One-step error correction for multipartite polarization entanglement

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One-step error correction for multipartite polarization entanglement

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We present two economical one-step error-correction protocols for multipartite polarization-entangled systems in a Greenberger-Horne-Zeilinger state. One uses the spatial entanglement to correct the errors in the polarization entanglement of an N-photon system, resorting to linear optical elements. The other uses frequency entanglement to correct the errors in the polarization entanglement of an N-photon system. The parties in quantum communication can obtain a maximally entangled state from each N-photon system transmitted with one step in these two protocols and both their success probabilities are 100% in principle. That is, they both work in a deterministic way and they do not consume the less-entangled photon systems largely, which is far different from conventional multipartite entanglement purification schemes. These features maybe make these two protocols more useful in the practical applications in long-distance quantum communication.

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I. INTRODUCTION

Entanglement is an important quantum resource for quantum information processing [1]. The powerful speedup of quantum computation resorts to multipartite entanglement. Long-distance quantum communication should resort to entanglement for setting up the quantum channel between remote locations, including quantum teleportation [2], quantum key distribution (QKD) [3–7], quantum dense coding [8–10], quantum secret sharing [11–13], and so on. In a practical transmission, an entangled quantum system will inevitably interact with its environment, which will degrade the entanglement of the system. In general, the interaction will make an entangled system be in a mixed state. The decoherence of entanglement in quantum system will affect quantum communication largely. For example, it will decrease the security of a QKD protocol if a maximally entangled state transmitted over a noisy channel becomes a mixed entangled state as a vicious eavesdropper can exploit the decoherence to hide her illegal action. The non-maximally entangled quantum channel will decrease the fidelity of quantum teleportation and quantum dense coding.

Entanglement purification [14–22] is an interesting tool for the parties in quantum communication to improve the fidelity of entangled quantum systems after they are transmitted over a practical channel. Its task is to obtain a subset of high-fidelity entangled quantum systems from a set of quantum systems in a mixed entangled state. In 1996, Bennett *et al.* [14] proposed an original entanglement purification protocol (EPP) to purify a Werner state [23], resorting to quantum controlled-not (CNOT) gates and bidirectional unitary operations. Subsequently, Deutsh *et al.* [15] improved this EPP with two additional specific unitary operations. In 2001, Pan *et al.* [16] presented an EPP with linear optical elements and an ideal entanglement source by sacrificing a half of the efficiency in the EPP by Bennett *et al.* However, this protocol decreases largely the difficulty for the implementation of EPP with current technology as it is impossible to construct a perfect CNOT gate based on linear optics. In 2002, Simon and Pan [17] proposed an EPP with a currently available parametric down-conversion (PDC) source, not an ideal single-pair entanglement source. This protocol exploits the spatial entanglement of a photon pair to purify its polarization entanglement. In 2003, Pan *et al.* [24] demonstrated this EPP in experiment. In 2008, an efficient EPP [18] based on a PDC source was proposed with cross-Kerr nonlinearity. It has the same efficiency as the EPP by Bennett *et al.* with perfect CNOT gates. Moreover, it can be repeated to get a high-fidelity entangled photon pairs from a practical entanglement source. However, the cross-Kerr nonlinearity is too small in nature, which increases the difficulty for the implementation of this EPP at present. In 2008, Xiao *et al.* [19] proposed an EPP with frequency entanglement.

Recently, the concept of deterministic entanglement purification was proposed [20] for two-photon entangled systems, which is far different from the conventional entanglement purification (CEPP) [14–19] as the former works in a deterministic way, while the latter works in a probabilistic way. In 2010, we introduced a two-step deterministic entanglement purification protocol (DEPP) [20] for entangled photon pairs, resorting to hyerentanglement. In this two-step DEPP, the spatial entanglement and the frequency entanglement of a quantum system are used to correct the bit-flip error and the phase-flip error in this quantum system, respectively. With two error-correction processes,

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the two parties in quantum communication, say Alice and Bob, can obtain a maximally entangled state from each entangled system in theory. Certainly, it is difficult to implement this two-step DEPP in experiment at present as it requires the hyperentanglement of photon systems and a nonlinear optical interaction. Subsequently, we proposed an one-step DEPP [21], resorting to the spatial entanglement of a practical PDC source and linear optical elements. With this one-step DEPP, Alice and Bob can in principle obtain a maximally entangled photon pair from each photon system with only one step. Simultaneously, Li [22] presented independently an interesting DEPP using spatial entanglement, called it also an one-step DEPP as it has the same role as the one-step DEPP in Ref. [21] although there are, in essence, some differences in explaining the principle of the detereministic entanglement purification. Although the physics in the one-step DEPP in Ref. [21] is more clear than that in Ref. [22], the one-step DEPP by Li is more economical than that in Ref. [21] as the former needs two polarizing beam splitters (PBSs), while the latter needs eight PBSs.

In essence, each of all existing EPPs [14–22] can be considered as a quantum error-correction protocol. In CEPPs, the process for error correction is divided into two steps which are repeated some times for improving the fidelity of entanglement largely. One step is used to correct the bit-flip error and the other is used correct the phase-flip error. That is, all the CEPPs [14–19] work in a probabilistic way for the correction of the errors in only one degree of freedom of photons. In DEPPs [20–22], the error correction is completed with one step. That is, they work in a deterministic way, but they should resort to at least another degree of freedom of photons.

By far, there have been several interesting EPPs [14–22] focusing on the bipartite entangled quantum systems, while the number of multipartite entanglement purification protocols (MEPP) [25–27] is very small as the structure of a multipartite quantum system is more complicated than that of a bipartite quantum system. Moreover, it is difficult to optimize a MEPP as its efficiency depends on a great number of parameters coming from the channel noise. In 1998, Murao *et al.* [25] proposed the first multipartite entanglement purification protocol (MEPP) to purify multipartite quantum system in a Greenberger-Horne-Zeilinger (GHZ) with CNOT gates, following some ideas in the EPP by Bennett et al [14]. In 2009, a MEPP based on cross-Kerr nonlinearities was proposed [26]. In this protocol, the cross-Kerr nonlinearity is used to construct a nondestructive quantum nondemolition detector (QND) [28] which has the functions of both a parity-check detector and a single-photon detector. With QNDs, the parties can obtain some high-fidelity GHZ states from an ensemble in a mixed entangled state by performing this MEPP repeatedly. In 2009, a multipartite electronic entanglement purification was proposed with charge detection [27]. All these three MEPPs work in a conventional way. That is, the parties can only obtain a subset of high-fidelity entangled states from an ensemble in a mixed less-entangled state by performing the MEPPs repeatedly and sacrificing a great deal of quantum resource.

In this paper, we will present two economical one-step error-correction protocols for multipartite polarization-entangled systems in a GHZ state. The first one-step multipartite polarization entanglement error-correction protocol (MPEECP) is based on spatial entanglement and simple linear optical elements. The parties in quantum communication can obtain a maximally entangled state from each system polluted by the channel noise on the polarization degree of freedom. Also, the number of the linear optical elements in this protocol is reduced to be a minimal one. Moreover, this protocol works in a deterministic way, which is far different from the polarization entanglement purification protocol for two-photon system by using the spatial entanglement [17] as the latter can only improve the fidelity of an ensemble in a mixed entangled state by repeating the protocol again and again. The second MPEECP uses the frequency entanglement to correct the errors in the polarization entanglement of an N-photon system and reduces the number of the channels for the transmission of the system. The success probabilities of these two economical one-step MPEECPs are in principle 100%. Compared with the conventional multipartite entanglement purification protocols [25–27], these two economical one-step MPEECPs reduces the quantum resource consumed largely. These advantages maybe make these two MPEECPs more useful in practical applications in long-distance quantum communication in future.

II. ECONOMIC ONE-STEP MPEECP WITH SPATIAL ENTANGLEMENT

As shown by Simon and Pan [17], an entangled state of a photon pair in two degrees of freedom can be written as

$$|\Psi\rangle = \frac{1}{2}(|a_1\rangle|b_1\rangle + |a_2\rangle|b_2\rangle)(|H\rangle|H\rangle + |V\rangle|V\rangle)_{ab}, \tag{1}$$

where $|H\rangle$ and $|V\rangle$ represent the horizonal and the vertical polarizations of photons, respectively. The subscripts a and b represent the two photons sent to the two parties, say Alice and Bob, respectively. a_1 (b_1) and a_2 (b_2) are the two spatial modes for the photon sent to Alice (Bob). As the spatial entanglement is far more stable than the polarization entanglement over an optical-fiber channel, Simon and Pan [17] exploited the spatial entanglement of a photon pair coming from a PDC source to purify the polarization of a photon pair. By controlling the phase stability,

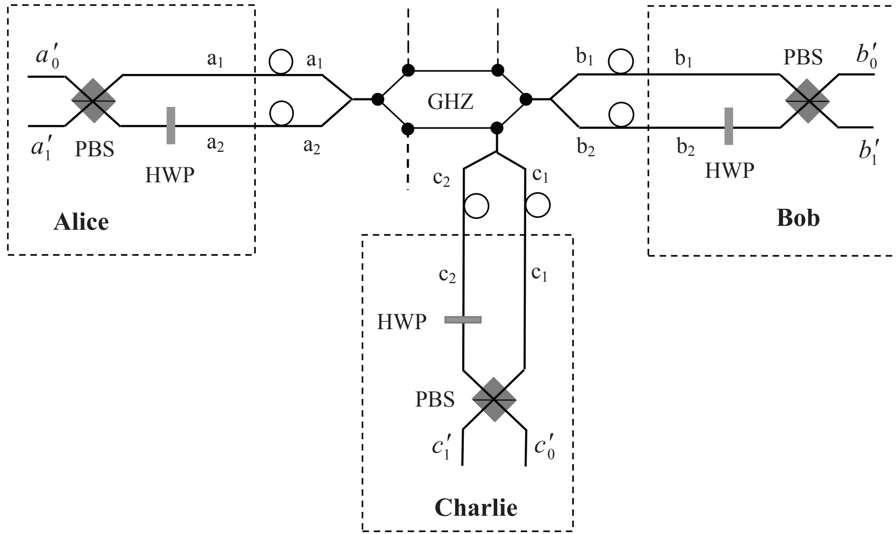

FIG. 1: Schematic illustration for the principle of the present economic one-step multipartite polarization entanglement error-correction protocol using spatial entanglement. PBS represents a polarizing beam splitter and it is used to transfer a $|H\rangle$ polarization photon and reflet a $|V\rangle$ polarization photon. HWP represents a half wave plate and it can convert $|H\rangle$ into $|V\rangle$, and $|V\rangle$ into $|H\rangle$. After the detection for each spatial mode, the parties in quantum communication can obtain a maximally entangled polarization state from each multi-photon system in a deterministic way in principle.

the EPP by Simon and Pan can improve the fidelity of an ensemble in a mixed entangled state, as shown in the experiment by Pan *et al.* [24].

The GHZ state of a multipartite entangled system composed of N two-level particles can be described as

$$|\Phi^+\rangle_N = \frac{1}{\sqrt{2}}(|HH\cdots H\rangle + |VV\cdots V\rangle)_{AB\cdots C}. \tag{2}$$

Here the subscripts A, B, $\cdots$, and C represent the photons sent to the parties Alice, Bob, $\cdots$, and Charlie, respectively. Certainly, there are other $2^N - 1$ GHZ state for an N-qubit system and can be written as $|\Phi^+\rangle_{ij\cdots k} = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle + |\bar{i}\bar{j}\cdots\bar{k}\rangle)_{AB\cdots C}$ and $|\Phi^-\rangle_{\bar{i}\bar{j}\cdots\bar{k}} = \frac{1}{\sqrt{2}}(|ij\cdots k\rangle - |\bar{i}\bar{j}\cdots\bar{k}\rangle)_{AB\cdots C}$. Here $i, j, k \in \{0, 1\}$, $\bar{i} = 1-i$, $\bar{j} = 1-j$, and $\bar{k} = 1-k$. $|0\rangle \equiv |H\rangle$ and $|1\rangle \equiv |V\rangle$. An entangled state of an N-photon system in the polarization and the spatial-mode degrees of freedom can be written as

$$|\Phi\rangle_s = \frac{1}{2}(|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |a_2\rangle|b_2\rangle\cdots|c_2\rangle)(|H\rangle|H\rangle\cdots|H\rangle + |V\rangle|V\rangle\cdots|V\rangle)_{AB\cdots C}. \tag{3}$$

Here $|\phi\rangle_s = \frac{1}{\sqrt{2}}(|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |a_2\rangle|b_2\rangle\cdots|c_2\rangle)$ represents the spatial entanglement of an N-photon system.

The principle of the present economic one-step MPEECP using spatial entanglement is shown in Fig. 1. Suppose that the original state of an N-photon system is $|\Phi\rangle_s$. After the photons suffer from the noise in the channels, the state of the system becomes

$$\rho = \rho_p \cdot \rho_s, \tag{4}$$

where ρ_p and ρ_s represent the states of the multi-photon systems in the polarization and the spatial degrees of freedom, respectively, i.e.,

$$\rho_s = |\phi\rangle_s\langle\phi|, \tag{5}$$

$$\rho_p = \sum_{i,j,\cdots,k} F_{ij\cdots k}|\Phi^+\rangle_{ij\cdots k}\langle\Phi^+| + F_{\bar{i}\bar{j}\cdots\bar{k}}|\Phi^-\rangle_{ij\cdots k}\langle\Phi^-|. \tag{6}$$

$$\sum_{i,j,\cdots,k} F_{ij\cdots k} + F_{\bar{i}\bar{j}\cdots\bar{k}} = 1. \tag{7}$$

Here $F_{ij\cdots k}$ ($F_{\bar{i}\bar{j}\cdots\bar{k}}$) is the probability that the N-photon system is in the state $|\Phi^+\rangle_{ij\cdots k}$ ($|\Phi^-\rangle_{ij\cdots k}$) after it is transmitted over a noisy channel. That is, the mixed state shown in Eq.(4) can be viewed as a probabilistic mixture of 2^N pure states. The system is in the state $|\Phi^+\rangle_{ij\cdots k}\cdot|\phi\rangle_s$ or the state $|\Phi^-\rangle_{ij\cdots k}\cdot|\phi\rangle_s$ with the probabilities $F_{ij\cdots k}$ and $F_{\bar{i}\bar{j}\cdots\bar{k}}$, respectively.

Let us first discuss the case that the N-photon system is in the state $|\Phi^+\rangle_s\equiv|\phi\rangle_s\cdot|\Phi^+\rangle_{ij\cdots k}=\frac{1}{\sqrt{2}}(|a_1\rangle|b_1\rangle\cdots|c_1\rangle+|a_2\rangle|b_2\rangle\cdots|c_2\rangle)\cdot\frac{1}{\sqrt{2}}(|ij\cdots k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle)_{AB\cdots C}$. After the setup shown in Fig.1, the evolution of the state $|\Phi^+\rangle_s$ can be described as

$$\begin{aligned}|\Phi^+\rangle_s \xrightarrow{HWPs} & \frac{1}{2}(|ij\cdots k\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle\\ & +|\bar{i}\bar{j}\cdots\bar{k}\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle+|ij\cdots k\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle)\\ \xrightarrow{PBSs} & \frac{1}{2}(|ij\cdots k\rangle|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle\\ & +|\bar{i}\bar{j}\cdots\bar{k}\rangle|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle+|ij\cdots k\rangle|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle)\\ = & \frac{1}{2}(|ij\cdots k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle)(|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle+|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle)_{AB\cdots C}.\end{aligned} \tag{8}$$

That is, the parties in quantum communication can determine the state of their N-photon system by postselection. If the photons emit from the outports $|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle$ or $|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle$ of the PBSs, the system is in the state $\frac{1}{\sqrt{2}}(|ij\cdots k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle)$. The parties can obtain the state $|\Phi^+\rangle_N=\frac{1}{\sqrt{2}}(|HH\cdots H\rangle+|VV\cdots V\rangle)_{AB\cdots C}$ with some unitary single-photon operations. For example, if the N photons emit from the outports $|a'_1\rangle|b'_0\rangle\cdots|c'_0\rangle$, the system is in the state $\frac{1}{\sqrt{2}}(|10\cdots 0\rangle+|01\cdots 1\rangle)_{AB\cdots C}=\frac{1}{\sqrt{2}}(|V\rangle|H\rangle\cdots|H\rangle+|H\rangle|V\rangle\cdots|V\rangle)_{AB\cdots C}$ and the parties can obtain the state $\frac{1}{\sqrt{2}}(|H\rangle|H\rangle\cdots|H\rangle+|V\rangle|V\rangle\cdots|V\rangle)_{AB\cdots C}$ with only a bit-flip operation $\sigma_x=|V\rangle\langle H|+|H\rangle\langle V|$ on the first qubit A.

When the N-photon system is in the state $|\Phi^-\rangle_s\equiv|\phi\rangle_s\cdot|\Phi^-\rangle_{ij\cdots k}=\frac{1}{\sqrt{2}}(|a_1\rangle|b_1\rangle\cdots|c_1\rangle+|a_2\rangle|b_2\rangle\cdots|c_2\rangle)\cdot\frac{1}{\sqrt{2}}(|ij\cdots k\rangle-|\bar{i}\bar{j}\cdots\bar{k}\rangle)_{AB\cdots C}$, one can obtain the similar result. After the setup shown in Fig.1, the evolution of the state $|\Phi^-\rangle_s$ can be described as

$$\begin{aligned}|\Phi^-\rangle_s \xrightarrow{HWPs} & \frac{1}{2}(|ij\cdots k\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle\\ & -|\bar{i}\bar{j}\cdots\bar{k}\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle-|ij\cdots k\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle)\\ \xrightarrow{PBSs} & \frac{1}{2}(|ij\cdots k\rangle|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle\\ & -|\bar{i}\bar{j}\cdots\bar{k}\rangle|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle-|ij\cdots k\rangle|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle)\\ = & \frac{1}{2}(|ij\cdots k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle)(|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle-|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle)_{AB\cdots C}.\end{aligned} \tag{9}$$

The parties in quantum communication can also determine the state of their N-photon system by postselection on the spatial modes of the N photons. If the photons emit from the outports $|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle$ or $|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle$ of the PBSs, the system is in the state $\frac{1}{\sqrt{2}}(|ij\cdots k\rangle+|\bar{i}\bar{j}\cdots\bar{k}\rangle)$. That is, the phase-flip error does not affect the result and the parties can obtain the state $|\Phi^+\rangle_N=\frac{1}{\sqrt{2}}(|HH\cdots H\rangle+|VV\cdots V\rangle)_{AB\cdots C}$ with some unitary single-photon operations.

III. ECONOMIC ONE-STEP MPEECP WITH FREQUENCY ENTANGLEMENT

Experimental results showed that the frequency of a photon is also far more stable than its polarization as previous experiments showed that the polarization entanglement is quite unsuitable for transmission over distances of more than a few kilometers in an optical fiber [7]. For example, Naik *et al.* demonstrated the Ekert protocol [3] by only a few meters [7, 29]. Also, they observed the quantum bit error rate (QBER) increase to 33% in the experiment implementation of the six-state protocol [30, 31]. For frequency coding [32–37], for example, the Besancon group performed a key distribution over a 20-km single-mode optical fiber spool. They recorded a QBER$_{opt}$ contribution of approximately 4%, and estimated that 2% could be attributed to the transmission of the central frequency by the Fabry-Perot cavity [37]. That is, the parties in quantum communication can also exploit the frequency entanglement of a multipartite entangled system to correct the errors in its polarization entanglement, similar to the case with a spatial entanglement. In this way, the parties can obtain the maximally entangled state $|\Phi^+\rangle_N$ with less channels than that with spatial entanglement.

An entangled state of an N-photon system in both the polarization and the frequency degrees of freedom can be written as

$$|\Psi\rangle_s = \frac{1}{2}(|\omega_1\rangle|\omega_1\rangle\cdots|\omega_1\rangle + |\omega_2\rangle|\omega_2\rangle\cdots|\omega_2\rangle)(|H\rangle|H\rangle\cdots|H\rangle + |V\rangle|V\rangle\cdots|V\rangle)_{AB\cdots C}, \tag{10}$$

where $|\omega_1\rangle$ and $|\omega_2\rangle$ are the two frequency modes for photons. $|\phi\rangle_f = \frac{1}{\sqrt{2}}(|\omega_1\rangle|\omega_1\rangle\cdots|\omega_1\rangle + |\omega_2\rangle|\omega_2\rangle\cdots|\omega_2\rangle)$ represents the frequency entanglement of an N-photon system.

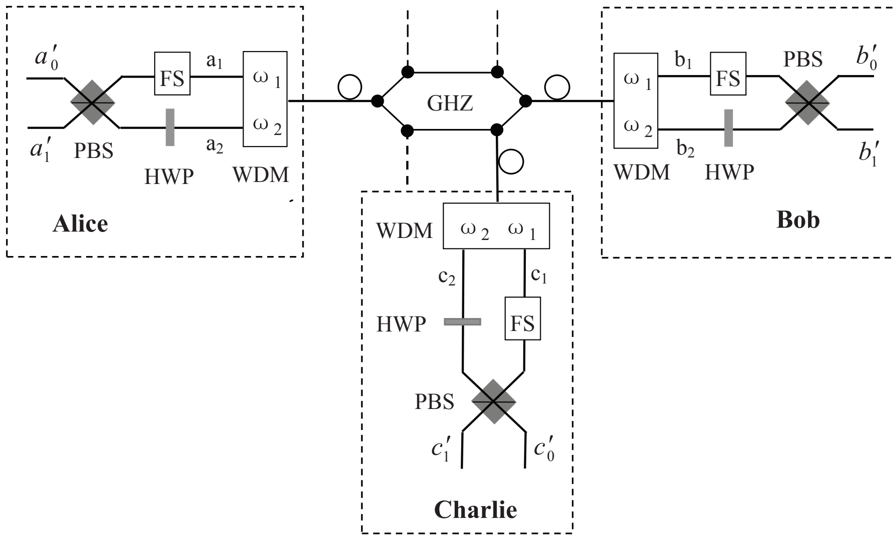

FIG. 2: Schematic illustration for the principle of the present economic one-step multipartite polarization entanglement error-correction protocol using frequency entanglement. WDM represents a polarization independent wavelength division multiplexer which will lead the photons to different spatial modes according to their frequencies. FS is a frequency shifter which is used to complete the frequency shift from ω_1 to ω_2.

The principle of the present economic one-step MPEECP using frequency entanglement is shown in Fig. 2. Suppose that the original state of an N-photon system is $|\Psi\rangle_s$. After the photons suffer from the noise in the channels, the state of the system becomes

$$\rho_s = \rho_p \cdot \rho_f, \tag{11}$$

where

$$\rho_f = |\phi\rangle_f\langle\phi|. \tag{12}$$

ρ_p represents the density matrix in the polarization degree of freedom shown in Eq. (6). The mixed state shown in Eq.(11) means that the system is in the state $|\Phi^+\rangle_{ij\cdots k}\cdot|\phi\rangle_f$ or the state $|\Phi^-\rangle_{ij\cdots k}\cdot|\phi\rangle_f$ with the probabilities $F_{ij\cdots k}$ and $F_{\bar{i}\bar{j}\cdots\bar{k}}$, respectively.

Similar to the case with the spatial entanglement of an N-photon system, if the system is in the state $|\Psi^+\rangle_s \equiv |\phi\rangle_f\cdot|\Phi^+\rangle_{ij\cdots k} = \frac{1}{\sqrt{2}}(|\omega_1\rangle|\omega_1\rangle\cdots|\omega_1\rangle + |\omega_2\rangle|\omega_2\rangle\cdots|\omega_2\rangle)\cdot\frac{1}{\sqrt{2}}(|ij\cdots k\rangle + |\bar{i}\bar{j}\cdots\bar{k}\rangle)_{AB\cdots C}$, after the setup shown in Fig.2, the evolution of the state $|\Psi^+\rangle_s$ can be described as

$$
\begin{aligned}
|\Psi^+\rangle_s \xrightarrow{WDMs} & \frac{1}{2}(|ij\cdots k\rangle + |\overline{ij}\cdots\bar{k}\rangle)_{AB\cdots C}\cdot(|a_1\rangle_{\omega_1}|b_1\rangle_{\omega_1}\cdots|c_1\rangle_{\omega_1} + |a_2\rangle_{\omega_2}|b_2\rangle_{\omega_2}\cdots|c_2\rangle_{\omega_2})\\
\xrightarrow{FSs} & \frac{1}{2}(|ij\cdots k\rangle + |\overline{ij}\cdots\bar{k}\rangle)_{AB\cdots C}\cdot(|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |a_2\rangle|b_2\rangle\cdots|c_2\rangle)\\
\xrightarrow{HWPs} & \frac{1}{2}(|ij\cdots k\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |\overline{ij}\cdots\bar{k}\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle\\
& +|\overline{ij}\cdots\bar{k}\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |ij\cdots k\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle)\\
\xrightarrow{PBSs} & \frac{1}{2}(|ij\cdots k\rangle|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle + |\overline{ij}\cdots\bar{k}\rangle|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle\\
& +|\overline{ij}\cdots\bar{k}\rangle|a'_{i}\rangle|b'_{j}\rangle\cdots|c'_{k}\rangle + |ij\cdots k\rangle|a'_{i}\rangle|b'_{j}\rangle\cdots|c'_{k}\rangle)\\
= & \frac{1}{2}(|ij\cdots k\rangle + |\overline{ij}\cdots\bar{k}\rangle)(|a'_{i}\rangle|b'_{j}\rangle\cdots|c'_{k}\rangle + |a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle)_{AB\cdots C}.
\end{aligned} \tag{13}
$$

That is, the parties in quantum communication can determine the state of their N-photon system by postselection on the spatial modes of the photons. If the photons emit from the outports $|a'_i\rangle|b'_j\rangle\cdots|c'_k\rangle$ or $|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle$ of the PBSs, the system is in the state $\frac{1}{\sqrt{2}}(|ij\cdots k\rangle + |\overline{ij}\cdots\bar{k}\rangle)$. The parties can obtain the state $|\Phi^+\rangle_N = \frac{1}{\sqrt{2}}(|HH\cdots H\rangle + |VV\cdots V\rangle)_{AB\cdots C}$ with some unitary single-photon operations.

When the N-photon system is in the state $|\Psi^-\rangle_s \equiv |\phi\rangle_f\cdot|\Phi^-\rangle_{ij\cdots k} = \frac{1}{\sqrt{2}}(|\omega_1\rangle|\omega_1\rangle\cdots|\omega_1\rangle + |\omega_2\rangle|\omega_2\rangle\cdots|\omega_2\rangle)\cdot\frac{1}{\sqrt{2}}(|ij\cdots k\rangle - |\overline{ij}\cdots\bar{k}\rangle)_{AB\cdots C}$, one can obtain the similar result. After the setup shown in Fig.2, the evolution of the state $|\Psi^-\rangle_s$ can be described as

$$
\begin{aligned}
|\Psi^-\rangle_s \xrightarrow{WDMs} & \frac{1}{2}(|ij\cdots k\rangle - |\overline{ij}\cdots\bar{k}\rangle)_{AB\cdots C}\cdot(|a_1\rangle_{\omega_1}|b_1\rangle_{\omega_1}\cdots|c_1\rangle_{\omega_1} + |a_2\rangle_{\omega_2}|b_2\rangle_{\omega_2}\cdots|c_2\rangle_{\omega_2})\\
\xrightarrow{FSs} & \frac{1}{2}(|ij\cdots k\rangle - |\overline{ij}\cdots\bar{k}\rangle)_{AB\cdots C}\cdot(|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |a_2\rangle|b_2\rangle\cdots|c_2\rangle)\\
\xrightarrow{HWPs} & \frac{1}{2}(|ij\cdots k\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle + |\overline{ij}\cdots\bar{k}\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle\\
& -|\overline{ij}\cdots\bar{k}\rangle|a_1\rangle|b_1\rangle\cdots|c_1\rangle - |ij\cdots k\rangle|a_2\rangle|b_2\rangle\cdots|c_2\rangle)\\
\xrightarrow{PBSs} & \frac{1}{2}(|ij\cdots k\rangle|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle + |\overline{ij}\cdots\bar{k}\rangle|a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle\\
& -|\overline{ij}\cdots\bar{k}\rangle|a'_{i}\rangle|b'_{j}\rangle\cdots|c'_{k}\rangle - |ij\cdots k\rangle|a'_{i}\rangle|b'_{j}\rangle\cdots|c'_{k}\rangle)\\
= & \frac{1}{2}(|ij\cdots k\rangle + |\overline{ij}\cdots\bar{k}\rangle)(|a'_{i}\rangle|b'_{j}\rangle\cdots|c'_{k}\rangle - |a'_{\bar{i}}\rangle|b'_{\bar{j}}\rangle\cdots|c'_{\bar{k}}\rangle)_{AB\cdots C}.
\end{aligned} \tag{14}
$$

That is, the phase-flip error does not affect the result yet, as the same as the case with spatial entanglement.

IV. DISCUSSION AND SUMMARY

It is interesting to compare these two economic one-step MPEECPs with the conventional multipartite entanglement purification protocols (CMEPPs) in Refs.[25, 26]. In Ref.[25], Murao *et al.* divided their CMEPP into two steps. One is used to purify the bit-flip errors and the other is used to purify the phase-flip errors, by resorting to CNOT gates. The parties in quantum communication can in principle improve the fidelity of an ensemble in a multipartite mixed entangled state by repeating these two steps again and again. The CMEPP in Ref.[26] is similar to that by Murao *et al.* but the CNOT gates are replaced with some parity-check detectors based on nonlinear optics. That is, the two existing CMEPPs [25, 26] for N-photon systems can be used to improve the fidelity of an ensemble in a mixed entangled state and they works in a probabilistic way. In these two economic one-step MPEECPs, the parties can in principle obtain a maximally entangled state from each system transmitted and they works in a completely deterministic way. On the other hand, the system after it is transmitted over a noisy channel in the CMEPPs [25, 26] should be in an entangled state in the polarization degree of freedom. In these two MPEECPs, they do not require the system to be entangled in the polarization degree of freedom but they require another degree of freedom of photons to keep the entanglement.

In essence, these two MPEECPs exploit the entanglement transformation between two degrees of freedom, not that between two systems, to correct the errors in the polarization entanglement, which is different from the CMEPPs

[25, 26]. In our two MPEECPs, the polarization degree of freedom of an N-photon system is polluted by the noisy channels. The noise may make the system be in a completely mixed state in the polarization degree of freedom. In our first MPEECP, the spatial entanglement is kept as spatial entanglement is far more stable than the polarization entanglement over an optical-fiber channel. So does the frequency entanglement in our second MPEECP. After the transmission over noisy channels, the parties transform the spatial entanglement into the polarization entanglement completely in our first MPEECP, by using some linear optical elements to remove the noise effect on the polarization first, which is in principle different from the EPP using spatial entanglement for two-photon systems in Ref. [17]. In our second MPEECP, the parties first exploit WDMs to transform the frequency entanglement of an N-photon system into its spatial entanglement. With FSs, the parties erase the distinguishability for the frequencies of their photons. In this way, the frequency entanglement is completely transformed into the spatial entanglement. The other precesses are the same as the case in our first MPEECP. Our second MPEECP only exploits a fiber channel for each party, not two channels in our first MPEECP. The parties can obtain this economization on the number of channels but they should resort to some nonlinear elements to complete this MPEECP. This is a tradoff between our two MPEECPs.

In summary, we have presented two economic one-step MPEECPs for an N-photon system in a GHZ state. In our first MPEECP, the parties exploit the spatial entanglement of an N-photon system to correct the errors in its polarization entanglement with some linear optical elements. Moreover, this protocol works in a deterministic way, which is far different from the polarization entanglement purification protocol for two-photon systems by using the spatial entanglement [17] as the latter can only improve the fidelity of an ensemble in a mixed entangled state by repeating the protocol again and again. In our second MPEECP, the parties exploit the frequency entanglement of an N-photon system to correct the errors in its polarization entanglement, resorting to some nonlinear optical elements but reducing a fiber channel for each party. As both the spatial entanglement and the frequency entanglement of an N-photon system are far more stable than the polarization entanglement over an optical-fiber channel, these two economic one-step MPEECPs will reduce a greet deal of quantum resource consumed as they both work in a deterministic way, not a probabilistic way. This advantage maybe make these two MPEECPs more useful in practical applications in long-distance quantum communication in future.

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