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Shandong Provincial Key Laboratory of Laser Polarization and Information Technology,

Department of physics, Qufu Normal university, Qufu 273165, China

Abstract

In this paper the entanglement and quantum phase transition of the anisotropic $s = 1/2$ XY model are studied by using the quantum renormalization group method. By solving the renormalization equations, we get the trivial fixed point and the untrivial fixed point which correspond to the phase of the system and the critical point, respectively. Then the concurrence between two blocks are calculated and it is found that when the number of the iterations of the renormalziation trends infinity, the concurrence develops two staturated values which are associated with two different phases, i.e., Ising-like and spin-fluid phases. We also investigate the first derivative of the concurrence, and find that there exists non-analytic behaviors at the quantum critical point, which directly associate with the divergence of the correlation length. Further insight, the scaling behaviors of the system are analyzed, it is shown that how the maximum value of the first derivative of the concurrence reaches the infinity and how the critical point is touched as the size of the system becomes large.

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[*]Corresponding author; Electronic address: kongxm@mail.qfnu.edu.cn(X.-M.Kong)

I. INTRODUCTION

The quantum entanglement, as one of the most intriguing features of quantum theory, has attracted much attention because its non-classical correlation can be regarded as an essential resource in quantum communication and information processing [1]. In view of the connection between the quantum entanglement and quantum correlation [2], the entanglement in some many-body systems has been widely investigated [3–5]. The entanglement, which is the main difference between quantum and classical systems, also plays an important role in the quantum phase transition (QPT). The QPT is induced by the change of an external parameter or coupling constant [6]. This change occurs at absolute zero temperature where the quantum fluctuations paly the dominant role and all the thermal fluctuations get frozen.

The entanglement between two nearest-neighbor spins in one-dimensional XY system with the transverse magnetic field was studied in Ref. [7]. The relation between the QPT and entanglement are studied in this paper and it is found that the system exists QPT and the scaling behavior in the vicinity of the critical point. After that, a great deal of efforts have been devoted to investigating the entanglement and QPT in some spin systems [8–11]. The XY model is exactly solved by the Jordan-Wigner transform in Ref. [8], in which the Ising model, as a special case of the XY model, exhibits QPT and a maximum value for the next-nearest-neighbor entanglement at the critical point. The entanglement properties and the spin squeezing of the ground state of mutually interacting spins 1/2 in a transverse magnetic field are analyzed and the system shows a cusplike singularity at the critical point in the thermodynamical limit [11]. To further discuss the QPT and entanglement, the renormalization group (RG) method is introduced, such as real space RG, Monte Carlo RG, and density-matrix RG [12–15]. The pairwise entanglement of the system is also discussed by means of quantum renormalization group (QRG) method [16, 17]. Very recently, the spin$-1/2$ Ising and Heisenberg models are studied by using the same method by a group of Iran and found that the systems exist QPT [18–21]. It is also shown that the nonanalytic behavior of the entanglement and the scaling behaviors closing to the quantum critical point are obtained.

The XY model is firstly discussed in Ref. [22], and then has been widely investigated recently [23–25]. In this paper, we study the quantum entanglement and QPT in the spin$-1/2$ XY model by the method of QRG. It is found that the system exists QPT between the spin-

fluid and Ising-like phases. Further insight, the nonanalytic behavior of the entanglement and the the scaling behavior of the system are gotten. The organization of this paper is as follows. In Sec. II we briefly introduce the RG method and obtain the fixed points of the XY model. The block-block entanglement is obtained in Sec. III and in Sec. IV we discuss the non-analytic and the scaling behaviors of the entanglement. Sec. V is devoted to the conclusions.

II.   RENORMALIZATION OF THE XY MODEL

Firstly, the QRG is introduced. Eliminating the degrees of freedom of the system followed by an iteration is the main idea of QRG. The aim of the iteration is that reduces the number of variables step by step until a more manageable situation is reached. For this idea, the kadanoff's block approach is implemented in this paper, because it is not only well suited to perform analytical calculations but also easy to be extended to the higher dimensions [26, 27]. We consider three sites as a block, the kadanoff's block approach is given in Fig. 1. In this way, the effective Hamiltonian $\left(H^{eff}\right)$ can be gotten which has structural similarities with the original Hamiltonian (H). The original Hilbert space is also replaced by a reduced Hilbert space which acts on the renormalized subspace [27, 28].

The Hamiltonian of the XY model on a periodic chain with N sites can be written as

$$H\left(J,\gamma\right)=\frac{J}{4}\sum_{i}^{N}\left[\left(1+\gamma\right)\sigma_i^x\sigma_{i+1}^x+\left(1-\gamma\right)\sigma_i^y\sigma_{i+1}^y\right],\tag{1}$$

where J is the exchange coupling constant, γ is the anisotropy parameter, and σ^α $(\alpha=x,y)$ are Pauli matrices. The XY model can be encompassed another two well-know spin models, i.e., the Ising model for $\gamma=1$ and the XX model for $\gamma=0$. For $0<\gamma\leq1$, it belongs to the Ising universality class [22].

By using the Kadanoff's block approach, Eq. (1) can be written as

$$H=H^A+H^{AA},\tag{2}$$

where H^A is the block Hamiltonian, H^{AA} is the interblock Hamiltonian. The specific forms of H^A and H^{AA} are

$$H^A=\sum_{L}^{N/3}h_L^A,\tag{3}$$

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$$H^{AA} = \frac{J}{4} \sum_{L}^{N/3} \left[(1+\gamma)\,\sigma^x_{L,3}\sigma^x_{L+1,1} + (1-\gamma)\,\sigma^y_{L,3}\sigma^y_{L+1,1} \right], \qquad (4)$$

where

$$h^A_L = \frac{J}{4} \left[(1+\gamma)\left(\sigma^x_{L,1}\sigma^x_{L,2} + \sigma^x_{L,2}\sigma^x_{L,3}\right) + (1-\gamma)\left(\sigma^y_{L,1}\sigma^y_{L,2} + \sigma^y_{L,2}\sigma^y_{L,3}\right) \right], \qquad (5)$$

which is the Lth block Hamiltonian.

In the terms of matrix product states [29], the Lth block Hamiltonian can be exactly diagonalized and solved. We can obtain four distinct eigenvalues which are doubly-degeneracy. Defining $|\uparrow\rangle$ and $|\downarrow\rangle$ as the eigenstates of σ^z, both degenerate ground states are given as follows:

$$|\Phi_0\rangle = \frac{1}{2\sqrt{1+\gamma^2}} \left(-\sqrt{1+\gamma^2}\,|\uparrow\uparrow\downarrow\rangle + \sqrt{2}\,|\uparrow\downarrow\uparrow\rangle - \sqrt{1+\gamma^2}\,|\downarrow\uparrow\uparrow\rangle + \sqrt{2}\gamma\,|\downarrow\downarrow\downarrow\rangle \right), \qquad (6)$$

$$\left|\Phi'_0\right\rangle = \frac{1}{2\sqrt{1+\gamma^2}} \left(-\sqrt{2}\gamma\,|\uparrow\uparrow\uparrow\rangle + \sqrt{1+\gamma^2}\,|\uparrow\downarrow\downarrow\rangle - \sqrt{2}\,|\downarrow\uparrow\downarrow\rangle + \sqrt{1+\gamma^2}\,|\downarrow\downarrow\uparrow\rangle \right). \qquad (7)$$

The energy corresponding the ground states is

$$E_0 = -\frac{J}{\sqrt{2}}\sqrt{1+\gamma^2}. \qquad (8)$$

For eliminating the higher energy of the system and retaining the lower, the projection operator T_0 is composed by the lowest energy eigenstates of the system. The relation between the original Hamiltonian and the effective Hamiltonian can be given by the projection operator [30], i.e., $H^{eff} = T_0^+ H T_0$, where T_0^+ is the hermitian operator of T_0. In the effective Hamiltonian, we only consider the first order correction in the perturbation theory. The effective Hamiltonian is

$$H^{eff} = H_0^{eff} + H_1^{eff} = T_0^+ H^A T_0 + T_0^+ H^{AA} T_0. \qquad (9)$$

The projection operator T_0 can be searched in a factorized form

$$T_0 = \prod_{i=1}^{N/3} T_0^L, \qquad (10)$$

where T_0^L is the Lth block, which is defined as

$$T_0^L = |\Uparrow\rangle_L \langle\Phi_0| + |\Downarrow\rangle_L \langle\Phi_0|. \qquad (11)$$

The $|\Uparrow\rangle_L$ and $|\Downarrow\rangle_L$ in the Eq. (11) are the renamed states of Lth block which can be seen as a new spin$-1/2$. The renormalization of Pauli matrices are given by

$$T_0^L \sigma^\alpha_{i,L} T_0^L = \eta^\alpha_i \sigma'^\alpha_L \quad (i=1,2,3; \quad \alpha = x,y), \qquad (12)$$

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where

$$\eta_1^x = \eta_3^x = \frac{1+\gamma}{\sqrt{2\left(1+\gamma^2\right)}}, \quad \eta_2^x = -\frac{(1+\gamma)^2}{2\left(1+\gamma^2\right)},$$

$$\eta_1^y = \eta_3^y = \frac{1-\gamma}{\sqrt{2\left(1+\gamma^2\right)}}, \quad \eta_2^y = -\frac{(1-\gamma)^2}{2\left(1+\gamma^2\right)}. \tag{13}$$

Then, the effective Hamiltonian of the renormalized chain can be gotten,

$$H^{eff} = \frac{J'}{4} \sum_L^{N/3} \left[\left(1+\gamma'\right)\sigma_L^x \sigma_{L+1}^x + \left(1-\gamma'\right)\sigma_L^y \sigma_{L+1}^y \right], \tag{14}$$

where

$$J' = J\frac{3\gamma^2+1}{2\left(1+\gamma^2\right)}, \quad \gamma' = \frac{\gamma^3+3\gamma}{3\gamma^2+1}. \tag{15}$$

In the above equations, not only do we get the nontrivial fixed point $\gamma = 0$, but also obtain the trivial fixed point $\gamma = 1$ which specifies the critical point of the system by solving $\gamma \equiv \gamma'$. When $\gamma \to 0$, the model falls into the universality class of XX model corresponding to a spin-fluid phase; while for $\gamma \to 1$, the system is in Ising-like phase. From above analyzing, it is found that there is a phase boundary that separates the spin-fluid phase $\gamma = 0$, from the Ising-like phase $0 < \gamma \leq 1$.

III.   ENTANGLEMENT ANALYSIS

We investigate the ground-state entanglement between two blocks of the XY chain by the concept of concurrence [31, 32] and demonstrate how the concurrence varies as the size of the block becomes large. We consider one of the degenerate ground states to define the pure state density matrix. Thus, the pure state density matrix is defined by

$$\rho = |\Phi_0\rangle \langle \Phi_0| , \tag{16}$$

where $|\Phi_0\rangle$ has been introduced in Eq. (6). The results is same if we consider the Eq. (7) to construct the density matrix. Because the concurrence is one measure of pairwise entanglement, we must trace over the degrees of freedom of one site in the block. Without loss of generality, we trace over the site 2. The reduced density matrix for the sites 1 and 3

can be obtained as,

$$\rho_{13} = \frac{1}{4(\gamma^2+1)} \begin{pmatrix} 2 & 0 & 0 & 2\gamma \\ 0 & \gamma^2+1 & \gamma^2+1 & 0 \\ 0 & \gamma^2+1 & \gamma^2+1 & 0 \\ 2\gamma & 0 & 0 & 2\gamma^2 \end{pmatrix}. \tag{17}$$

C_{13} denotes the concurrence of the sites 1 and 3 which is defined as

$$C_{13} = \mathrm{Max}\left\{ \lambda_4^{1/2} - \lambda_3^{1/2} - \lambda_2^{1/2} - \lambda_1^{1/2}, \; 0 \right\}, \tag{18}$$

where the $\lambda_k \, (k=1,2,3,4)$ are the eigenvalues of $\widehat{R} = \rho_{13}\widetilde{\rho_{13}}$ [where $\widetilde{\rho_{13}} = (\sigma_1^y \otimes \sigma_3^y)\rho_{13}^* (\sigma_1^y \otimes \sigma_3^y)$] in ascending order. The eigenvalues of $\widehat{R}$ can be exactly solved which are

$$\lambda_1 = \lambda_2 = 0, \; \lambda_3 = \frac{\gamma^2}{(1+\gamma^2)^2}, \; \lambda_4 = \frac{1}{4}. \tag{19}$$

We substitute the above equations into Eq. (18), then C_{13} can be gotten which is the function of γ. The renormalization of γ defines the evolution of concurrence on the increasing of the size of the system. For better to discuss the concurrence, we set C_{13} as a function of g, where

$$g = (1+\gamma)/(1-\gamma). \tag{20}$$

C_{13} versus different QRG iterations is plotted in Fig. 2.

For different QRG steps, the plots of C_{13} versus different QRG steps cross each other at the critical point. In the thermodynamic limit, the concurrence develops two saturated values which is nonzero for $g_c = 1$ and zero for $0 \le g < 1$ and $g > 1$. At the point $g_c = 1$, the system exists quantum correlation because C_{13} is a nonzero constant. The infinite chain can be effectively described by a three-site model with the renormalized coupling constants. In this case, the quantum fluctuations play an important role and destroy any long-range order of the system. Nonzero C_{13} verifies that the system is entangled where the ground state is characterized by a gapless excitation and algebraic decay of spin correlations. While for $0 \le g < 1$ and $g > 1$, the model is gap with magnetic long-range order, the nontrivial points correspond to two Ising phases ordered in y direction ($g \to 0$) and x direction ($g \to \infty$), respectively. It can be seen that the Néel phase is the dominant phase of the system.

IV. NON-ANALYTIC AND SCALING BEHAVIOR

The system can occur QPT because of the non-analyticity property of the block-block entanglement. The non-analyticity behavior can be also accompanied by a scaling behavior because of the diverging of the correlation length. In this section, we show the behaviors of the concurrence and the QPT in the XY model. By using of QRG method, a large system $(N = 3^{n+1})$ can be effectively described by three-site block with the renormalized coupling constants after the nth iteration of RG. Thus, the entanglement between two renormalized sites represents the entanglement between two blocks of the system each containing $N/3$ sites.

The first derivative of concurrence is analyzed and shows a singular behavior when the size of the system is infinity. All these dates have been shown in Fig. 3. It is easy to see that the first derivative of the concurrence is discontinuous at $g_c = 1$, while C_{13} is continuous. This indicates that the QPT of the system is the second-order QPT [33]. The scaling behavior of the maximum of $y = dC_{13}/dg$ versus N is plotted in Fig. 4 which shows a linear behavior of $\ln(y_{\max})$ versus $\ln(N)$. In order to show the intuitionistic scaling behavior, the numerical results are given. The position of maximum (g_m) of y touches the critical point as the size of the system increases. This is plotted in the Fig. 5. It is found that there are the relation of $g_m = g_c - N^{-\theta}$ with $\theta = 0.98$. The exponent θ which is called entanglement exponent is directly related to the correlation length exponent closing to the critical point. The singular behavior of the concurrence and the scaling behavior of the system depend on the entanglement exponent, which is the reciprocal of correlation length exponent, i.e., $\theta = 1/\nu$. The emerging singularity connects to the universality class of the model. As the critical point is approached in the limit of large size, the correlation length covers the whole system. That is to say, the RG implementation of entanglement truly captures the critical behavior of the XY model in the vicinity of the critical point.

V. CONCLUSIONS

In this paper, we discuss the entanglement and QPT in the anisotropic $s = 1/2$ XY model by the quantum renormalization group method. The concurrence as one measure of the quantum correlation is investigated. The critical behavior of the system is obtained by

the renormalization of the lattice. As the number of RG iterations reaches the infinity, the system occurs QPT between the spin-fluid and the Ising-like phases which correspond to two different fixed values of the concurrence at the critical point and both sides of it respectively. The diverging behavior of the first derivative of the concurrence is accompanied by the scaling behavior in the vicinity of the critical point. Further insight, the scaling behavior is investigated and characterizes how the critical point of the model is touched as the size of system increases. In the thermodynamic limit, the non-analytic behavior of entanglement is correlated with the diverging of the correlation length at the critical point.

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Figure captions:

Fig. 1. The kadanoff's block renormalization group approach of a chain where we consider three sites as a block.

Fig. 2. Representation of the evolution of the concurrence versus g in terms of QRG iterations.

Fig. 3. First derivative of concurrence and its manifestation toward diverging as the number of QRG iterations increases (Fig. 2).

Fig. 4. The logarithm of the absolute value of maximum, $\ln(|dC_{13}/dg|_{g_{\max}}|)$, versus the logarithm of chain size, $\ln(N)$, which is linear and shows a scaling behavior. Each point corresponds to the maximum value of a single plot of Fig. 3.

Fig. 5. The scaling behavior of $g_{\max}$ in terms of system size (N) where $g_{\max}$ is the position of maximum in Fig. 3.
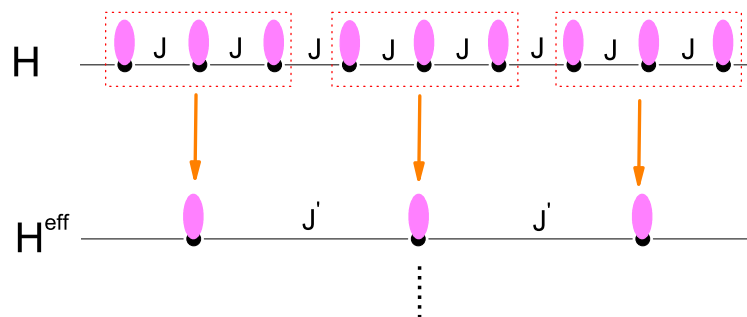
Figure 1   AD10700   07Apr2011
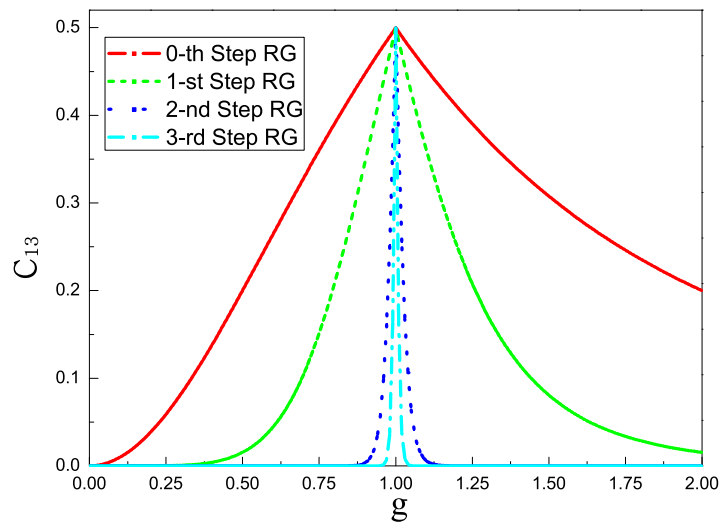
Figure 2   AD10700   07Apr2011

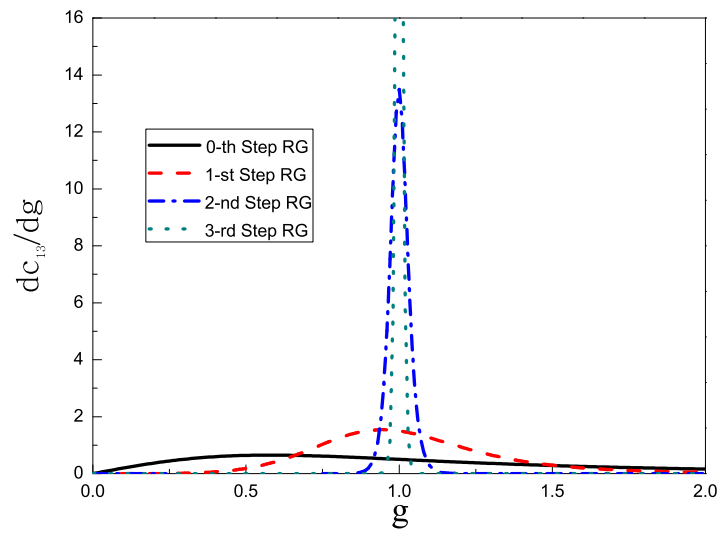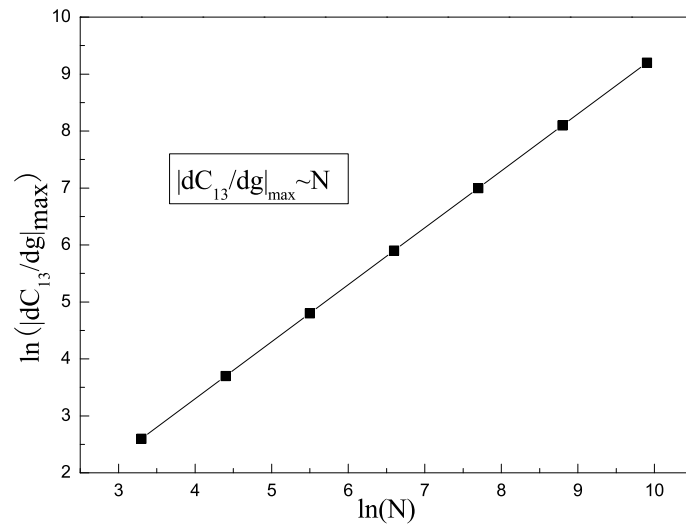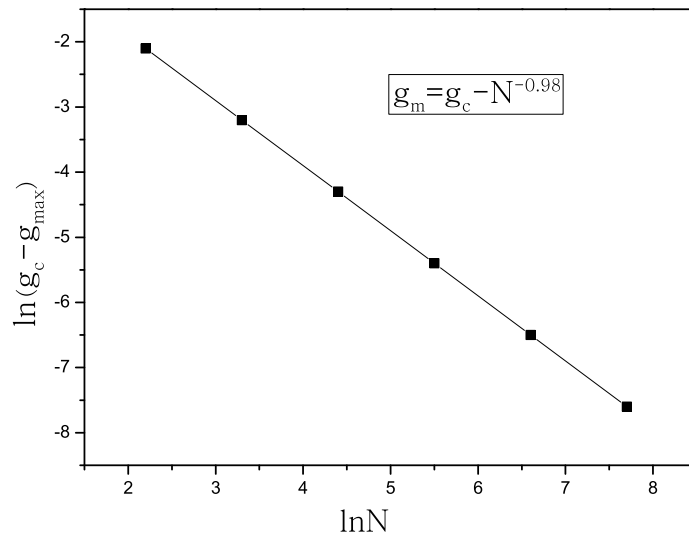Figure 3   AD10700   07Apr2011

Figure 4   AD10700   07Apr2011

Figure 5   AD10700   07Apr2011