Origin of maximal symmetry breaking in even PT-symmetric lattices

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Origin of maximal symmetry breaking in even $\mathcal{PT}$-symmetric lattices

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By investigating a parity and time-reversal ($\mathcal{PT}$) symmetric, N-site lattice with impurities $\pm i\gamma$ and hopping amplitudes $t_0(t_b)$ for regions outside (between) the impurity locations, we probe the inter-impurity-distance dependence of the critical impurity strength and the origin of maximal $\mathcal{PT}$-symmetry breaking that occurs when the impurities are nearest neighbors. Through a simple and exact derivation, we prove that the critical impurity strength is equal to the hopping amplitude between the impurities, $\gamma_c = t_b$, and the simultaneous emergence of N complex eigenvalues is a robust feature of any $\mathcal{PT}$-symmetric hopping profile. Our results show that the threshold strength γ_c can be widely tuned by a small change in the global profile of the lattice, and thus have experimental implications.

Introduction: The discovery of "complex extension of quantum mechanics" by Bender and coworkers [1, 2] set in motion extensive mathematical [3–5] and theoretical investigations [6] of non-Hermitian Hamiltonians $H_{\mathcal{PT}} = \hat{K} + \hat{V}$ that are symmetric with respect to combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) operations. Such continuum or lattice Hamiltonians [7–10] usually consist of a Hermitian kinetic energy part, $\hat{K} = \hat{K}^\dagger$, and a non-Hermitian, $\mathcal{PT}$-symmetric potential part, $\hat{V} = \mathcal{PT}\hat{V}\mathcal{PT} \neq \hat{V}^\dagger$. Although it is not Hermitian $H_{\mathcal{PT}}$ has purely real eigenvalues $E = E^*$ over a range of parameters, and its eigenfunctions are simultaneous eigenfunctions of the combined $\mathcal{PT}$-operation; this range is defined as the $\mathcal{PT}$-symmetric region. The breaking of $\mathcal{PT}$-symmetry, along with the attendant non-reciprocal behavior, was recently observed in two coupled optical waveguides [11, 12] and has ignited further interest in $\mathcal{PT}$-symmetric lattice models. These evanescently coupled waveguides provide an excellent realization [13] of an ideal, one-dimensional lattice with tunable hopping [14], disorder [15], and non-Hermitian, on-site, impurity potentials [16, 17].

Recently nonuniform lattices with site-dependent hopping $t_\alpha(k) = t_0\left[k(N-k)\right]^{\alpha/2}$ and a pair of imaginary impurities $\pm i\gamma$ at positions $(m, \bar{m})$ have been extensively explored [17–20], where $\bar{m} = N + 1 - m$ and $N \gg 1$ is the number of lattice sites. The $\mathcal{PT}$-symmetric phase in such a lattice is robust when $\alpha \geq 0$, the loss and gain impurities $\pm i\gamma$ are closest to each other, and $\gamma \leq \gamma_c$ where the critical impurity strength is proportional to the bandwidth of the clean lattice, $\gamma_c \propto 4t_0(N/2)^\alpha$. For a generic impurity position m, when the impurity strength $\gamma > \gamma_c(m)$ increases the number of complex eigenvalues increases sequentially from four to $N-1$ when N is odd and to N when it is even. *In an exceptional contrast*, when $m = N/2$ - nearest neighbor impurities on an even lattice - all eigenvalues simultaneously become complex at the onset of $\mathcal{PT}$-symmetry breaking. This maximal symmetry breaking is accompanied by unique signatures in the time-evolution of a wavepacket [20].

These results raise the following questions: Is this exceptional behavior limited to lattices with α-dependent hopping or is it generic? Which factors truly determine the critical impurity strength $\gamma_c(N/2)$ in the exceptional case? In the general case, how does the critical impurity strength $\gamma_c(m)$ for arbitrary impurity positions $(m, \bar{m})$ depend upon the lattice parameters?

In this Brief Report, we investigate an N-site lattice with impurities $\pm i\gamma$ at positions $(m, \bar{m})$ and a constant hopping amplitude $t_0(t_b)$ for sites outside (between) the parity-symmetric impurity locations. Our two salient results are as follows: (i) When $t_b \gg t_0$, the critical impurity strength $\gamma_c(m) \to t_b$ irrespective of the impurity position m and whether N is even or odd. When $t_b < t_0$, the critical impurity strength $\gamma_c(m) \sim t_b^\eta$ where the exponent $\eta(d) \sim d$ increases monotonically with the distance $d = \bar{m} - m = N + 1 - 2m$ between the impurities, irrespective of whether N is even or odd. (ii) For an even lattice, when $m = N/2$, we analytically prove that all eigenvalues simultaneously become complex when $\gamma > \gamma_c(N/2) = t_b$. *This robust result is true for any symmetric distribution of real hopping amplitudes.* Thus, the $\mathcal{PT}$-symmetry breaking threshold can be substantially tuned without significant changes in the global hopping-amplitude profile of the lattice, and the exceptional nature of the $m = N/2$ case is due to the ability to partition the system into two, and exactly two, pieces.

Tight-binding Model: We start with the Hamiltonian for a one-dimensional, tight-binding, non-uniform lattice

$$H_{\mathcal{PT}} = -\sum_{i=1}^{N-1} t(i)\left(a^\dagger_{i+1}a_i + a^\dagger_i a_{i+1}\right) + i\gamma\left(a^\dagger_m a_m - a^\dagger_{\bar{m}} a_{\bar{m}}\right), \tag{1}$$

where $a^\dagger_n(a_n)$ is the creation (annihilation) operator for a state localized at site n, and the hopping function is given by $t(i) = t_b > 0$ for $m \leq i \leq \bar{m} - 1$, and $t(i) = t_0 > 0$ otherwise. This Hamiltonian continuously extrapolates from that for a lattice of length $d = N + 1 - 2m$ with impurities at its end when $t_b \gg t_0$, to that of a pair of disconnected lattices, one with the gain impurity and the other with the loss impurity, when $t_b \ll t_0$. Note

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that the critical impurity strengths in these two limits are known [17, 21]. Due to the constant hopping amplitude outside or between the impurity locations, an arbitrary eigenfunction $|\psi\rangle = \sum_{n=1}^{N} \psi(n) a_n^\dagger |0\rangle$ with energy E can be expressed using the Bethe *ansatz* as

$$\psi(n) = \begin{cases} A\sin(kn), & 1 \le n \le m, \\ P\sin(k'n) + Q\cos(k'n), & m < n < \bar{m}, \\ B\sin(k\bar{n}), & \bar{m} \le n \le N. \end{cases} \quad (2)$$

Here $E(k,k') = -2t_0\cos(k) = -2t_b\cos(k')$ defines the relation between the quasimomenta k, k'. In the $\mathcal{PT}$-symmetric phase, the energy spectrum of Eq.(1) is particle-hole symmetric [22], and the eigenenergies satisfy $|E| \lesssim 2\max(t_0, t_b)$. Note that the relative phases of $\psi(n)$ are the same at different points within each of the three regions, although there may be a phase difference between wavefunctions in different regions. Therefore, without loss of generality, *we may choose* $\psi(n)$ *to be real for* $1 \le n \le m$. By considering the eigenvalue equation $H_{\mathcal{PT}}|\psi\rangle = E|\psi\rangle$ at points $m, m+1$ and their reflection counterparts, it follows that the quasimomenta (k, k') obey the equation [21]

$$\begin{aligned} M(k,k') \quad &\equiv \left[\sin^2\left[k(m+1)\right] + \Gamma^2\sin^2(km)\right] \\ &\times \sin\left[k'(N+1-2m)\right] + T_b^2\sin^2(km) \\ &\times \sin\left[k'(N-1-2m)\right] - 2T_b\sin(km) \\ &\times \sin\left[k(m+1)\right]\sin\left[k'(N-2m)\right] = 0, \end{aligned} \quad (3)$$

where $\Gamma = \gamma/t_0$ and $T_b = t_b/t_0$ denote the dimensionless impurity strength and hopping amplitude respectively. Note that when $2\min(t_0,t_b) < |E| \le 2\max(t_0,t_b)$, k is real and k' is purely imaginary (or vice versa), whereas for $|E| \le 2\min(t_0,t_b)$, both k, k' are real. Thus, Eq.(3) represents two distinct equations in these two cases.

The right-hand panel in Fig. 1 shows the dimensionless critical impurity strength $\Gamma_c(d) = \gamma_c(m)/t_0$ as a function of $T_b = t_b/t_0 \ge 1$ for various inter-impurity-distances $d = N + 1 - 2m$ in an $N = 20$ lattice; we obtain similar results for an odd lattice. Note that the distance between $\mathcal{PT}$-symmetric impurities is odd when N is even and *vice versa*. We find that $\gamma_c \to t_b$ quickly for $t_b/t_0 > 1$; when $t_b/t_0 \gg 1$, the lattice reduces to a uniform one with $d+1$ sites, impurities at its end points, and the result $\gamma_c = t_b$ is expected [21]. The left-hand panel shows $\Gamma_c(d)$ vs. T_b on a logarithmic scale in $N = 20$ and $N = 21$ lattices for $T_b < 1$. As the distance d between the impurities increases, corresponding critical impurity strength decreases as a power-law, $\Gamma_c(d) \propto T_b^{\eta(d)}$ where the exponent $\eta(d) \sim d$. This behavior can be qualitatively understood as follows: the system is in the $\mathcal{PT}$-symmetric region if the frequency $\sim \gamma/t_0$ at which particles are created at the gain-impurity site m is lower than rate at which these excess particles can hop over to the loss-impurity site, where they are absorbed at frequency $\sim \gamma/t_0$. Since t_b is the hopping amplitude at sites between the impurities, it follows that the effective frequency of hopping from the gain- to the loss-site decreases with d as T_b^d.

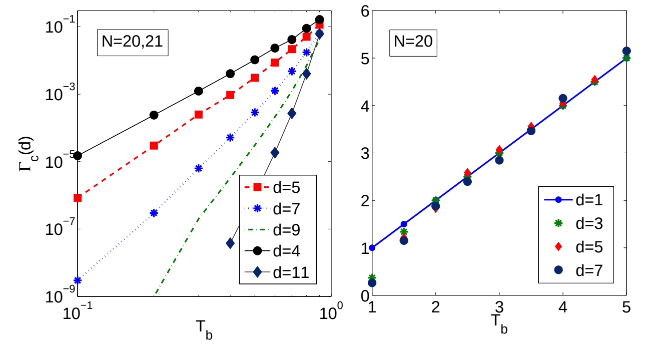

FIG. 1. (color online) a) Left-hand panel shows dimensionless critical impurity strength $\Gamma_c(d) = \gamma_c/t_0$ as a function of dimensionless hopping amplitude $0 < T_b = t_b/t_0 < 1$ for various distances d between $\mathcal{PT}$-symmetric impurities in $N = 20, 21$ lattices; note that d must be odd when N is even and *vice versa*. It follows that $\Gamma_c(d) \propto T_b^{\eta(d)} \to 0$, as is expected on physical grounds, irrespective of the whether N is even or odd. b) Right-hand panel shows the critical impurity strength $\Gamma_c(d)$ as a function of $T_b \ge 1$ for various values of d in an $N = 20$ lattice. Although at $T_b = 1$, the critical strength $\Gamma_c(d)$ reduces with distance d between the impurities, for $T_b \ge 2$ the critical impurity strength $\Gamma_c \to T_b$ ($\gamma_c \to t_b$) irrespective of d and N.

Indeed, when $t_b/t_0 \ll 1$, the system is divided into two, non $\mathcal{PT}$-symmetric, uniform lattices, one with the loss impurity and the other with the gain. It follows, then, that $\gamma_c \to 0$ as $t_b/t_0 \to 0$. We remind the reader that except when $d = 1$ (right-hand panel in Fig. 1) - *the exceptional case* - the $\mathcal{PT}$-symmetry breaks sequentially with the emergence of four complex eigenvalues.

Origin of Maximal Symmetry Breaking: Now let us consider the exceptional case in an even lattice, $m = N/2$, where Eq.(3) reduces to

$$t_0^2\sin^2\left[k\left(\frac{N}{2}+1\right)\right] = \left(t_b^2 - \gamma^2\right)\sin^2\left(\frac{kN}{2}\right). \quad (4)$$

It follows from Eq.(4) that the $\mathcal{PT}$-symmetry breaks maximally when $\gamma > \gamma_c(N/2) = t_b$ and is accompanied by the simultaneous emergence of N *complex* (not purely imaginary) quasimomenta and eigenenergies. Since the bandwidth of the clean lattice is determined by both hoppings (t_0, t_b), it follows that the critical impurity strength is independent of the lattice bandwidth.

To generalize this result, we consider the system with an arbitrary, $\mathcal{PT}$-symmetric, position-dependent hopping profile $t_k = t_{N-k}$ and real energy eigenvalues. In the $\mathcal{PT}$-symmetric region, the coefficients of an eigenfunction $|\phi\rangle = \sum_{m=1}^{N} \phi(m)|m\rangle$ satisfy $\phi(\bar{m}) = z\phi^*(m)$ where $z = e^{i\chi}$ is a complex number of unit modulus; this follows from the constraint $(\mathcal{PT})^2|\phi\rangle = |\phi\rangle$. Since the hopping and eigenvalues are real, the eigenvalue difference equations imply that for any eigenfunction $|\phi\rangle$, we can choose the coefficients $\phi(k)$ to be real for $1 \le k \le m$.

A real eigenvalue ϵ and the (real) coefficients $\alpha = \phi(N/2)$ and $\beta = \phi(N/2-1)$ of its corresponding eigenfunction $|\phi_\epsilon\rangle = \sum_{i=1}^{N} \phi(i)|i\rangle$ satisfy

$$\det \begin{bmatrix} t_{N/2-1}\beta + (\epsilon - i\gamma)\alpha & t_{N/2}\alpha \\ t_{N/2}\alpha & t_{N/2-1}\beta + (\epsilon + i\gamma)\alpha \end{bmatrix} = 0, \tag{5}$$

where we have used the $\mathcal{PT}$-symmetric nature of eigenfunctions to deduce that $\phi(N/2+1) = e^{i\chi}\alpha$, $\phi(N/2+2) = e^{i\chi}\beta$. Thus, when $\gamma > \gamma_c = t_{N/2} = t_b$, the eigenvalue ϵ must become complex. *Since this is true for all eigenfunctions, the $\mathcal{PT}$-symmetry breaks maximally and the critical impurity strength is equal to the hopping between the nearest-neighbor impurities.* Note that when the eigenvalue ϵ becomes complex, the corresponding eigenfunction $|\phi\rangle$ does not remain $\mathcal{PT}$-symmetric, $\phi(\bar{m}) \neq z\phi^*(m)$, and thus the eigenfunction coefficients $\phi(k)$ for $1 \leq k \leq m$ cannot be chosen as real; instead $\mathcal{PT}|\phi\rangle$ is an eigenfunction of the Hamiltonian with eigenvalue $\epsilon^* \neq \epsilon$. Our robust result also explains the fragile nature of $\mathcal{PT}$-symmetric phase in lattices with hopping function $t_\alpha(k)$ for $\alpha < 0$ [20]: in this case, the lattice bandwidth $\Delta_\alpha \sim N^{-|\alpha|/2}$ whereas the hopping amplitude between the two nearest-neighbor impurities scales as $t_b \sim N^{-|\alpha|}$. Therefore the critical impurity strength $\gamma_c/\Delta_\alpha \sim N^{-|\alpha|/2} \to 0$ as $N \to \infty$. A similar analysis for closest impurities in an odd-N lattice shows that, due to the presence of a lattice site between the two impurity positions $m = (N-1)/2$ and $\bar{m} = (N+3)/2$, the corresponding critical impurity strength γ_c depends on the details of the eigenfunction.

Thus, the maximal symmetry breaking only occurs in an even, $\mathcal{PT}$-symmetric lattice with nearest-neighbor impurities, and its origin is the ability to naturally partition such a lattice into exactly two components.

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