Ultracold fermions in real or fictitious magnetic fields: BCS-BEC evolution and type-I–type-II transition

M. Iskin and C. A. R. Sá de Melo

Phys. Rev. A **83**, 045602 — Published 26 April 2011

DOI: 10.1103/PhysRevA.83.045602

Ultra-cold fermions in real or fictitious magnetic fields: The BCS-BEC evolution and the type-I–type-II transition

M. Iskin[1] and C. A. R. Sá de Melo[2]
[1]Department of Physics, Koç University, Rumelifeneri Yolu, 34450 Sariyer, Istanbul, Turkey.
[2]School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA.
(Dated: March 2, 2011)

We study ultra-cold neutral fermion superfluids in the presence of fictitious magnetic fields, as well as charged fermion superfluids in the presence of real magnetic fields. Charged fermion superfluids undergo a phase transition from type-I to type-II superfluidity, where the magnetic properties of the superfluid change from being a perfect diamagnet without vortices to a partial diamagnet with the emergence of the Abrikosov vortex lattice. The transition from type-I to type-II superfluidity is tuned by changing the scattering parameter (interaction) for fixed density. We also find that neutral fermion superfluids such as ^{6}Li and ^{40}K are extreme type-II superfluids, and that they are more robust to the penetration of a fictitious magnetic field in the BCS-BEC crossover region near unitarity, where the critical fictitious magnetic field reaches a maximum as a function of the scattering parameter (interaction).

PACS numbers: 03.75.Ss, 03.75.Hh, 05.30.Fk

A key experiment in the verification that neutral Fermi superfluids can evolve from the Bardeen-Cooper-Schrieffer (BCS) to the Bose-Einstein condensation (BEC) regime was the observation of quantized vortices throughout the BCS-BEC evolution upon rotation of the atomic cloud [1]. This observation had a very dramatic impact beyond the atomic physics community, because it showed that superfluidity of Cooper pairs and of tightly bound bosonic molecules for s-wave pairing are the manifestation of the same type of physics. The key tool that permitted such realization is the tunability of the interaction between fermions through the use of Feshbach resonances. The same kind of tunability does not exist in ^{3}He, the standard condensed matter neutral superfluid, or in superconductors. The situation is even worse in neutron and proton superfluids, which are thought to exist in the core of neutron stars.

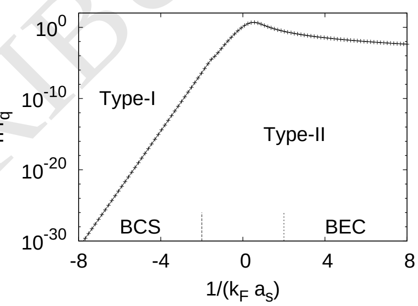

FIG. 1: Universal phase diagram of the dimensionless fermion density nr_q^3 versus scattering parameter $1/(k_F a_s)$, where $r_q = q^2/(mc_0^2)$ is the classical radius of a fermion with mass m and charge q, c_0 is the speed of light, k_F is the Fermi momentum and a_s is the scattering length. The dotted line separates regions of type-I and type-II superfluidity.

Very recently, a new technique has been developed that permitted the production of fictitious magnetic fields which can couple to neutral bosonic atoms [2, 3]. These fictitious magnetic fields are produced through an all optical Raman process, couple to a fictitious charge, but produce real effects like the creation of vortices in the superfluid state of bosons. In principle, the same technique can be applied to ultra-cold fermions, which coupled with the control over interaction using Feshbach resonances allows the exploration of superfluidity not only as a function of interaction, but also as a function of fictitious magnetic field. It is in anticipation of similar experiments involving ultracold fermions that we address in this manuscript the effects of fictitious magnetic fields on fermion superfluids as a function of interaction.

Unlike neutral superfluids, standard condensed matter charged superfluids (superconductors) can be of two types [4]. Many superconductors are now known to be type-II (including heavy fermions, organics, and high-T_c cuprates), where the application of an external magnetic field beyond the lower critical field H_{c_1} leads to a non-uniform superfluid phase, which appears in the form of the Abrikosov vortex lattice, until a second critical field H_{c_2} is reached, when the system becomes normal. Other charged superfluids are known to be type-I and do not allow the magnetic field to penetrate the sample. These systems are perfect diamagnets until the critical field H_c is reached, where the charged superfluid becomes normal. The parameter that characterizes the type of charged superfluid is the Ginzburg-Landau parameter $\kappa = \lambda/\xi$ corresponding to the ratio between the penetration depth λ of the magnetic field into the sample and the coherence length ξ of the charged superfluid, such that type-I superfluids have $\kappa < 1/\sqrt{2}$ and type-II have $\kappa > 1/\sqrt{2}$.

In this manuscript, we study neutral fermion superfluids in the presence of fictitious magnetic fields and charged fermion superfluids in the presence of real magnetic fields as a function of interaction (scattering parameter). We show that throughout the crossover region between BCS and BEC superfluidity both ^{6}Li and ^{40}K are extreme type-II superfluids, and for charged superfluids we find a phase transition from type-I to type-II superfluidity for fermions of density $n = k_F^3/(3\pi^2)$ interacting via a contact potential characterized by the interaction parameter $1/(k_F a_s)$. As shown in Fig. 1, the phase boundary in the density n versus interaction parameter $1/(k_F a_s)$ occurs when the critical value $\kappa_c = 1/\sqrt{2}$ is crossed. In the literature of charged superfluids the transition from type-I to type-II was thought possible when induced by disorder and was described microscopically *only* in the BCS limit [5–8]. In contrast, here we show that, microscopically, a clean (no disorder) charged superfluid can exhibit a type-I–type-II transition induced by interactions. The phase diagram shown in Fig. 1 has a wider applicability to include standard charged superfluids (like superconductors of condensed matter physics) and even proton superfluidity in nuclei or neutral stars, as long as the interactions can be described by a contact potential with corresponding scattering length a_s. In addition, we indicate that neutral (charged) superfluids are more robust to the penetration of ficitious (real) magnetic fields near unitarity, where the critical fictitious (real) magnetic fields reach a maximum as a function of the scattering parameter.

To describe the transition from type-I to type-II superfluidity as a function of the interaction parameter and the properties of neutral (charged) superfluids in the presence of fictitious (real) magnetic fields during the BCS-BEC evolution [9–11] for s-wave superfluids in three dimensions, we start with the Hamiltonian density

$$\bar{\mathcal{H}}(\mathbf{r}) = \sum_\sigma \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m} - \mu \right) \psi_\sigma(\mathbf{r}) + \hat{U}(\mathbf{r}), \quad (1)$$

where $\hat{U}(\mathbf{r}) = \int d\mathbf{r}' V(\mathbf{r},\mathbf{r}') \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}') \psi_\downarrow(\mathbf{r}') \psi_\uparrow(\mathbf{r})$ contains the attractive contact interaction potential $V(\mathbf{r},\mathbf{r}') = -g\delta(\mathbf{r}-\mathbf{r}')$, and $\psi_\sigma^\dagger(\mathbf{r})$ is the creation operator of fermions with mass m and spin σ. Notice that g has dimensions of energy times volume. To make progress, we rewrite the Hamiltonian $\mathcal{H} = \int d\mathbf{r} \bar{\mathcal{H}}(\mathbf{r})$ from real space to momentum space

$$\mathcal{H} = \sum_{\mathbf{k},\sigma} \xi_\mathbf{k} \psi_{\mathbf{k},\sigma}^\dagger \psi_{\mathbf{k},\sigma} - g \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} b_{\mathbf{k},\mathbf{q}}^\dagger b_{\mathbf{k}',\mathbf{q}}, \quad (2)$$

where $b_{\mathbf{k},\mathbf{q}}^\dagger = \psi_{\mathbf{k}+\mathbf{q}/2,\uparrow}^\dagger \psi_{-\mathbf{k}+\mathbf{q}/2,\downarrow}^\dagger$ creates a fermion pair with center of mass momentum $\mathbf{q}$ and relative momentum $2\mathbf{k}$, $\xi_\mathbf{k} = \epsilon_\mathbf{k} - \mu$ is the kinetic energy term with $\epsilon_\mathbf{k} = \hbar^2 k^2/(2m)$ and μ is the chemical potential.

Integration over the fermion fields [12] leads to the order parameter equation

$$\frac{1}{g} = \frac{1}{L^3} \sum_\mathbf{k} \frac{\tanh\left[\xi_\mathbf{k}/(2T_c)\right]}{2\xi_\mathbf{k}} \quad (3)$$

at the critical temperature T_c, where the order parameter vanishes. Here L^3 is the sample volume. The interaction g can be written in terms of the scattering length a_s leading to $1/g = -m/\left(4\pi\hbar^2 a_s\right) + (1/L^3)\sum_\mathbf{k}\left[1/(2\epsilon_\mathbf{k})\right]$. The second self-consistency relation is the number equation

$$N = \sum_{\mathbf{k},\sigma} \mathrm{f}(\xi_\mathbf{k}) + T_c \sum_q \frac{\partial\left[\ln(L^3\mathcal{K}/T_c)\right]}{\partial\mu} \quad (4)$$

where $\mathrm{f}(\xi_\mathbf{k})$ is the Fermi function, and

$$\mathcal{K}^{-1} = \frac{1}{g} - \frac{1}{L^3} \sum_\mathbf{k} \frac{1 - \mathrm{f}(\xi_{\mathbf{k}+\mathbf{q}/2}) - \mathrm{f}(\xi_{-\mathbf{k}+\mathbf{q}/2})}{\xi_{\mathbf{k}+\mathbf{q}/2} + \xi_{-\mathbf{k}+\mathbf{q}/2} - i\hbar\omega} \quad (5)$$

is the pair propagator, and ω is the Matsubara frequency for bosons.

The effective action is $TS_{\rm eff}/\hbar = \sum_q \mathcal{K}^{-1}(q)|\Delta(q)|^2 + \frac{b}{2L^3}\sum_{q_1,q_2,q_3} \Delta(q_1)\Delta^*(q_2)\Delta(q_3)\Delta^*(q_1-q_2+q_3)$ in terms of the order parameter $\Delta(q)$, where $q = (\mathbf{q}, i\omega)$. To study thermodynamic properties, we take $i\hbar\omega = 0$, or equivalently, $\Delta(\mathbf{r},\tau) \equiv \Delta(\mathbf{r})$, leading to the effective Lagrangian density

$$\mathcal{L}_{\rm eff} = a|\Delta|^2 + \sum_{i,j} \frac{\hbar^2 c_{ij}}{2m} \nabla_i \bar{\Delta} \nabla_j \Delta + \frac{b}{2}|\Delta|^4.$$

Using the notation $X_\mathbf{k} = \tanh[\xi_\mathbf{k}/(2T)]$ and $Y_\mathbf{k} = \mathrm{sech}^2[\xi_\mathbf{k}/(2T)]$, the coefficients of the Lagrangian density are $a(\mu,T) = \frac{1}{g} - \frac{1}{L^3}\sum_\mathbf{k} \frac{X_\mathbf{k}}{2\xi_\mathbf{k}}$ for the constant term,

$$L^3 c_{ij}(\mu_c, T_c) = \sum_\mathbf{k} \left[\left(\frac{X_\mathbf{k}}{8\xi_\mathbf{k}^2} - \frac{Y_\mathbf{k}}{16\xi_\mathbf{k} T_c} \right) \delta_{ij} + \frac{X_\mathbf{k} Y_\mathbf{k}}{T_c^2} \frac{\hbar^2 k_i k_j}{16 m \xi_\mathbf{k}} \right]$$

for the coefficient of the gradient terms $\nabla_i \bar{\Delta} \nabla_j \Delta$, and $L^3 b(\mu_c, T_c) = \sum_\mathbf{k} \left(\frac{X_\mathbf{k}}{4\xi_\mathbf{k}^3} - \frac{Y_\mathbf{k}}{8\xi_\mathbf{k}^2 T_c} \right)$ for the coefficient of the non-linear quartic term. Notice that $c_{ij} = c\delta_{ij}$ for s-wave superfluids.

In general, near T_c, $a(\mu,T) = -a_0\epsilon(T)$, where $a_0 = T_c\left[\partial a/\partial T\right]_{T_c}$ and $\epsilon(T) = (1 - T/T_c)$. In the BCS limit of $1/(k_F a_s) \to -\infty$, $L^3 a_0 = \mathcal{D}_F$, where $\mathcal{D}_F = m k_F L^3/(2\pi^2\hbar^2)$ is the density of single particle states per spin channel at the Fermi energy ϵ_F. Also the coefficient of the quartic term is $L^3 b = \left[7\zeta(3)/(8\pi^2 T_c^2)\right]\mathcal{D}_F$, while the coefficient of the gradient term is $L^3 c = \left[7\zeta(3)/(12\pi^2 T_c^2)\right]\mathcal{D}_F \epsilon_F$. Here, the zeta function $\zeta(3) = 1.202$, while the critical temperature $T_c = \left(8e^{\gamma-2}/\pi\right)\epsilon_F \exp\left[-\pi/(2k_F|a_s|)\right]$, with $e^\gamma \approx 1.781$, and the chemical potential $\mu = \epsilon_F$. However, in the BEC limit of $1/(k_F a_s) \to +\infty$, $L^3 a_0 = \mathcal{D}_F \epsilon_F/(4|\mu|)$. Correspondingly the coefficient of the quartic term is $L^3 b = (\pi/32)\,\mathcal{D}_F/(|\mu|\sqrt{\epsilon_F|\mu|})$, and the coefficient of the gradient term is $L^3 c = (\pi/16)\,\mathcal{D}_F/\sqrt{\epsilon_F|\mu|}$. In this case, $T_c \approx 0.218\epsilon_F$ and $\mu = E_b/2$, where $E_b = -\hbar^2/(m a_s^2)$ is the two-particle binding energy in vacuum.

Next, we scale the order parameter to $\psi(\mathbf{r}) = \sqrt{c}\Delta(\mathbf{r})$ and introduce an external (real or fictitious) magnetic field via the vector potential $\mathbf{A}(\mathbf{r})$, using the substitution $\nabla_i \to \nabla_i - 2iqA_i/(\hbar c_0)$, where q is the real or fictitious particle charge and c_0 is the speed of light. The difference in free energy density between the charged superfluid and its normal state in the presence of magnetic fields takes the Ginzburg-Landau form

$$\mathcal{F}_{GL} = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m}\left|\left(-i\nabla - \frac{2q}{\hbar c_0}\mathbf{A}\right)\psi\right|^2 + \frac{|\mathbf{H}|^2}{8\pi}$$

where $\mathbf{H} = \nabla \times \mathbf{A}$ is the real or fictitious microscopic magnetic field. The parameter $\alpha = a/c$ changes sign at $T = T_c$, however $\beta = b/c^2$ is always positive guaranteeing the stability of the theory. It is also useful to define the flux quantum $\Phi_q = \pi\hbar c_0/q$, which will be used below.

Minimization of $\mathcal{F}_{GL}$ with respect to ψ and $\mathbf{A}$ lead the order parameter equation

$$\alpha\psi + \beta|\psi|^2\psi + \frac{\hbar^2}{2m}\left(-i\nabla - \frac{2q}{\hbar c_0}\mathbf{A}\right)^2\psi = 0 \qquad (6)$$

and to the current density

$$\mathbf{j} = -\frac{\hbar q}{im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{4q^2}{mc_0}|\psi|^2\mathbf{A}. \qquad (7)$$

Using the relation $\nabla\times\mathbf{H} = 4\pi\mathbf{j}/c_0$ and taking the curl of the current density leads to the London equation $\lambda^2\nabla\times(\nabla\times\mathbf{H}) + \mathbf{H} = 0$ where $\lambda = \sqrt{mc_0^2/(16\pi q^2|\psi|^2)}$ is the magnetic penetration depth. Since $|\psi|^2 = |\alpha|/\beta = |a|c/b$ in weak magnetic fields, the penetration depth becomes $\lambda(T) = \lambda_{GL}|\epsilon(T)|^{-1/2}$, where $\lambda_{GL} = \sqrt{b/(16\pi r_q a_0 c)}$. Here, $r_q = q^2/(mc_0^2)$ is the classical radius of a fermion with mass m and charge q in CGS units. In terms of the classical radius of the electron $r_e = e^2/m_e c_0^2 \approx 2.82\times10^{-13}$ cm, we write $r_q = (q/e)^2(m_e/m)r_e$. Since $|\psi|^2$ plays the role of the superfluid density n_s, we may write $|\psi|^2 = |\psi_0|^2|\epsilon(T)| = n_s = n_{s,0}|\epsilon(T)|$, where $n_{s,0} = |\psi_0|^2 = a_0c/b$ is the temperature independent prefactor. This observation allows us to write $k_F\lambda_{GL} = \sqrt{[3\pi/(16k_Fr_q)]\,(n/n_{s,0})}$. The prefactor $n_{s,0}$ reflects a zero temperature extrapolation of the superfluid density n_s, however, in a Galilean invariant system we must have $n_{s,0} \approx n/2$. Indeed, in the BCS limit $n_{s,0} = n/2$ such that $k_F\lambda_{GL} = \sqrt{3\pi/(8k_Fr_q)}$, while in the BEC limit $n_{s,0} = 3n/8$ leads to $k_F\lambda_{GL} = \sqrt{\pi/(2k_Fr_q)}$. The BCS value of $k_F\lambda_{GL}$ is slightly smaller than its BEC value, however throughout the BCS-BEC evolution, $k_F\lambda_{GL}$ does not change substantially.

The coherence length can be extracted from Eq. (6) as $\xi(T) = \hbar/\sqrt{2m|\alpha(T)|}$ leading to $\xi(T) = \xi_{GL}|\epsilon(T)|^{-1/2}$ where $\xi_{GL} = \hbar\sqrt{c/(2ma_0)}$. Unlike the penetration depth, the coherence length ξ_{GL} changes substantially during the BCS-BEC evolution. In the BCS regime, $k_F\xi_{GL} = \sqrt{7\zeta(3)/(12\pi^2)}(\epsilon_F/T_c)$ is very large, and in terms of k_Fa_s becomes $k_F\xi_{GL} = A\exp[\pi/(2k_F|a_s|)]$, where $A = \sqrt{7\zeta(3)/(12\pi^2)}(\pi e^{2-\gamma}/8)$. In the BEC regime, $k_F\xi_{GL} = \sqrt{\pi/4}\,(|\mu|/\epsilon_F)^{1/4}$ is also very large, and in terms of k_Fa_s becomes $k_F\xi_{GL} = \sqrt{\pi/4}/\sqrt{k_Fa_s}$. However, $k_F\xi_{GL}$ passes through a minimun in the intermediate regime, where $k_F\xi_{GL} \approx \mathcal{O}(1)$.

As discovered by Abrikosov [4], the parameter $\kappa = \lambda(T)/\xi(T)$ is of fundamental importance in the characterization of the magnetic properties of charged superfluids. When $\kappa < \kappa_c = 1/\sqrt{2}$ the charged superfluid is a perfect diamagnet (type-I), which does not allow the magnetic field to penetrate. When $\kappa > \kappa_c$, the charged superfluid allows the penetration of magnetic field in the superfluid state in the form of vortices (type-II). Since the temperature dependence of $\lambda(T)$ and $\xi(T)$ is exactly the same, the parameter

$$\kappa = \frac{\lambda(T)}{\xi(T)} = \frac{\lambda_{GL}}{\xi_{GL}} = \sqrt{\frac{mb}{8\pi r_q\hbar^2c^2}} \qquad (8)$$

is independent of temperature. Notice that $\kappa\sqrt{k_Fr_q} = \sqrt{s/(16\pi)}$, where $s = k_F^3b/(\epsilon_Fc^2)$ is a dimensionless parameter which is a function of $1/(k_Fa_s)$ only. In Fig. 2, we show the evolution of κ as a function of the scattering parameter $1/(k_Fa_s)$. In the BCS limit, $\kappa\sqrt{k_Fr_q} = \sqrt{9\pi^3/[14\zeta(3)]}(T_c/\epsilon_F)$ corresponds to $\kappa\sqrt{k_Fr_q} = B\exp[-\pi/(2k_F|a_s|)]$, where $B = \sqrt{9\pi^3/[14\zeta(3)]}(8e^{\gamma-2}/\pi)$. While in the BEC limit, $\kappa\sqrt{k_Fr_q} = \sqrt{2}\,(\epsilon_F/|\mu|)^{1/4}$ corresponds to $\kappa\sqrt{k_Fr_q} = \sqrt{2}\sqrt{k_Fa_s}$. Notice the maximum of κ in the vicinity of unitarity and $\mu = 0$ ($1/[k_Fa_s] = 0.554$).

Unfortunately, in current experiments for neutral atoms in fictitious fields, only the product qH is controlled, instead of H alone [2]. So it is useful to think of ultra-cold superfluids like ^{6}Li or ^{40}K as having charge $q \to 0$, but with qH finite. In this sense, these neutral superfluids are extreme type-II with $\kappa \to \infty$ throughout the BCS-BEC evolution [13].

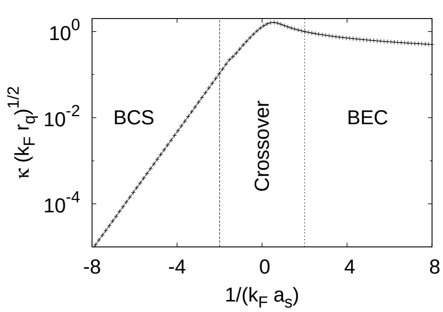

FIG. 2: Universal plot of the Ginzburg-Landau parameter κ versus scattering parameter $1/(k_Fa_s)$, where r_q is the classical radius of a fermion with mass m and charge q.

To obtain the phase diagram shown in Fig. 1, we set

$\kappa = \kappa_c$ in Eq. (8) and extract the fermion density n as a function of $1/(k_F a_s)$, which leads to $n r_q^3 = s^3/(1536\pi^5)$. In Fig. 1, κ is higher (lower) than κ_c below (above) the critical line indicating a type-II (type-I) charged superfluid phase. For fixed density n, a phase transition from type-I to type-II charged superfluid occurs, as the interaction parameter $1/(k_F a_s)$ increases. Electron superfluids with 10^{21} cm^{-3} $\leq n \leq 10^{23}$ cm^{-3} have $2.24 \times 10^{-27} < n r_q^3 < 2.24 \times 10^{-25}$, and the transition between type-I and type-II occurs in the interval $-8 < 1/(k_F a_s) < -4$, while proton superfluids in nuclear matter, with $r_q^3 \approx 3.60 \times 10^{-48}$ cm^{-3} and 10^{37} cm$^{-3} \leq n \leq 10^{38}$ cm^{-3}, have a type-I–type-II transition in the range $-4 < 1/(k_F a_s) < 0$.

For type-I charged superfluids there is only the thermodynamic critical field $H_c(T)$ determined by the condition $H_c^2(T)/(8\pi) = F_n - F_s$, where F_n (F_s) is the Helmholtz free energy for the normal (superfluid) state. For a uniform superfluid state the energy difference is $F_n - F_s = \alpha^2/(2\beta) = a^2/(2b)$ leading to $H_c(T) = H_{c,0}|\epsilon(T)| = |a(T)|\sqrt{4\pi/b}$, where $H_{c,0} = a_0\sqrt{4\pi/b}$ is the temperature independent prefactor, which gives an estimate of the zero temperature critical field. We define the dimensionless thermodynamic critical field $\widetilde{H}_{c,0} = H_{c,0}/H_{k_F}$, where $H_{k_F} = \Phi_q k_F^2$. Notice that $\widetilde{H}_{c,0} = \hbar\omega_{c,0}/(2\pi\epsilon_F)$, where $\omega_{c,0} = |q|H_{c,0}/(mc_0)$ is the cyclotron frequency at $H_{c,0}$. Using the asymptotic expressions for a_0 and b, we obtain $\widetilde{H}_{c,0} = \sqrt{k_F r_q}\sqrt{4/[7\pi\zeta(3)]}(T_c/\epsilon_F)$ in the BCS regime, which can be rewritten as $\widetilde{H}_{c,0} = \sqrt{k_F r_q} C \exp\left[-\pi/(2k_F|a_s|)\right]$, with $C = \sqrt{4/[7\pi\zeta(3)]}(8e^{\gamma-2}/\pi)$. While we obtain $\widetilde{H}_{c,0} = \sqrt{k_F r_q}(1/\pi^2)(\epsilon_F/|\mu|)^{1/4}$ in the BEC regime, which can be rewritten as $\widetilde{H}_{c,0} = \sqrt{k_F r_q}(1/\pi^2)\sqrt{k_F a_s}$. The field $\widetilde{H}_{c,0}$ reaches a maximum near unitary and $\mu = 0$, thus indicating that type-I superfluids are most robust to the penetration of magnetic fields in that same region.

For type-II superfluids there are two critical fields. The first is called $H_{c_1}(T)$ and separates the perfect-diamagnet Meissner phase from the non-uniform phase exhibiting vortices. The second is called $H_{c_2}(T)$ and separates the non-uniform phase exhibiting vortices from the normal state. Since ^{6}Li and ^{40}K are extreme type-II superfluids with $\kappa \to \infty$, then $H_{c_1}(T) \to 0$, and thus we concentrate on the results for $H_{c_2}(T)$. The calculation of $H_{c_2}(T)$ is performed by linearizing Eq. (6) $-\hbar^2\left(\nabla - i2\pi\mathbf{A}/\Phi_q\right)^2\psi + 2m\alpha(T)\psi = 0$. Using the Landau gauge $\mathbf{A} = Hx\hat{\mathbf{y}}$, the momentum components k_y and k_z are good quantum numbers and the solution for ψ becomes $\psi_{n,k_y,k_z}(x,y,z) = e^{(ik_y y + ik_z z)}u_n(x)$, which substituted in the previous equation leads to the one-dimensional *Schrödinger* equation $\left[-\hbar^2/(2m)d^2/dx^2 + m\omega_s^2(x-x_0)^2/2\right]u_n(x) = \epsilon_n u_n(x)$, where $x_0 = \Phi_q k_y/(2\pi H)$ is the equilibrium position of the harmonic potential, $\omega_s = 2|q|H/(mc_0)$ is the harmonic potential frequency, and $\epsilon_n = |\alpha(T)| - \hbar^2 k_z^2/(2m) = \hbar\omega_s(n + 1/2)$ is the eigenvalue. The highest magnetic field at which superconductivity nucleates occurs for $n = 0$ and $k_z = 0$ leading to the condition $|\alpha(T)| = \hbar\omega_s/2$. Isolating the magnetic field from the harmonic potential frequency leads to $H_{c_2}(T) = [\Phi_q/(2\pi)]2m|\alpha(T)|/\hbar^2$, which can be finally expressed in terms of the coherence length $\xi(T)$ as $H_{c_2}(T) = \Phi_q/\left[2\pi\xi^2(T)\right]$. Substituting $\xi(T) = \xi_{GL}|\epsilon(T)|^{-1/2}$, we write $H_{c_2}(T) = H_{c_2,0}|\epsilon(T)|$, where $H_{c_2,0} = \Phi_q/(2\pi\xi_{GL}^2)$. Using again the reference field $H_{k_F} = \Phi_q k_F^2$, we obtain the dimensionless upper critical field $\widetilde{H}_{c_2,0} = H_{c_2,0}/H_{k_F} = 1/(2\pi k_F^2\xi_{GL}^2)$. This expression is equivalent to the ratio $\hbar\omega_{c_2,0}/(2\pi\epsilon_F)$, where $\omega_{c_2,0} = qH_{c_2,0}/(mc_0)$ is the cyclotron frequency at $H_{c_2,0}$. In the BCS limit, $\widetilde{H}_{c_2,0} = [6\pi/7\zeta(3)](T_c/\epsilon_F)^2$, which in terms of the scattering parameter $1/(k_F a_s)$ becomes $\widetilde{H}_{c_2,0} = D\exp[-\pi/(k_F|a_s|)]$ with $D = 256e^{2\gamma-4}/[7\pi\zeta(3)]$. In the BEC limit, $\widetilde{H}_{c_2,0} = (2/\pi^2)\sqrt{\epsilon_F/|\mu|}$ which in terms of the scattering parameter $1/(k_F a_s)$ becomes $\widetilde{H}_{c_2,0} = (2/\pi^2)k_F a_s$. Since $k_F\xi_{GL}$ reaches a minimum in the region near unitarity and $\mu = 0$, it is clear that $\widetilde{H}_{c_2,0}$ has a maximum there, where type-II superfluids are most robust to the presence of real or fictitious magnetic fields.

Before concluding, we note that the quantum regime $\hbar\omega_c \geq 2\pi T$ (where Landau level quantization is important) can be reached experimentally for ^{6}Li and ^{40}K while preserving superfluidity, since they are extreme type-II superfluids in the BCS-BEC crossover region.

In conclusion, we have analyzed the effects of real or fictitious magnetic fields during the BCS-BEC evolution of s-wave superfluids with direct application to ultracold fermionic atoms. We have shown that a transition from type-I to type-II charged superfluidity occurs as the Ginzburg-Landau paramater crosses its critical value $\kappa_c = 1/\sqrt{2}$ in the density versus interaction phase diagram of fermions of charge q and mass m. We have shown that ^{6}Li and ^{40}K in fictitious magnetic fields are extreme type-II superfluids. Finally, we have indicated that the critical magnetic fields (real or fictitious) depend strongly on the scattering parameter $1/(k_F a_s)$ and reach a maximum in a region near unitarity, where superfluidity is more robust to their penetration.

CSdM and MI would like to thank NSF (DMR-0709584) and ARO (W911NF-09-1-0220), and Marie Curie IRG (FP7-PEOPLE-IRG-2010-268239) and TÜBİTAK, respectively, for support.

[1] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, Nature **435**, 1047 (2005).
[2] Y. J. Lin, R. L. Compton, K. Jimenez-Garcia, J. V. Porto, and I. B. Spielman, Nature **462**, 628 (2009).

[3] I. B. Spielman, Phys. Rev. A **79**, 063613 (2009).
[4] A. A. Abrikosov, Sov. Phys. JETP **5**, 1174 (1957).
[5] A. A. Abrikosov, and L. P. Gorkov, Sov. Phys. JETP **8**, 1090 (1958).
[6] L. P. Gorkov, Sov. Phys. JETP **9**, 1364 (1959); Sov. Phys. JETP **10**, 998 (1960).
[7] E. Helfand, and N. R. Werthamer, Phys. Rev. **147**, 288 (1966).
[8] A. A. Abrikosov, in "Fundamentals of the theory of metals", p. 368, North Holland, (1988).
[9] D. M. Eagles, Phys. Rev. **186**, 456 (1969).
[10] A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, edited by A. Peralski and R. Przystawa (Springer-Verlag, Berlin 1980).
[11] P. Nozieres and S. Schmitt-Rink, J. Low. Temp. Phys. **59**, 195 (1985).
[12] C. A. R. Sá de Melo, M. Randeria, J. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
[13] It may be possible that placing ultra-cold neutral fermions in a cavity with Raman beams could help create the screening necessary for the type-I–type-II transition.