (46)$$

where

$$\begin{aligned} F_1^0 &= -nkT \ln \left[ \frac{mR^2 |\omega| e^{\frac{\hbar |\omega|}{2kT}} \text{Csch} \left[ \frac{\hbar |\omega| \alpha^2}{2kT} \right]}{4\hbar \alpha^2} \right], \\ F_1^\theta &= -\frac{kmnT |\omega| \theta}{4\hbar \alpha^2} + \frac{1}{8} \text{Coth} \left[ \frac{\hbar |\omega| \alpha^2}{2kT} \right] mn |\omega|^2 \theta, \\ F_1^{\bar{\theta}} &= -\frac{knT \bar{\theta}}{\hbar m |\omega| \alpha^2} + \frac{\text{Coth} \left[ \frac{\hbar |\omega| \alpha^2}{2kT} \right] n \bar{\theta}}{2m}. \end{aligned} \quad (47)$$

In high temperature approximation, the free energy on NC phase space is

$$\begin{aligned} F_{-1} &= Tkn \left( \ln \left[ \frac{1}{T} \right] - \ln \left[ \frac{kmR^2}{2\hbar^2 \alpha^4} \right] \right) + \frac{\hbar n |\omega|}{2} + \frac{1}{T} \left( \frac{n\hbar^2 |\omega|^2 \alpha^4}{24k} - \frac{\hbar mn |\omega|^3 \alpha^2 \theta}{48k} - \frac{n\hbar |\omega| \alpha^2 \bar{\theta}}{12km} \right), \\ F_1 &= Tkn \left( \ln \left[ \frac{1}{T} \right] - \ln \left[ \frac{kmR^2}{2\hbar^2 \alpha^4} \right] \right) - \frac{\hbar n |\omega|}{2} + \frac{1}{T} \left( \frac{n\hbar^2 |\omega|^2 \alpha^4}{24k} + \frac{\hbar mn |\omega|^3 \alpha^2 \theta}{48k} + \frac{n\hbar |\omega| \alpha^2 \bar{\theta}}{12km} \right). \end{aligned} \quad (48)$$

Further, if we restrict the values  $\theta$ ,  $\bar{\theta}$  and  $\alpha$ , we can get the free energies on NC space and commutative space, respectively,

$$\begin{aligned} F_{-1}^{NC \text{ Space}} &= Tkn(\ln[\frac{1}{T}] - \ln[\frac{kmR^2}{2\hbar^2}]) + \frac{\hbar n|\omega|}{2} + \frac{1}{T}(\frac{n\hbar^2|\omega|^2}{24k} - \frac{\hbar mn|\omega|^3\theta}{48k}), \\ F_1^{NC \text{ Space}} &= Tkn(\ln[\frac{1}{T}] - \ln[\frac{kmR^2}{2\hbar^2}]) - \frac{\hbar n|\omega|}{2} + \frac{1}{T}(\frac{n\hbar^2|\omega|^2}{24k} + \frac{\hbar mn|\omega|^3\theta}{48k}) \end{aligned} \quad (49)$$

and

$$\begin{aligned} F_{-1}^C \text{ Space} &= Tkn(\ln[\frac{1}{T}] - \ln[\frac{kmR^2}{2\hbar^2}]) + \frac{\hbar n|\omega|}{2} + \frac{1}{T}\frac{n\hbar^2|\omega|^2}{24k}, \\ F_1^C \text{ Space} &= Tkn(\ln[\frac{1}{T}] - \ln[\frac{kmR^2}{2\hbar^2}]) - \frac{\hbar n|\omega|}{2} + \frac{1}{T}\frac{n\hbar^2|\omega|^2}{24k} \end{aligned} \quad (50)$$

## (2). Zero Temperature Limit

According to Eqs. (38)and(39), if we consider  $T \rightarrow 0$ , we will find the free energy on NC phase space is

$$\begin{aligned} F_{-1} &\rightarrow \frac{\hbar n|\omega|}{2} + \frac{\hbar n|\omega|\alpha^2}{2} - \frac{1}{8}mn|\omega|^2\theta - \frac{n\bar{\theta}}{2m}, \\ F_1 &\rightarrow -\frac{\hbar n|\omega|}{2} + \frac{\hbar n|\omega|\alpha^2}{2} + \frac{1}{8}mn|\omega|^2\theta + \frac{n\bar{\theta}}{2m}. \end{aligned} \quad (51)$$

The results on NC space and commutative space are

$$\begin{aligned} F_{-1}^{NC \text{ Space}} &\rightarrow \hbar n|\omega| - \frac{1}{8}mn|\omega|^2\theta, \\ F_1^{NC \text{ Space}} &\rightarrow \frac{1}{8}mn|\omega|^2\theta \end{aligned} \quad (52)$$

and

$$\begin{aligned} F_{-1}^C \text{ Space} &\rightarrow \hbar n|\omega|, \\ F_1^C \text{ Space} &\rightarrow 0. \end{aligned} \quad (53)$$

Here we emphasize the two points for the two particular cases of the temperature. (i)The free energy of the Anandan system can be restricted to the expressions of the AC system and the HMW system by making the replacement  $|\omega| \rightarrow |\omega_{AC}|$  and  $|\omega| \rightarrow |\omega_{HMW}|$  respectively; (ii) For a given  $\sigma$ , the free energies have some difference among NC phase space, NC space and commutative space which may provide some clues to verify the presence of NC situation in future. For example, the free energy tends to zero for  $\sigma = 1$  on commutative space in zero temperature limit, but this phenomenon doesn't occur on NC situation.

## 5. REVOLUTION DIRECTION AND MAXWELL DUALITY

As we know, a coherent state represents a way that is as close as possible to classical localization. In the sections above. we introduced the sign  $\sigma$  which describes the revolution direction of the corresponding classical motion in fact. Here we use the most classical quantum state, the coherent state, to clearly catch the revolution direction  $\sigma$ . The mean coordinates of the state  $|z_x, z_y\rangle$  related to  $\sigma$  is given by Eq.(35). For  $\sigma = -1$ , the wave packet centroid of the coherent state  $|z_x, z_y\rangle$  moves anticlockwise with the radius  $|Az_x^*|$  and the frequency  $|\bar{\omega}|$ . For  $\sigma = 1$ , it moves clockwise with the radius  $|Bz_y|$ and the same frequency  $|\bar{\omega}|$ . So here  $\sigma = -1$  describes anticlockwise revolution and  $\sigma = 1$  describes clockwise revolution, which is all right for the Anandan system, the AC system and the HMW system on commutative space, NC space and NC phase space.

On any given space among commutative space, NC space and NC phase space, for a given revolution direction, Landau like levels of the Anandan system are invariant and the levels between the AC system and the HMW system become each other under Maxwell dual transformations,

$$\begin{aligned} \rho_e &\rightarrow \rho_m, & d &\rightarrow u, \\ \rho_m &\rightarrow -\rho_e, & u &\rightarrow -d. \end{aligned} \quad (54)$$



This result can be explained by the invariance of the Anandan Hamiltonian and the interchangeability between the AC Hamiltonian and the HMW Hamiltonian under Maxwell dual transformations on the corresponding space. Thus Landau like levels of the AC system and the HMW system are similar under the same revolution direction. In Ref.[14], Ribeiro *et al.* think that Landau like levels of the HMW system have the same form as the levels of the AC system with the opposite direction. But that's not true. For example, in terms of their paper, the AC system ( $u\rho_e > 0$ ) and the HMW system ( $d\rho_m < 0$ ) should have the same form of the levels and opposite directions. But in fact the cases of  $u\rho_e > 0$  and  $d\rho_m < 0$  describe the same revolution direction as shown in the movement of the wave packet centroid of the coherent state  $|z_x, z_y\rangle$  for  $\sigma = 1$  above. The reason of the mistake is that the definition of the cyclotron frequencies in their paper break Maxwell duality. However, in our paper, the cyclotron frequencies  $\omega_{AC} = \sigma \frac{|u\rho_e|}{mc^2} = \frac{u\rho_e}{mc^2}$  and  $\omega_{HMW} = \sigma \frac{|d\rho_m|}{mc^2} = \frac{-d\rho_m}{mc^2}$  will change each other under Maxwell dual transformations.

## 6. SUPERSYMMETRY

Now, we have known landau like levels of the Anandan system on commutative space, NC space and NC phase space can be all divided into two classes labelled by the revolution direction  $\sigma$ . In this section we will utilize supersymmetry to study the difference of the system between commutative space and NC situation by transforming the Hamiltonian (16) again. We introduce the new annihilation and creation operators

$$\begin{aligned}\hat{b}' &= \frac{1}{\sqrt{2M\hbar|\bar{\omega}|}}(\Pi_x + i\sigma \Pi_y), \\ \hat{b}'^+ &= \frac{1}{\sqrt{2M\hbar|\bar{\omega}|}}(\Pi_x - i\sigma \Pi_y),\end{aligned}\tag{55}$$

where  $\Pi_x = p_x + \frac{2c^2\bar{\theta} + 2\alpha^2\hbar(u\rho_e - d\rho_m)}{4\alpha^2\hbar c^2 + (u\rho_e - d\rho_m)\bar{\theta}}y$  and  $\Pi_y = p_y - \frac{2c^2\bar{\theta} + 2\alpha^2\hbar(u\rho_e - d\rho_m)}{4\alpha^2\hbar c^2 + (u\rho_e - d\rho_m)\bar{\theta}}x$ . Thus the Hamiltonian (16) can be expressed as

$$H = \hbar|\bar{\omega}|(\hat{b}'^+\hat{b}' + \frac{1}{2}) - \frac{\hbar\sigma|\omega|}{2}.\tag{56}$$

On commutative space,  $|\bar{\omega}|$  becomes  $|\omega|$ , and  $\hat{b}'$  becomes  $\hat{b}$  where  $\hat{b} = \frac{1}{\sqrt{2m\hbar|\omega|}}(p_x + \frac{u\rho_e - d\rho_m}{2c^2}y + i\sigma p_y - i\sigma \frac{u\rho_e - d\rho_m}{2c^2}x)$ . So the Hamiltonian on this space is

$$H_C = \hbar|\omega|(\hat{b}^+\hat{b} + \frac{1}{2} - \frac{\sigma}{2}).\tag{57}$$

Here we introduce fermionic annihilation and creation operators  $\hat{d}, \hat{d}^+$  and suppose the eigenvalue of  $[\hat{d}, \hat{d}^+]$  is  $\sigma$ . Thus the Hamiltonian  $H_C$  can be expressed as  $H_C = \hbar|\omega|(\hat{b}^+\hat{b} + \hat{d}^+\hat{d})$ . The supercharge can be defined as

$$Q = \sqrt{\hbar|\omega|}\hat{b}\hat{d}^+\tag{58}$$

which together with the Hamiltonian  $H_C$  close the surperalgebra

$$Q^2 = (Q^+)^2 = 0, \quad H_C = \{Q, Q^+\}.\tag{59}$$

The extended Fock states can be defined as  $|n_b, n_d, \kappa\rangle$  where  $n_b, n_d$  are the eigenvalues of the number operators  $\hat{b}^+\hat{b}, \hat{d}^+\hat{d}$  and  $\kappa$  is a good quantum number. These eigenstates can be related by the good supersymmetry transformation

$$|n_b - 1, 1, \kappa\rangle = \frac{1}{\sqrt{E_{n_b}}}Q|n_b, 0, \kappa\rangle, \quad |n_b + 1, 0, \kappa\rangle = \frac{1}{\sqrt{E_{n_b+1}}}Q^+|n_b, 1, \kappa\rangle\tag{60}$$

and

$$Q|0, 0, \kappa\rangle = Q^+|0, 0, \kappa\rangle = 0,\tag{61}$$

where  $E_{n_b}$  is the energy eigenvalues of the states  $|n_b, 0, \kappa\rangle, |n_b - 1, 1, \kappa\rangle$  and  $E_{n_b+1}$  is the ones of the states  $|n_b, 1, \kappa\rangle, |n_b + 1, 0, \kappa\rangle$ . According to the Hamiltonian (56), the energy eigenvalues on NC phase space is

$$E_{n', \sigma} = \hbar|\bar{\omega}|(n' + \frac{1}{2}) - \frac{\hbar\sigma|\omega|}{2}\tag{62}$$

where  $n'$  is the eigenvalue of the number operator  $\hat{b}'^\dagger \hat{b}'$ . And the corresponding eigenstates are  $|n', \kappa'\rangle_\sigma$  where  $\kappa'$  is a good quantum number. Obviously, for different  $\sigma$ , the energy eigenvalues are not equal, that is,  $E_{n'_1, 1} \neq E_{n'_2, -1}$ . Thus the two sets of eigenstates labelled by  $\sigma$  can not be related by a supersymmetry transformation on NC phase space. The phenomenon also exists on NC space. The difference resulted from supersymmetry between commutative space and NC situation also occurs for the AC system and the HMW system.

## 6. CONCLUSION

In this letter we study the Anandan system in a special electromagnetic field, where we find unlike the cases of the AC system and the HMW system the torques on the magnetic dipole and the electric dipole don't vanish. We obtain the Landau analog levels and eigenstates on commutative space, NC space and NC phase space. We study some statistical properties of such free atom gas with the Anandan interaction and present the expressions of some thermodynamic quantities related to revolution direction. Some simple formulae of the free energy in two cases of temperature on the three spaces are also obtained. Some difference of the free energy among NC phase space, NC space and commutative space may provide us some clues to verify the presence of NC situation in future. The relation between the value of  $\sigma$  and revolution direction is presented clearly. We find Landau like levels of the Anandan system are invariant and the levels between the AC system and the HMW system become each other under Maxwell dual transformations on the three spaces. And point the mistake of Ribeiro *et al.*, Landau like levels of the HMW system have the same form as the levels of the AC system with the opposite direction. We find the two sets of eigenstates labelled by  $\sigma$  can be related by a supersymmetry transformation on commutative space, but the phenomenon don't occur on NC situation. Some results relevant to the AC system and the HMW system are also obtained by restricting the magnetic dipole and the electric dipole.

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