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Casimir Invariants for Systems Undergoing Collective Motion

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Dicke states are an important class of states which exhibit collective behavior in many-body systems. They are interesting because (1) the decay rates of these states can be quite different from a set of independently evolving particles, and (2) a particular class of these states are decoherence-free/noiseless with respect to a set of errors. These noiseless states, or more generally subsystems, avoid certain types of errors in quantum information processing devices. Here we provide a method for determining a set of transformations of these states which leave the states in their subsystems but still enable them to evolve in particular ways. For subsystems of particles undergoing collective motions, these transformations can be calculated by using essentially the same construction which is used to determine the famous Casimir invariants for quantum systems. Such invariants can be used to determine a complete set of commuting observables for a class of Dicke states as well as identify possible logical operations for decoherence-free/noiseless subsystems. Our method is quite general and provides results for cases where the constituent particles have more than two internal states.

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I. INTRODUCTION

Decoherence-free/noiseless subsystems (DFSs) are now part of an arsenal of weapons used to prevent errors in quantum information processing and storage [1–6]. (For reviews see [7, 8].) DFSs are subsystems which are immune to certain types of errors. The most common type found in the literature is a DFS which is immune to collective errors. Quantum systems undergoing unitary collective transformations were studied earlier by Dicke in a different context [9]. Dicke described the evolution, and decay rates, of states undergoing collective motion and gave a description of which kinds of states behaved quite differently from those which are effectively independent.

There are several types of states which are now called Dicke states. One such set corresponds to a set of particles which undergo a collective motion, are distinguishable, and do not interact with each other. These states are unchanged by particle interchange, or more generally, the interchange of particular constituents [10]. One particularly clear example is a gas interacting with an external field which has a wavelength significantly longer than the container confining the particles. These are conditions for collective motion, i.e., the external field interacts in the same way with each particle. In this case, if the size of the container $\sim R$ and the wavelength of the field is λ, then the "Dicke limit" $\lambda \gg R$ is said to be satisfied. In this limit, when the external influence gives rise to errors in a quantum computing device, the errors are called collective, whether they describe an evolution of each particle which is unitary or not.

Since errors are the greatest obstacle to building a fully functional quantum computing device, any method which aids in the prevention of errors is quite important. However, for the practical use of a DFS/NS for quantum information processing one requires the ability to perform universal computing on these states. This requires finding evolutions which do not take the states out of the protected subspace during gating operations [5]. We refer to such operations as being compatible with the DFS structure. In the physical systems considered by Dicke, one could imagine evolutions of the states which do not change the essential features of the state (energy or total angular momentum quantum numbers), but are indeed nontrivial evolutions. In the case of quantum information processing, these enable quantum computing in a DFS and for some important cases they enable universal quantum computing on a subspace even when it is not possible on the entire physical space. This, in fact, is perhaps more important than the noise prevention properties for which these states were originally intended.

In both the early analysis of Dicke states and also quantum computing applications, primarily only two internal states of the constituents were considered. However, three or more internal states of an atom could certainly become important in various experiments and could also arise in particle physics where more than two degrees of freedom are associated with both flavor and color symmetries. Recent experiments [11–13] and proposed experiments [14–16] have provided explicit constructions for these so-called Dicke states using a variety of physical systems.

Here we consider collections of particles undergoing some collective motions, for example collective errors, and ask the following question. What Hamiltonians give rise to evolutions which are compatible with these motions? *Our results are not restricted to any particular number of internal states for each of the constituents, nor* *are they restricted to any number of particles.* We then answer the question by using a construction analogous to Casimir's construction of invariants for Lie algebras and Lie groups. These are the same invariants which are used to label a complete set of commuting observables and thus identify the largest set of simultaneously measurable quantities for a quantum system.

The paper will be outlined as follows. In Section II we provide the motivation for this work and discuss the collective motion of Dicke states. We then review the standard Casimir construction for single-particle invariants in Section III. Section IV extends the construction to sets of N particles each with d internal states. Section V discusses the physical implications of our results and provides examples. In particular, we discuss how these invariants can be used to manipulate decoherence-free or noiseless subsystems in such as to preserve the integrity of the encodings and also how they relate to the original collective Dicke states. Section VI concludes.

II. MOTIVATION

We will begin by providing the motivation behind this work. The main objective of this paper is to provide a methodical procedure for identifying the Hamiltonians which generate non-trivial, symmetry-preserving evolutions for a system which is undergoing some type of collective motion. Since these evolutions are important for manipulating DFS/NS encodings, we will briefly summarize an important result regarding a compatibility condition placed on such Hamiltonians. As stated before, DFS/NS states correspond to a particular class of Dicke states. This being the case, we will begin our discussion with an introduction to the collective motion of Dicke states.

A. Dicke States

In Ref. [9] Dicke examined the spontaneous radiation of photons emitted from a gas consisting of two-level particles. Gasses of both small and large extent were treated separately, the scale being determined relative to the wavelength λ of an externally applied field. Taking R to be the spatial extent of the container, the two cases correspond to $\lambda \gg R$ or $\lambda \ll R$. In both cases it was assumed that there was insufficient overlap of the wave functions of separate particles to require symmetrization of the states. It was also assumed that each particle coupled to the common radiation field via an electric dipole interaction. In general, the interaction energy of the αth particle with the field can be written as

$$H_I^{(\alpha)} = -\mathbf{A}(\mathbf{r}_\alpha) \cdot (\mathbf{e_1}\sigma_x^{(\alpha)} + \mathbf{e_2}\sigma_y^{(\alpha)}), \tag{II.1}$$

for some constant real vectors $\mathbf{e_1}$ and $\mathbf{e_2}$.

In the case of a gas confined to a small region of space the vector potential can effectively be considered an independent function of the spatial coordinates $\mathbf{r}_\alpha$. In this approximation the total interaction energy becomes

$$H_I = c_1 \sum_\alpha \sigma_x^{(\alpha)} + c_2 \sum_\alpha \sigma_y^{(\alpha)}, \tag{II.2}$$

where c_1 and c_2 denote constants. There are two degrees of freedom associated with the internal energy of any given particle. The energy eigenvalues of the jth particle, corresponding to the diagonal operator $\sigma_z^{(\alpha)}$, take on the values $\pm\hbar\omega/2$. The sum of all internal particle energies, together with the translational energy of the gas H_0 and the interaction with the field, provides a complete description of a gaseous system consisting of mutually non-interacting particles.

The Hamiltonian for this system can be broken up into two parts,

$$H = H_0 + \left(c_1 \sum_\alpha \sigma_x^{(\alpha)} + c_2 \sum_\alpha \sigma_y^{(\alpha)} + \hbar\omega/2 \sum_\alpha \sigma_z^{(\alpha)} \right), \tag{II.3}$$

where the first part describes the translational energy of the system and thus depends solely on the spatial positions $\mathbf{r}_\alpha$ while the second is a quantity independent of these coordinates. As a result, these two parts commute implying the existence of simultaneous eigenfunctions of the two contributions. Let us denote these energy eigenstates

$$\psi_{pq} = U_p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, \Phi_q, \tag{II.4}$$

where U_p depends on the spatial coordinates and Φ_q is a function of the internal coordinates. The operators $S_i = \sum_\alpha \sigma_i^{(\alpha)}$ $(i = x, y, z)$ not only individually commute with the spatially independent quantity $S^2 = S_x^2 + S_y^2 + S_z^2$, but also satisfy the same commutation relations (up to a multiplicative scaling factor) as the three components of angular momentum. In other words, they form a representation of the SO(3) algebra. Stationary states of this system can therefore be identified with those eigenstates that conserve the square of the total angular momentum operator, i.e., $\Phi_q \equiv \Phi_{jm}$, with $S^2\Phi_{jm} = j(j+1)\Phi_{jm}$ and $|m| \leq j \leq N/2$. Consequently, the stationary states of a gaseous system confined to a small region of space can be expressed as

$$\psi_{pjm} = U_p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, \Phi_{jm}. \tag{II.5}$$

Since the individual particles which form the gas all experience a common interaction with the radiation field, the system as a whole evolves in a collective manner. However, while this collective motion is occurring on these states, they may still undergo other non-trivial evolutions that conserve energy and angular momentum. Furthermore, the number of internal states is not restricted to two, but can be arbitrary. Many internal states may be undergoing simultaneous transitions to other internal states, collectively, while still undergoing this evolution. We will now consider these higher-dimensional Dicke states in the context of decoherence-free/noiseless subsystem encodings which are actually invariant under these collective motions. We will restate a sufficient condition placed on the Hamiltonians which generate non-trivial evolutions of these systems yet remain compatible with the encoding. Although the argument follows the usual treatment regarding the compatibility of transformations of a collective DFS/NS, it applies to the present case as well since DFS/NS states suitable for quantum information processing correspond to degenerate Dicke states.

B. DFS/NS-Compatible Gates

Let us suppose that the Dicke states corresponding to a collective DFS/NS are spanned by the set $\{|\lambda\rangle \otimes |\mu\rangle\}$, with $\lambda = 1, \ldots, d$ and $\mu = 1, \ldots, n$. Here the $|\lambda\rangle$'s distinguish a particular basis state of an encoded d-state system and the $|\mu\rangle$'s label the n orthogonal elements which span each qudit dimension. When acted upon by the collective errors S_j these DFS/NS states have the property that

$$S_j \, |\lambda\rangle \otimes |\mu\rangle = \sum_{\mu'=1}^{n} M_{\mu\mu',j} \, |\lambda\rangle \otimes |\mu'\rangle \, . \tag{II.6}$$

In other words, these encoded qudit states remain unaffected by the presence of such noise since they map every $|\lambda\rangle$ to itself. One can parametrize the collective errors using a set of time-independent complex numbers $\{v_j\}$,

$$D(v_1, v_2, ...) = \exp\left[\sum_j v_j S_j\right] . \tag{II.7}$$

The DFS/NS states are not the only accessible states inherent to a system. There are some orthogonal to these which cannot protect against collective noise. When information is leaked into these regions of the systems Hilbert space it may be permanently lost. Gates which are used to manipulate the state of an encoded qudit should therefore operate in a manner such that they map DFS/NS states to other DFS/NS states. It can be shown that a sufficient condition for a transformation $U = \exp(-iHt)$ to satisfy this compatibility requirement is that

$$UD(v_1, v_2, \ldots)U^\dagger = D(v_1', v_2', \ldots), \tag{II.8}$$

or, equivalently

$$\sum_j v_j U S_j U^\dagger = \sum_j v_j' S_j. \tag{II.9}$$

Taking the derivative of both sides of this equation with respect to time yields a sufficient condition for a Hamiltonian to generate a compatible transformation

$$[H, S_j] = 0, \quad \forall S_j. \tag{II.10}$$

Since the Casimir operators for the algebra $\mathcal{A}$ satisfy this condition, they can be used to generate nondissapative transformations of a DFS/NS encoding. We will discuss the Casimir construction for single-particle invariants next. The generalization to higher-dimensional systems will then follow.

III. CASIMIR CONSTRUCTION FOR COLLECTIVE ERRORS

A Casimir Operator is a member of the center of the universal enveloping algebra meaning such an operator will commute with every element of the universal enveloping algebra. For matrix representations of quantum evolutions, which we will consider here, the universal enveloping algebra is the algebra of all products of Lie algebra basis elements. It is most important for our purposes that the Casimir operators commute with every generator of the Lie algebra and the collective errors form a representation of the Lie algebra (which is the algebra of Hermitian matrices). Once we find such invariants, we will have a set of Hamiltonians which commute with collective errors and are therefore compatible transformations. We begin by reviewing the construction of Casimir invariants.

Let a basis for the Lie algebra of SU(d) (hereafter denoted $\mathcal{L}(\mathrm{SU(d)})$) be given by a set $\{\lambda_i\}$ with the normalization and properties described in the Appendix. The Casimir operators of SU(d) are known. The most familiar, the quadratic Casimir, is proportional to the sum of the squares of the elements,

$$C_2 \propto \sum_i \lambda_i \lambda_i. \tag{III.1}$$

This along with all other Casimir operators can be obtained using the formula [17, 18]

$$I_n = \mathrm{Tr}(\mathrm{ad}_{\lambda_{a_1}} \circ \mathrm{ad}_{\lambda_{a_2}} \circ \cdots \circ \mathrm{ad}_{\lambda_{a_n}})\lambda_{a_1}\lambda_{a_2}...\lambda_{a_n}. \tag{III.2}$$

For example,

$$C_2 = \sum_{a_1,a_2,b_1,b_2} f_{a_1,b_1,b_2} f_{a_2,b_2,b_1} \lambda_{a_1}\lambda_{a_2}, \tag{III.3}$$

which reduces to Eq. (III.1) using Eq. (A.7). It turns out that the formula given in Eq. (III.2) does not produce independent invariants for the collective errors. However, the independent invariants can be obtained [17] and may be written in terms of the totally symmetric d-tensor. For example, the cubic Casimir invariant is

$$C_3 = \sum_{ijk} d_{ijk} \lambda_i \lambda_j \lambda_k. \tag{III.4}$$

Higher order Casimir operators can be constructed using the general formulation

$$C_n = \sum_{i_1,i_2,\ldots,i_n} d_{i_1,i_2,i_3} d_{i_3,i_4,i_5}, \ldots d_{i_{n-4},i_{n-3},i_{n-2}} \times d_{i_{n-2},i_{n-1},i_n} \lambda_{i_1}\lambda_{i_2}\lambda_{i_4}\ldots\lambda_{i_{n-1}}\lambda_{i_n} \quad \text{(III.5)}$$

The sum is over all elements of the algebra.

To show independence, one may begin with Eq. (III.2) and reduce the expressions using the identities in the appendix. Here our objective is to find a set of operators which commute with the set of collective motions. A basis for these motions is given by the set of operators of the form

$$S_j = \sum_\alpha \lambda_j^{(\alpha)}, \quad \text{(III.6)}$$

where the sum is taken over the particles in the system. These types of operators also form a basis for the collective errors acting on a DFS/NS and linear combinations give the stabilizer elements. (See Sec. II B for the definition and discussion.) An element of the algebra (with real coefficients) which commutes with these provides the Hamiltonians which are compatible with a DFS/NS.

IV. EXPLICIT FORMS FOR THE INVARIANTS

In this section we will find a set of independent operations for which each element of the set commutes with all members of the algebra formed by the S_j. Denote the algebra of the S_j by $\mathcal{A}$.

Note that the Casimir operators formed from the elements S_j form a representation of $\mathcal{L}$(SU(d)) if the λ_i do [19]. Therefore these are invariants of the algebra $\mathcal{A}$, i.e. they commute with elements of this algebra. However, this is not an irreducible algebra. Thus the construction must rely on the identification of the irreducible components.

To proceed, we first calculate the Casimir invariants of $\mathcal{L}$(SU(d)). Then, noting that linear combinations of these invariants are also invariants, we extract reducible components of the invariants. From a physical perspective, this means identifying n-body interactions which are contained within the m-body interactions where n≤m.

The quadratic Casimir operator for the algebra $\mathcal{A}$ is

$$J_2 = \sum_{i,j,k,l} f_{ijk} f_{kli} S_j S_l \propto \sum_j S_j S_j. \quad \text{(IV.1)}$$

Expanding this in terms of the basis elements $\{\lambda_i\}$ gives

$$J_2 \propto \sum_i \left(\sum_\alpha \lambda_i^{(\alpha)}\right)^2 = \sum_i \left(\sum_\alpha (\lambda_i^{(\alpha)})^2 + 2\sum_{\alpha<\beta} \lambda_i^{(\alpha)}\lambda_i^{(\beta)}\right). \quad \text{(IV.2)}$$

Note that the first term of the last expression is the sum of single-particle Casimir invariants. This allows us to infer that the second term in Eq. (IV.2) is also an invariant quantity. Furthermore, the only nontrivial contributions appearing in the commutator $\left[\sum_i \lambda_i^{(\alpha)}\lambda_i^{(\beta)}, S_l\right]$ have the form

$$[\lambda_i^{(\alpha)}, \lambda_j^{(\alpha)}]\lambda_i^{(\beta)} + \lambda_i^{(\alpha)}[\lambda_i^{(\beta)}, \lambda_j^{(\beta)}], \quad \text{(IV.3)}$$

with all other terms vanishing. Since this can be rewritten as

$$2if_{ijk}(\lambda_k^{(\alpha)}\lambda_i^{(\beta)} - \lambda_k^{(\alpha)}\lambda_i^{(\beta)}) = 0, \quad \text{(IV.4)}$$

we find that

$$I_2^{(\alpha,\beta)} = \sum_i \lambda_i^{(\alpha)}\lambda_i^{(\beta)} \quad \text{(IV.5)}$$

is also an independent invariant for each pair (α, β).

Now consider

$$J_3 = \sum f_{ijk} f_{klm} f_{mni} S_j S_l S_n = \sum f_{ijk} f_{klm} f_{mni} \left(\sum_\alpha \lambda_j^{(\alpha)}\right)\left(\sum_\beta \lambda_l^{(\beta)}\right)\left(\sum_\gamma \lambda_n^{(\gamma)}\right). \quad \text{(IV.6)}$$

Expanding the sums over the particle (Greek) indices, and reducing the results, three types of terms are obtained. First, if all three superscripts are the same, for example $\lambda_i^{(\alpha)}\lambda_j^{(\alpha)}\lambda_k^{(\alpha)}$, the term reduces to the quadratic Casimir invariant for particle α. Since any linear combination of invariants is invariant, the sum of all terms having this form is also invariant. Second, if two are the same, e.g. $\lambda_i^{(\alpha)}\lambda_j^{(\alpha)}\lambda_k^{(\beta)}$, then the result reduces to $I_2^{(\alpha,\beta)}$, thus terms of this form are also invariant quantities. Third, if all three are different, we obtain

$$I_3^{(\alpha,\beta,\gamma)} = \sum_{ijk} f_{ijk} \lambda_i^{(\alpha)}\lambda_j^{(\beta)}\lambda_k^{(\gamma)}, \quad \text{(IV.7)}$$

as an independent invariant. Notice this case is different from the ordinary Casimir construction where no such independent invariant arises for a term of the form of J_3.

Defining and expanding J_4 produces one new invariant,

$$I_4^{(\alpha,\beta,\gamma)} = \sum_{ijk} d_{ijk} \lambda_i^{(\alpha)}\lambda_j^{(\beta)}\lambda_k^{(\gamma)}. \quad \text{(IV.8)}$$

Continuing with this will iteratively produce a set of independent invariants for collective motions of particles. For three qutrits this set, I_2, I_3, I_4 is complete [20].

V. APPLICATIONS

After the motivation in the introduction and the Casimir construction, we now consider more explicitly the implications of our findings. First, we shall connect our findings to the familiar three-qubit DFS/NS where the exchange interaction is well known to allow for universal computation. We will then discuss the implications of these results for the case of a three qudit encoding, with a particular emphasis on the ability of these operations to generate universal quantum computation.

A. Qubits and the Exchange Interaction

We now consider the three-qubit DFS/NS. A collection of three qubits is smallest set of qubits which can protect against an arbitrary collective error (dephasing and rotations). In this case a basis for the Lie algebra of SU(2) can be taken to be the set of Pauli matrices $\{\sigma_i\}$. Collective errors span the set $\{S_j\}$, where $S_j = \sum_\alpha \sigma_j^{(\alpha)}$ for $j = x, y, z$. The quadratic Casimir operator for the algebra of the S_j can be expanded using Levi-Civita coefficients

$$J_2 = \sum_{i,j,k,l} \epsilon_{i,j,k}\epsilon_{k,l,i} \left(\sum_{(\alpha)} \sigma_j^{(\alpha)}\right)\left(\sum_{(\beta)} \sigma_l^{(\beta)}\right), \quad \text{(V.1)}$$

which reduces to -2$\sum S_i^2$. To show that this invariant can be decomposed into a sum of two-body exchange interactions one can follow the prescription outlined in Sec. IV. It is easy to check that the decomposition is given by $J_2 = -18\mathbb{1} - 4\sum_{i<j} E_{ij}$, where $E_{ij} = \sum \sigma_l^{(i)}\sigma_l^{(j)}$ is the two-qubit exchange Hamiltonian for qubits i and j.

It is known that the exchange interaction alone can be used to perform universal quantum computation on a three-qubit DFS/NS. Section IV describes a method for showing that each individual exchange Hamiltonian E_{ij} is an invariant quantity and thus can be used to reliably manipulate DFS/NS encodings. Although these results regarding the two-qubit exchange interaction are not new, the method for determining these invariants is quite general and be applied to a wide variety of DFS/NS encodings. Our next example highlights this point as we consider a three-qudit DFS/NS encoding of arbitrary dimension d.

B. Three Qudits

As mentioned earlier, a basis for the collective errors is given by the set

$$S_i = \sum_\alpha \lambda_i^{(\alpha)}, \quad \text{(V.2)}$$

where the subscript indicates the type of error and the superscript labels the particle on which the operator acts. The invariants I_2, I_3, and I_4 not only commute with every element of this set, but can also be used to form a representation of the Lie algebra of $SU(2)$ [20]. It has been shown that the encoded, or logical analogues of the Pauli matrices acting on an encoded qubit can be given in terms of these invariants by the relations

$$\bar{X} = \frac{1}{2\sqrt{3}}\left[I_2^{(2,3)} - I_2^{(1,3)}\right], \;\; \bar{Y} = \frac{I_3}{2\sqrt{3}}, \quad \text{(V.3)}$$

and

$$\bar{Z} = \left[I_2^{(2,3)} + I_2^{(1,3)} - 2I_2^{(1,2)}\right]/6. \quad \text{(V.4)}$$

All three of these generators can be expressed in terms of two body interactions since I_3 can be decomposed into products of I_2. In fact, the invariant I_2 alone suffices to perform universal computation using encoded qubits that are comprised of three physical qudits since they are able to generate any single qubit rotation, and can also be combined in such a way as to implement an entangling CNOT gate as well. This is due to the fact that the states which were used in Ref. [21] for the CNOT are also present in the expansions of the logical states encoded into qudits having $d \geq 3$.

In addition, the invariant $I_2^{(\alpha,\beta)}$ can also be used to perform the generalized exchange interaction between the states $|p\rangle^{(\alpha)}|q\rangle^{(\beta)}$ associated with particles α and β since it has been shown in Ref. [20] that

$$\exp\left[-i(\pi/4)\sum_j \lambda_j \otimes \lambda_j\right]|pq\rangle = -i\exp(\pi i/2d)\,|qp\rangle\,, \quad \text{(V.5)}$$

for $p, q = 1, 2, \ldots, d$.

Clearly these are linear combinations of the two-body interactions which are comprised of the invariants $I_2^{(\alpha,\beta)}$. Three-body and higher order interactions are less often experimentally controllable, but are also, in principle, viable candidates for quantum gates. For example the logical Y interaction for qudits is proportional to I_3.

C. Quantum Dots

Consider a spin quantum dot in an effective spin bath. The magnetic moment of the trapped electron can interact with the environmental spin moments, magnetic moments of nuclei, in such a way as to cause fluctuations in the electron's state, thus resulting in decoherence. If the environment is somehow driven to a zero-collective spin state (a collection of many-body singlet states), the spin will interact less strongly with its environment and this decoherence mechanism can be suppressed. There have been numerous theoretical approaches to the generation of large scale many-body singlet states (see, for example Ref. [24]) although a physical implementation remains technically challenging. These singlet states will almost certainly span a DFS/NS due to the fact that degeneracies in the singlets will almost certainly occur in a large collection of spins. In this case a discussion regarding the collective motion of such states is warranted.

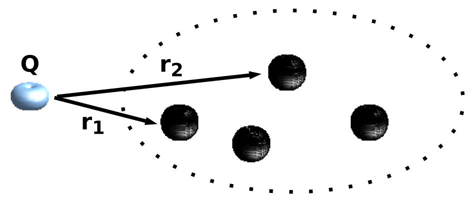

FIG. 1: Example of a spin singlet state (within the dotted ellipse) which has a non-zero spin coupling with a quantum dot (labeled Q). The environmental spins may be evolving under the DFS-compatible evolutions while remaining in a singlet state. Although the total spin is zero, the spins do interact due to the difference in distance between the spins comprising the singlet.

The DFS/NS compatible evolutions we have focused on here can occur in these spin baths in such a way as to preserve the singlet state structure yet can influence the state of the quantum dot due to the fluctuations in the states. For example, in the case of the four-qubit DFS, there are four spins which can be in one of two different singlet states [5]. One of these can represent the logical zero of an encoded qubit and the other a logical one. The DFS-compatible Hamiltonians can take the logical zero state and produce a logical one state thus changing the state of particular spins while leaving the set in a singlet state or superposition of singlet states. Since the nuclear spins are in a fixed location, some will be nearer to the electron in the quantum dot and some will be farther away. When they change states the dipole-dipole interaction will also change as a result of the variation in distance from neighboring spins. The dipole-dipole coupling drops off like $\sim 1/r^3$, so the spins which are nearer will have a greater influence even if the collection as a whole is in a singlet state. (See Fig. 1.) Even for the ideal case where the total collective spin of the environment is zero, the fact that there exists these collective motions which preserve the angular momentum of the system can have a detrimental effect on the state. An analysis of these effects will therefore be necessary in order to fully characterize the benefit of putting environmental spins into a set of singlet states.

VI. CONCLUSIONS

For quantum systems containing many particles, each having a number of internal states, the system could be in a vast array of possible states corresponding to a large Hilbert space dimension. The evolution of such states can be fairly simple however, as in the case of a system undergoing collective motion. Such motions occur, for example, when $\lambda \gg R$ so that each particle feels the same field. If states, or subsystems, of a collection of particles are invariant under collective motions, they are decoherence-free, or noiseless with respect to any collective operation, unitary or not. This leads to the promising method for error prevention–encoding in one of these subspaces to avoid collective errors. To take advantage of such an encoding for the purposes of quantum information processing, one requires a complete set of logical operations to be performed on these subsystems which is compatible with the encoding. We have provided a way in which to find the set of Hamiltonians for this purpose.

However, we also note that since collective motions commute with the operators we have presented here, the Casimir invariants may be measured while the system undergoes these collective errors. This allows one to describe the system by the values of these operators. Indeed one of the original motivations for studying these invariants was to find a complete set of commuting observables to completely specify a quantum system. (See for example Ref. [22] and references therein.) Not all of the invariants presented here will commute with each other, but they each commute with the collective motions. A subset of these invariant operators which also mutually commute will help provide a complete set of commuting operators along with the energy and total angular momentum.

Our work is quite general and can be applied to any set of d-state systems undergoing collective motions. Therefore, we have extended the Dicke-state description explicitly to the general case leading the way to the description of sets of particles undergoing collective motions and their manipulation when the particles have more than two internal states.

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Appendix A: The algebra of $SU(d)$

We have chosen the following convention for the normalization of the algebra of Hermitian matrices which are generators of $SU(d)$.

$$\mathrm{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}. \tag{A.1}$$

The commutation and anti-commutation relations of the matrices representing the basis for the Lie algebra can be summarized using the following equation:

$$\lambda_i \lambda_j = \frac{2}{d}\delta_{ij} + i f_{ijk}\lambda_k + d_{ijk}\lambda_k, \tag{A.2}$$

where here, and throughout this appendix, a sum over repeated indices is understood. The sums are written explicitly for clarity only in a few cases.

As with any Lie algebra we have the Jacobi identity:

$$f_{ilm}f_{jkl} + f_{jlm}f_{kil} + f_{klm}f_{ijl} = 0. \quad (A.3)$$

There is also a Jacobi-like identity,

$$f_{ilm}d_{jkl} + f_{jlm}d_{kil} + f_{klm}d_{ijl} = 0, \quad (A.4)$$

which was given by Macfarlane, et al. [23].

The following identities, also provided in [23], are useful

$$\begin{aligned} d_{iik} &= 0, && (A.5)\\ d_{ijk}f_{ljk} &= 0, && (A.6)\\ f_{ijk}f_{ljk} &= d\delta_{il}, && (A.7)\\ d_{ijk}d_{ljk} &= \frac{d^2-4}{d}\delta_{il}, && (A.8) \end{aligned}$$

and

$$f_{ijm}f_{klm} = \frac{2}{d}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (d_{ikm}d_{jlm} - d_{jkm}d_{ilm}) \quad (A.9)$$

and finally

$$\begin{aligned} d_{piq}d_{qjr}f_{rkp} &= \frac{d^2-4}{2d}f_{ijk}, && (A.10)\\ d_{piq}d_{qjr}d_{rkp} &= \frac{d^2-12}{2d}d_{ijk}. && (A.11) \end{aligned}$$

The proofs of these are fairly straight-forward, but we omit them here.

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