Quantum interference in timed Dicke basis and its effect on bipartite entanglement

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Quantum interference in timed Dicke basis and its effect on bipartite entanglement

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We analyze the effect of position dependent excitation phase on the properties of entanglement between two qubits formed in atomic systems. We show that the excitation phase induces a vacuum mediated quantum interference in the system that affects the dynamical behavior of entanglement between the qubits. It is also found that the quantum interference leads to a coherent population transfer between the symmetric and antisymmetric states which can considerably modify the dynamics of two-qubit entanglement and can even prevent finite time disentanglement (sudden death) under certain conditions.

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I. INTRODUCTION

Quantum interference (QI), an intriguing consequences of the superposition principle has led to numerous fascinating phenomena such as coherent population trapping [1], lasing without inversion [2], electromagnetically induced transparency [3], and quantum entanglement [4–6]. The application of QI in generation of bipartite entanglement both in discrete [7, 8] and continuous-variable [9, 10] settings has been the focus of current investigation. Note that bipartite entanglement involving two atoms, extensively used for implementations of various quantum information protocols [11–16], is known to be quite fragile in the face of decoherence [11, 17]. In view of this, in the past few years considerable effort has been devoted to the study of dynamical aspect of two atom entanglement in presence of decoherence [18–26]. In one such study [18] it was found that in contrary to the adverse effect of spontaneous emission on atomic entanglement [24], cooperative spontaneous emission in two atom systems can generate entanglement among the atoms. It is worth mentioning here that the problem of cooperative spontaneous emission first addressed by Dicke [27] is known to exhibit several counter-intuitive phenomena in two atom systems [28] namely, directed spontaneous emission [29], Lamb shift [30], single photon Dicke superradiance [31] and others. In recent times, with the discovery of atom like behavior of semiconductor quantum dots [33–35] and their utilization towards solid state quantum computing [33–36], we have a new class of systems where the phenomenon of cooperative spontaneous emission can be of immense importance from the context of quantum information sciences.

Recently, Ooi *et al*. [37] studied the effect of position dependant excitation phase on the population dynamics, intensity and spatial and angular correlations for two two-level atoms interacting via their dipoles. The results show that the excitation phase considerably modifies the dynamics of the system. Later, Das *et al*. [38] investigated the effect of the position dependent excitation phase on the Dicke cooperative emission spectrum. A strong quantum correlation among the atoms was reported in presence of the excitation phase. This was attributed to a vacuum mediated QI generated in the two atom system in presence of the position dependent excitation phase. The result of [38] qualitatively indicates that the spatial variation of the excitation phase can affect the generation and evolution of entanglement in the system. It may be added that, such vacuum mediated QI and its effect has been earlier studied in atomic systems [28, 39–41]. Further, a recent work has predicted how one can use such QI to protect bipartite entanglement [42]. While these earlier works utilize the quantum interference that comes about due to the configuration of the atomic system, we are motivated at studying the effect of quantum interference induced by the position dependent excitation phase.

To understand the effects of such QI on the two atom entanglement, we in this paper perform a systematic study of the time evolution of entanglement measure for two strongly dipole coupled atoms undergoing a cooperative spontaneous emission. We consider various initial quantum states in which the two atoms can be prepared and explore the effects on the dynamical behavior of entanglement that results as a consequence of the quantum interference. We explicitly take into account the position dependent excitation of the atoms by introducing *timed Dicke basis* [29]. It is important to understand that the entanglement in a two atom system crucially depends on the cooperative decay rates, the initial conditions, and the dipole-dipole interactions [21], all of which gets modified due to the quantum interference. It is worth mentioning here that QI arising from position dependent phase in such timed Dicke basis was explored in a recent study in context to population dynamics and photon correlation studies in two atom systems [37]. We however in this current work are interested in investigating the effect of such QI on the entanglement of two atoms. For instance, for the system initially prepared in the symmetric timed Dicke state, a coherence between the symmetric and antisymmetric states is dynamically generated as a result of the QI between the two pathways to the ground state. This coherence leads to considerably slow decay of entanglement.

The organization of the paper is as follows. In Sec. II we discuss our model and write down the dynamical equation for our system using a master equation approach. Then in Sec. III we discuss the entanglement measure and derive generalized

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analytical expressions in the timed Dicke basis for the two atom system. In Sec. IV, we then consider two initial conditions: for the atoms prepared in pure states and mixed state and show explicitly that the vacuum mediated QI induced by excitation phase can lead to considerable modification of two atom entanglement behavior. We provide analytical and numerical results in support of our propositions. Finally, we summarize our results in Sec. V.

II. MODEL AND EQUATIONS OF EVOLUTION

We consider a system of two qubits formed by the excited states $|e_i\rangle$ and ground states $|g_i\rangle$ $(i = 1, 2)$ of two identical two level atoms. The qubits are fixed at positions r_1 and r_2 and the inter-atomic distance is less than the wavelength of the radiation field, λ. We further assume that the qubits are coupled to one another by a dipole-dipole interaction and are coupled to the environment via an interaction with a common vacuum reservoir. The time evolution of the density operator for such a two-qubit system can be treated in a master equation framework and is given by [28]

$$\frac{d}{dt}\rho = -i\omega_0 \sum_{i=1}^{2}[\sigma_i^z, \rho] - i\sum_{i\neq j}^{2}\Omega_{ij}[\sigma_i^\dagger \sigma_j, \rho] - \sum_{i,j=1}^{2}\gamma_{ij}(\rho\sigma_i^\dagger\sigma_j + \sigma_i^\dagger\sigma_j\rho - 2\sigma_j\rho\sigma_i^\dagger), \quad (1)$$

where ω_0 is the atomic transition frequency, $\sigma_i^z = (\sigma_i^\dagger\sigma_i - \sigma_i\sigma_i^\dagger)/2$ is the energy operator with $\sigma_i^\dagger(\sigma_i)$ being the raising (lowering) operator for ith atom, Ω_{ij} and γ_{ij} for $i \neq j$ are respectively, the dipole-dipole interaction term and the cooperative decay rate given by

$$\Omega_{ij} = \frac{3}{2}\gamma\Bigg[(1-3\cos^2\theta)\left(\frac{\sin(k_0r_{ij})}{(k_0r_{ij})^2} + \frac{\cos(k_0r_{ij})}{(k_0r_{ij})^3}\right) - (1-\cos^2\theta)\frac{\sin(k_0r_{ij})}{k_0r_{ij}}\Bigg] \quad (2)$$

and

$$\gamma_{ij} = \frac{3}{2}\gamma\Bigg[(1-\cos^2\theta)\frac{\sin(k_0r_{ij})}{k_0r_{ij}} + (1-3\cos^2\theta)\left(\frac{\cos(k_0r_{ij})}{(k_0r_{ij})^2} - \frac{\sin(k_0r_{ij})}{(k_0r_{ij})^3}\right)\Bigg], \quad (3)$$

where $2\gamma \equiv 2\gamma_{11} = 2\gamma_{22} = 2|\vec{\wp}_{eg}|^2\omega^3/3\pi\varepsilon_0\hbar c^3$ is the spontaneous decay rate of the individual qubits. $\vec{\wp}_{eg}$ is the dipole moment, $k_0 = 2\pi/\lambda$ with λ being the wavelength of the emitted radiation and θ is the angle between the direction of the dipole moment and the line joining the ith and the jth qubits, and $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the interqubit distance. In this paper we assume that the orientation of the dipole moment is random and hence Eqs. (2) and (3) simplifies considerably and take the form

$$\Omega_{ij} = -\gamma\cos(k_0r_{ij})/k_0r_{ij}, \quad (4)$$

$$\gamma_{ij} = \gamma\sin(k_0r_{ij})/k_0r_{ij}. \quad (5)$$

We next consider the preparation of initial state of the qubits. For this purpose we assume that the qubits interacts with a very weak laser field (almost at a single photon level) propagating with a wave-vector $\mathbf{k}_0$. The interaction with the weak field can lead to a resonant single photon absorption process. It is important to note that we consider the direction of the wave-vector to be different to that of the inter-qubit axis. This thus generate a position dependent excitation phase of the qubits when ever a photon is absorbed. The excitation process, with the laser field treated classically and in the rotating wave approximation, can be described by the Hamiltonian

$$V = -\hbar\Omega\sum_{j=1}^{2}(\sigma_j^+ e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\, e^{-i(\nu_0-\omega_0)t} + \text{adj.}), \quad (6)$$

where $\Omega = \mathbf{d}_{e_1g_1}\cdot\mathcal{E}/\hbar = \mathbf{d}_{e_2g_2}\cdot\mathcal{E}/\hbar$ is the Rabi frequency and ν_0 is the angular frequency of the incident radiation. Note that the position dependent phase factors in the Hamiltonian would substantially affect the dynamical behavior of the correlation in the two qubit system. We except that this in turn will lead to modification of entanglement among the qubits. The investigation of any such modification in the entanglement feature is the key focus of this paper. In order to investigate the effect of position dependent excitation phase on the dynamics it proves to be convenient to work in a basis defined by the phase factors. Such a basis was introduced in Ref. [29] in context to directed spontaneous emission from an ensemble of atoms and is also known as the *timed Dicke basis*. To this end, for our system of two qubits there are four timed Dicke states:

$$|e\rangle = |e_1e_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_1 + i\mathbf{k}_0\cdot\mathbf{r}_2}, \quad (7)$$

$$|s\rangle = \frac{1}{\sqrt{2}}(|e_1g_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_1} + |g_1e_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_2}), \quad (8)$$

$$|a\rangle = \frac{1}{\sqrt{2}}(|e_1g_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_1} - |g_1e_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_2}), \quad (9)$$

$$|g\rangle = |g_1g_2\rangle. \quad (10)$$

In terms of this basis the equations of evolution for the elements of the density operator are:

$$\dot{\rho}_{ee} = -4\gamma\rho_{ee}, \quad (11a)$$

$$\dot{\rho}_{es} = -[3\gamma + \gamma_{12}\cos\varphi + i(\omega_0 - \Omega_{12}\cos\varphi)]\rho_{es} + i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{ea}, \quad (11b)$$

$$\dot{\rho}_{ea} = -[3\gamma - \gamma_{12}\cos\varphi + i(\omega_0 + \Omega_{12}\cos\varphi)]\rho_{ea} + i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{es}, \quad (11c)$$

$$\dot{\rho}_{eg} = -2(\gamma + i\omega_0)\rho_{eg}, \quad (11d)$$

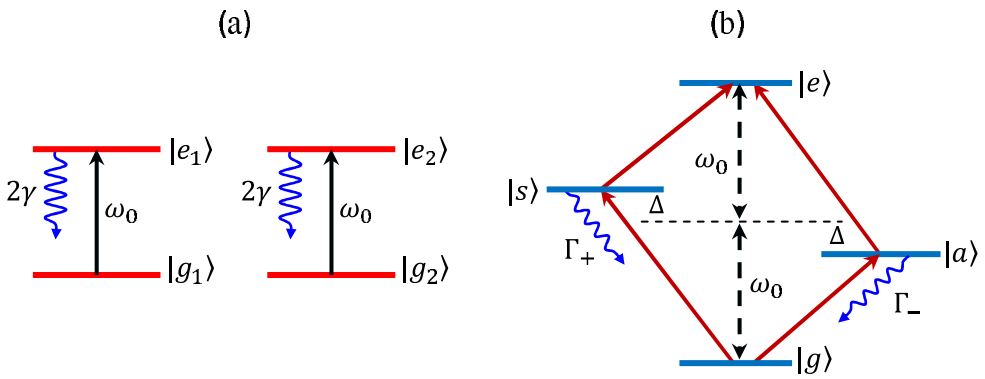

FIG. 1: (Color Online)Energy level diagram for two two-level atoms in bare basis (a) and in the timed Dicke basis (b). The frequency shift $\Delta = \Omega_{12}\cos\varphi$ occurs as a result of dipole-dipole interaction between the two atoms. The collective states $|s\rangle$ and $|a\rangle$ decays at a rate of Γ_+ and Γ_- respectively, where $\Gamma_\pm = 2(\gamma \pm \gamma_{12}\cos\varphi)$.

$$\dot{\rho}_{ss} = -2(\gamma + \gamma_{12}\cos\varphi)\rho_{ss} - i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{as} + i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{sa} + 2(\gamma + \gamma_{12}\cos\varphi)\rho_{ee}, \tag{11e}$$

$$\dot{\rho}_{aa} = -2(\gamma - \gamma_{12}\cos\varphi)\rho_{aa} - i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{as} + i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{sa} + 2(\gamma - \gamma_{12}\cos\varphi)\rho_{ee}, \tag{11f}$$

$$\dot{\rho}_{as} = -2(\gamma - i\Omega_{12}\cos\varphi)\rho_{as} + i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{ss} + i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{aa} - 2i\gamma_{12}\sin\varphi\rho_{ee}, \tag{11g}$$

$$\dot{\rho}_{gs} = -[\gamma + \gamma_{12}\cos\varphi - i(\omega_0 + \Omega_{12}\cos\varphi)]\rho_{gs} + i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{ga} + 2(\gamma + \gamma_{12}\cos\varphi)\rho_{se} + 2i\gamma_{12}\sin\varphi\rho_{ae}, \tag{11h}$$

$$\dot{\rho}_{ga} = -[\gamma - \gamma_{12}\cos\varphi - i(\omega_0 - \Omega_{12}\cos\varphi)])\rho_{ga} - i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{gs} - 2(\gamma - \gamma_{12}\cos\varphi)\rho_{ae} + 2i\gamma_{12}\sin\varphi\rho_{se}, \tag{11i}$$

$$\dot{\rho}_{gg} = 2(\gamma + \gamma_{12}\cos\varphi)\rho_{ss} + 2(\gamma - \gamma_{12}\cos\varphi)\rho_{aa} + 2i\gamma_{12}\sin\varphi(\rho_{as} - \rho_{sa}), \tag{11j}$$

where $\varphi = \mathbf{k}_0 \cdot (\mathbf{r}_i - \mathbf{r}_j) = \frac{2\pi}{\lambda} r_{ij}\cos\xi$ with ξ being the angle between the laser propagation direction and the line joining the two atoms.

Inspection of Eqs. (11b) and (11c) shows that the presence of atom-atom interaction gives rise to collective level shift (Lamb shift) which is as a result of repeated emission and absorption of short-lived virtual photons [30–32]. The level shift arises due to the atom-atom interaction only occurs in states $|s\rangle$ and $|a\rangle$. The other collective states $|e\rangle$ and $|g\rangle$ do not see any level shift due to this interaction as per Eq. (11d). That is, the energy difference between the $|e\rangle$ and $|g\rangle$ remain $2\omega_0$. The state $|s\rangle$ is shifted up while the state $|a\rangle$ is shifted down by an equal amount $\Delta = \Omega_{12}\cos\varphi$ from the single photon resonance line as shown in Fig. 1b. It is interesting to note that one can manipulate the level shift by only orienting the laser field appropriately with respect to the line joining the two atoms. For example, $\varphi = \pi/2$, i.e., when the angle between the laser propagation direction and the line joining the two atoms is $\xi = \pi/3$ and the interatomic distance equal to half of the radiation wavelength, $r_{12} = \lambda/2$, the level shift vanishes. Thus it is possible to control the level shift by applying a laser field in a particular direction without turning off the dipole-dipole interaction.

Further, we note that the transition probability from the excited state $|e\rangle$ to the one photon states, $|s\rangle$ and $|a\rangle$, is the sum of the probability of each transition. Since it is the probability, and not the probability amplitudes that adds up we don't expect quantum interference phenomenon to occur. However, the transition probability from the one photon states, $|s\rangle$ and $|a\rangle$ to the ground state $|g\rangle$ is obtained by squaring the sum of the amplitude of each transition. When there is a coherence between the two states ($|s\rangle$ and $|a\rangle$), this can lead to quantum interference yielding coherent population transfer between $|s\rangle$ and $|a\rangle$. Indeed, the populations in $|s\rangle$ and $|a\rangle$ is coupled to the coherence ρ_{as} as per Eqs. (11e)-(11g). It is worth to note that this coupling disappears when the direction of propagation of the laser field is perpendicular to the interqubit axis $\xi = \pi/2$ ($\varphi = 0$). Therefore, we see that in the presence of a position dependent excitation phase φ quantum interference is induced in the system. In this paper we hence explore to what extent the quantum interference developed in the system affects the dynamical properties of the bipartite entanglement between the two qubits.

III. ENTANGLEMENT MEASURE

In general a state of a quantum system is said to be entangled when the density operator of the composite system cannot factorize into that of the individual subsystems. There are several entanglement measures for two-qubit system in the lit-

erature. However, we use the concurrence, a widely used entanglement monotone, for our purpose. The concurrence, first introduced by Wooters [43], is defined by

$$\mathcal{C}(t) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \tag{12}$$

where $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. $\{\lambda i\}$ are the eigenvalues of the matrix $\rho\tilde{\rho}$ in which $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$ with σ_y being the Pauli matrix. The concurrence takes values ranging from 0 to 1. For maximally entangled state $\mathcal{C}(t) = 1$ and for separable state $\mathcal{C}(t) = 0$.

In general, for a dissipative system, without any external driving field, the density matrix of the qubits system has the form

$$\rho(t) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{13}$$

in the following basis set

$$\begin{aligned} |1\rangle &= |e_1 e_2\rangle e^{i\mathbf{k}_0\cdot(\mathbf{r}_1+\mathbf{r}_2)} \\ |2\rangle &= |e_1 g_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_1} \\ |3\rangle &= |g_1 e_2\rangle e^{i\mathbf{k}_0\cdot\mathbf{r}_2} \\ |4\rangle &= |g_1 g_2\rangle. \end{aligned} \tag{14}$$

Note that for a quantum state initially prepared in a block form of (13), the time-evolved density matrix will have the same block form, i.e., the zeros remain zero and the nonzero components evolve in time [21, 26]. We next proceed to calculate the concurrence for the qubits system initially prepared in the form of (13). To do so, one has to determine the matrix $\tilde{\rho}$ in the basis where ρ is expressed. Using the definition of the density matrix $\tilde{\rho}$, we obtain

$$\tilde{\rho}(t) = \begin{pmatrix} \rho_{44} & 0 & 0 & \rho_{14} \\ 0 & \rho_{33} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{22} & 0 \\ \rho_{41} & 0 & 0 & \rho_{11} \end{pmatrix}. \tag{15}$$

Thus the square root of the eigenvalues of the matrix $\rho\tilde{\rho}$ are:

$$\{\sqrt{\lambda_i}\} = \{\sqrt{\rho_{22}\rho_{33}} \pm |\rho_{23}|, \sqrt{\rho_{11}\rho_{44}} \pm |\rho_{14}|\}. \tag{16}$$

There are two possible expressions for the concurrence, depending on the values of the eigenvalues. The first case is that when $|\rho_{23}| + \sqrt{\rho_{22}\rho_{33}}$ be the largest eigenvalue. This leads to a concurrence

$$\mathcal{C}_1(t) = 2(|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}). \tag{17}$$

While if $|\rho_{14}| + \sqrt{\rho_{11}\rho_{44}}$ is the largest eigenvalue then the concurrence takes the form

$$\mathcal{C}_2(t) = 2(|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}). \tag{18}$$

Depending on the initial condition used, one of the concurrence expressions suffices to measure the entanglement present in the qubits system. Further, inspection of (17) and (18) shows that $\mathcal{C}_1(t)$ would be positive and hence the measure of entanglement when the one photon coherence is larger the square root of the product of the populations in the excited and ground states. On the other hand, for $\mathcal{C}_2(t)$ to be a measure of entanglement for the system the two photon coherence should be greater than the square root of the product of the population in one photon excited states.

In order to gain insight into the physics it is convenient to express the concurrences in terms of timed Dicke basis introduced earlier. To do so, one has to apply a unitary transformation $U\rho U^\dagger$ on the density matrix given by (13). The matrix U is given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

The elements of the density matrix $U\rho U^\dagger$ is related to that of ρ by

$$\begin{aligned} \rho_{ee} &= \rho_{11} \\ \rho_{eg} &= \rho_{14} \\ \rho_{aa} &= \frac{1}{2}(\rho_{22} + \rho_{33} - (\rho_{23} + \rho_{32})) \\ \rho_{ss} &= \frac{1}{2}(\rho_{22} + \rho_{33} + \rho_{23} + \rho_{32}) \\ \rho_{as} &= \frac{1}{2}(\rho_{22} - \rho_{33} + \rho_{23} - \rho_{32}) \\ \rho_{sa} &= \frac{1}{2}(\rho_{22} - \rho_{33} - (\rho_{23} - \rho_{32})). \end{aligned} \tag{20}$$

Therefore the concurrence can be expressed in terms of the timed Dicke basis as

$$\mathcal{C}(t) = \max(0, \mathcal{C}_1(t), \mathcal{C}_2(t)), \tag{21}$$

where

$$\mathcal{C}_1(t) = \sqrt{(\rho_{ss} - \rho_{aa})^2 + 4[\mathrm{Im}(\rho_{as})]^2} - 2\sqrt{\rho_{ee}\rho_{gg}} \tag{22}$$

$$\mathcal{C}_2(t) = 2|\rho_{eg}| - \sqrt{(\rho_{ss} + \rho_{aa})^2 + 4[\mathrm{Re}(\rho_{as})]^2}. \tag{23}$$

This expression for concurrence will be used in the following section to study the dynamical evolution of entanglement in the two-qubit system by considering various initial conditions.

IV. ENTANGLEMENT DYNAMICS OF TWO IDENTICAL QUBITS

Using the entanglement measure introduced in the previous section we study the effect of the position dependent excitation phase by considering pure and mixed state as initial conditions.

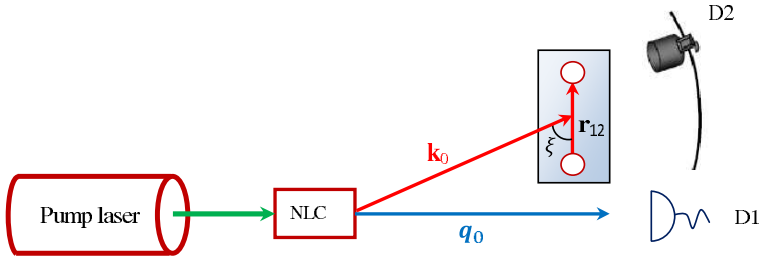

FIG. 2: (Color online) A scheme illustrating a proposed method to prepare symmetric timed Dicke state $|s\rangle$. A similar scheme has been proposed to excite one atom in a cloud of N atoms [29]. The nonlinear crystal (NLC) down converts the a pump photon into signal-idler pair. A click on detector (D1) indicates generation of the pair and hence no click on the second detector (D2)–assuming a perfect detector–means the other photon ($\mathbf{k}_0$) conditionally excite one of the atoms.

A. Initial pure states

In the two-qubit system one might consider a pure separable or entangled state as an initial condition. For instance, for pure separable state, one can take the two atom excited state, $|e\rangle$. Even though this is unentangled state at the initial time, the interaction of the atoms with the environment leads to weak transient entanglement [18, 21]. The effect of quantum interference induced by position dependent excitation phase is unimportant in this case and thus we rather focus on pure entangled state as an initial condition.

We take the initial state of the two-qubit system to be the symmetric state $|s\rangle$. This state is a pure maximally entangled state and can be prepared using correlated pair of photons generated from a parametric down conversion process in which one of the photons is sent to a detector (D1) and the other is directed towards the atoms. A click on the detector (D1) tells us that the other photon is sent to the atoms. If the second detector (D2) registrars a count then no atom is excited. However, if D1 shows a click and D2 does not then we know that one of the atoms is excited, but we don't know which one (see Fig. 2). This leads to a superposition state $|s\rangle$. Recently, Thiel *et al.* [44] proposed a method to prepare all the symmetric states using a linear optical tools. We seek to investigate the dynamics of entanglement of the two qubits as they interact with the environment and with each other via their electric-dipoles.

In terms of the timed Dicke basis the initial density matrix has only one nonzero element namely $\rho_{ss}(0) = 1$; all the rest of matrix elements are zero. Since there is no initial two photon coherence $\rho_{eg}(0) = 0$, according to Eq. (11d), it remains zero all the time. As a consequence the expression given by (23) will be negative and hence cannot be used as entanglement measure. Moreover, it is easy to see that for initial condition we considered, $\rho_{ee}(t) = 0$. In view of this expression (22) takes the form

$$\mathcal{C}_1(t) = \sqrt{(\rho_{ss} - \rho_{aa})^2 + 4[\mathrm{Im}(\rho_{as})]^2} > 0 \qquad (24)$$

and thus the concurrence can be written as

$$\mathcal{C}(t) = \max(0, \mathcal{C}_1(t)). \qquad (25)$$

This expression shows that the concurrence is unity at $t = 0$ as it should be.

Disregarding the relative phase shift ($\varphi = 0$) the solutions of the elements of the density matrix in Eq. (24) turn out to be $\rho_{ss}(t) = \exp[-2(\gamma + \gamma_{12})t]$, $\rho_{aa}(t) = 0$, and $\rho_{as} = 0$, which leads to

$$\mathcal{C}(t) = \max(0, e^{-2(\gamma+\gamma_{12})t}). \qquad (26)$$

We immediately see that the concurrence depends only on the symmetric state population, $\rho_{ss}(t)$. As there is no single photon coherence generated in this case, population transfer between the $|s\rangle$ and $|a\rangle$ does not occur. As a result the initial entanglement experiences an enhanced decay due to the collective decay rate (γ_{12}) and goes asymptotically to zero as $t \to \infty$. For nonidentical atoms, however, even though the entanglement has the same behavior as identical atoms at the initial time, it exhibits revival at later times [18]. Here the detuning plays an important role in creating coherence between the symmetric and antisymmetric states, which is the basis for entanglement in the two-qubit system. In the following we rather show, by taking into account the spatial phase dependence of the atomic states, that quantum interference in the system leads to a population transfer between the symmetric and antisymmetric states and hence generation of coherence (ρ_{as}).

The phase shift that an atom experiences during the excitation process contain physical information about the excited atom. For example, in the phase factor associated with an excited atom $\exp(i\mathbf{k} \cdot \mathbf{r}_j)$ the term $\mathbf{k} \cdot \mathbf{r}_j = \omega_0 \hat{n} \cdot \mathbf{r}_j/c \equiv \omega_0 t_i$ indicates that the atom located at position r_j is excited at different times t_i. This has been discussed in the context of directed spontaneous emission and collective Lamb shift in recent years [29, 30]. Here we present how this phase factor can be used to improve the entanglement at later times.

In one photon subspace [$\rho_{ss}(0) = 1$] and for nonzero spatial excitation phase the important equations read [38]

$$\begin{aligned}\dot{\rho}_{ss} =& -2(\gamma + \gamma_{12}\cos\varphi)\rho_{ss} - i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{as} \\ &+ i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{sa}, \end{aligned} \qquad (27)$$

$$\begin{aligned}\dot{\rho}_{aa} =& -2(\gamma - \gamma_{12}\cos\varphi)\rho_{aa} - i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{as} \\ &+ i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{sa}, \end{aligned} \qquad (28)$$

$$\begin{aligned}\dot{\rho}_{as} =& -2(\gamma - i\Omega_{12}\cos\varphi)\rho_{as} + i\sin\varphi(\gamma_{12} + i\Omega_{12})\rho_{ss} \\ &+ i\sin\varphi(\gamma_{12} - i\Omega_{12})\rho_{aa}. \end{aligned} \qquad (29)$$

These equations fully describe the dynamical behavior of the population transfer between the symmetric and antisymmetric states and the coherence developed between them under the given initial condition. We particularly note that the coherence critically depends on the spatial excitation phase (φ). It is not difficult to see from these equations that for a laser propagating perpendicular to the interqubit axis ($\varphi = 0$) there will be no coherence, which in turn implies the initial population in the symmetric state directly decays to the ground state without ever being transferred to the antisymmetric state. In this decay process the maximum entanglement present at the initial time will be washed out in short time. Therefore, for this particular initial condition, one has to play around with the spatial excitation phase to avoid enhanced decay of the entanglement.

Using the analytical solutions of the Eqs. (27)-(29) the concurrence can be expressed as

$$\mathcal{C}(t) = \max(0, \mathcal{C}_1(t)), \tag{30}$$

where

$$\begin{aligned}\mathcal{C}_1(t) =& e^{-2\gamma t}[(\cos\varphi\cosh 2\gamma_{12}t - \sinh 2\gamma_{12}t)^2 \\ &+ \sin^2\varphi\cos^2 2\Omega_{12}t]^{1/2}.\end{aligned} \tag{31}$$

Inspection of (31) shows that the presence of the excitation phase brings in the dipole-dipole interaction (Ω_{12}) into the dynamics. This is in contrast with the case where $\varphi = 0$ in which the concurrence is independent of the interatomic interaction. Note that it is the initial preparation of the state that determines the dynamical behavior of the two-qubit system. To better understand to what extent the excitation phase modifies the concurrence and hence the entanglement between the qubits, we graphically present the concurrence in Fig. 3.

In Fig. 3 we show the evolution of the concurrence as a function of the angle between the direction of propagation of the laser and the line joining the two atoms (ξ) for the two-qubit system prepared initially in the symmetric state $|s\rangle$ and for interatomic distance $r_{12} = \lambda/8$. As we have discussed earlier the concurrence corresponding to $\varphi = 0$ exhibits a sharp decrease and ultimately goes to zero for $t \to \infty$. The situation for nonzero excitation phase is different; the concurrence sharply diminishes during the decay time of the symmetric state $[2\gamma + \gamma_{12}\cos\varphi]^{-1}$ and shows a bit of revival and decays slowly before it goes to zero at $t \to \infty$. This can be understood by looking at the inset of the Fig. 3, where we plotted the time evolution of populations in the symmetric and antisymmetric states. As can be clearly seen from this inset that for $\varphi \neq 0 (\xi = 0)$, quantum interference leads to coherent transfer of population from the initially populated state $|s\rangle$ to antisymmetric state $|a\rangle$ [37] and hence generation of coherence between these levels as illustrated in Fig. 4. This coherence is responsible for the entanglement observed between the qubits.

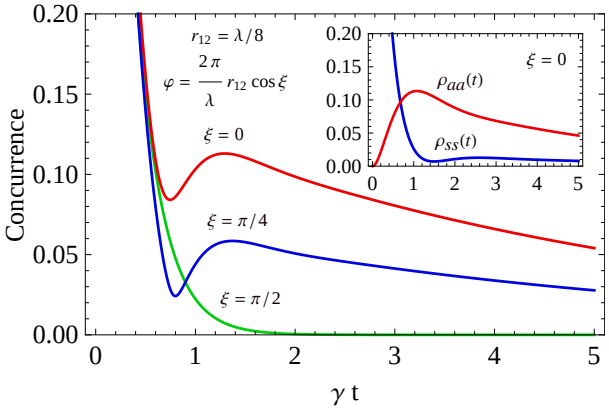

FIG. 3: (Color online) Plots of the time evolution of concurrence $\mathcal{C}_1(t)$ with initial condition $\rho_{ss}(0) = 1$, for interatomic distance $r_{12} = \lambda/8$ ($\gamma_{12}/\gamma = 0.9, \Omega_{12}/\gamma = -0.9$) for different values of ξ-the angle between the direction of propagation of the laser and the line joining the two atoms. The inset shows the populations in the symmetric and antisymmetric states for $\xi = 0 (\varphi = \pi/4)$.

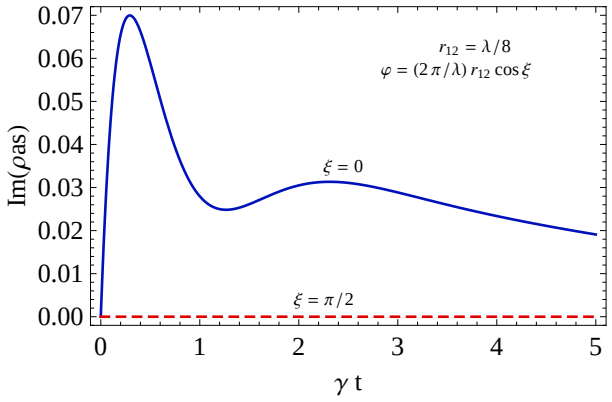

FIG. 4: (Color online) Plots the imaginary part of coherence between the symmetric and antisymmetric states ρ_{as} with initial condition $\rho_{ss}(0) = 1$, for interatomic distance $r_{12} = \lambda/8$ ($\gamma_{12}/\gamma = 0.9, \Omega_{12}/\gamma = -0.9$) and for $\xi = 0$.

B. Initial mixed state

We next consider the two qubits initially prepared in a mixed entangled state [24] given by the density matrix

$$\rho(0) = \frac{1}{3}(a|1\rangle\langle 1| + (1-a)|4\rangle\langle 4| + (b+c)|\Phi\rangle\langle\Phi|) \tag{32}$$

in which $|\Phi\rangle = \frac{1}{\sqrt{b+c}}(\sqrt{b}|2\rangle + e^{i\chi}\sqrt{c}|3\rangle)$ and the normalization condition reads $(1+b+c)/3 = 1$. Here a, b, c and χ are independent parameters which determine the initial state of the two entangled qubits. Note that the above state is a form of generalized Werner state. The initial condition given by (32) can be written in the basis of (13) as

$$\rho(0) = \frac{1}{3}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix} \tag{33}$$

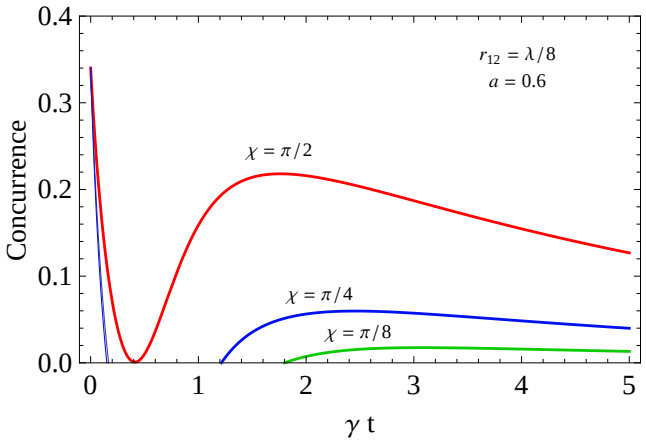

FIG. 5: (Color online)Plots of the time evolution of concurrence $\mathcal{C}(t)$ with initial condition $b = c = |z| = 1$ and for $a = 0.6$, $r_{12} = \lambda/8$, and for different values of the initial phase χ

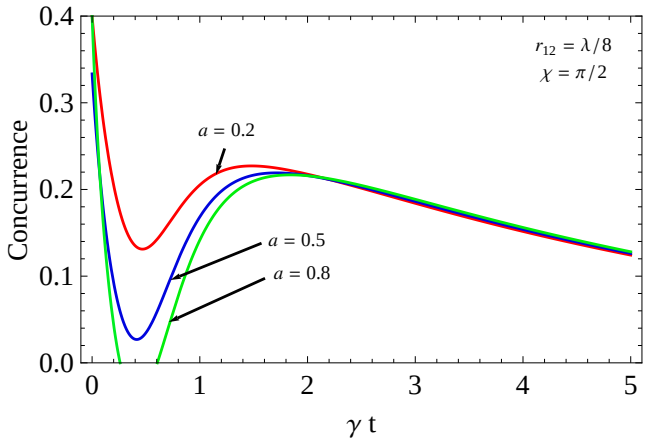

FIG. 6: (Color online)Plots of the time evolution of concurrence with initial condition $b = c = |z| = 1$ and for $\chi = \pi/2$, $r_{12} = \lambda/8$ ($\gamma_{12}/\gamma = 0.9, \Omega_{12}/\gamma = -0.9$), for different values of the initial populations a.

where $z = \sqrt{bc}\, e^{i\chi}$ is some initial single photon coherence in the system and χ is the respective phase of the coherence. Now applying the transformation given by (20), the initial density matrix elements for $b = c = |z| = 1$ become

$$\rho_{ee}(0) = a/3, \rho_{aa}(0) = (1 - \cos\chi)/3, \rho_{gg}(0) = (1 - a)/3$$
$$\rho_{ss}(0) = (1 + \cos\chi)/3, \rho_{as}(0) = \frac{i}{3}\sin\chi.$$

Since $\rho_{es}(0) = \rho_{ea}(0) = \rho_{gs}(0) = \rho_{ga}(0) = 0$, the form of the initial density matrix remain the same, i.e., all the zero elements remain zero and the all the rest evolves in time. Under this scenario the expression given by (23) will be negative and hence cannot be an entanglement measure for the qubit system. Therefore, (22) is the only candidate left to quantify the entanglement between the qubits. For $\varphi = 0$ the system of equations governing the dynamics of the two qubits can be solved analytically. Using these solutions, solved under the initial condition (33), the expression that describes the entanglement between the qubits, $\mathcal{C}_1(t)$, turns out to be

$$\mathcal{C}_1(t) = \frac{2}{3}e^{-2\gamma t}\Big\{\Big[(\cos\chi\cosh 2\gamma_{12}t - \sinh 2\gamma_{12}t + a\eta_1(t))^2 + \sin^2\chi\cosh 2\Omega_{12}t\Big]^{1/2} - \sqrt{3a(1-\eta_2(t))}\Big\}, \quad (34)$$

where

$$\eta_1(t) = \frac{(\gamma^2+\gamma_{12}^2)}{\gamma_{12}^2-\gamma^2}\sinh 2\gamma_{12}t + \frac{2\gamma\gamma_{12}}{\gamma_{12}^2-\gamma^2}(e^{-2\gamma t} - \cosh 2\gamma_{12}t), \quad (35)$$

$$\eta_2(t) = \frac{a}{3}e^{-4\gamma t} + \frac{2}{3}e^{-2\gamma t}\Big[\cosh 2\gamma_{12}t - \cos\chi\sinh 2\gamma_{12}t + a\frac{(\gamma^2+\gamma_{12}^2)}{\gamma_{12}^2-\gamma^2}(e^{-2\gamma t} - \cosh 2\gamma_{12}t) - a\frac{2\gamma\gamma_{12}}{\gamma_{12}^2-\gamma^2}\sinh 2\gamma_{12}t)\Big]. \quad (36)$$

We immediately see from this result that the concurrence depends on the parameters a, which characterizes the initial populations of the doubly excited state *i.e.* when both the qubits are excited and on the phase parameter χ which determines the initial populations in the symmetric and antisymmetric states as well as the coherence between them. If we consider that the qubits are coupled independently to their respective vacuum environment ($\gamma_{12} = 0$) and are well separated in position ($r_{12} \gg \lambda$) so that the dipole-dipole interaction ($\Omega_{12} \to 0$), $\mathcal{C}_1(t)$ reduces to

$$\mathcal{C}_1(t) = \frac{2}{3}e^{-2\gamma t}\Big[1 - \sqrt{a(1 - a + 2\alpha^2 + \alpha^4 a)}\Big]$$

where $\alpha(t) = \sqrt{1 - \exp(-2\gamma t)}$. Note that $\mathcal{C}_1(t)$ is independent of the initial phase χ. This coincides with the earlier results of Yu and Eberly [24].

In the following we study the dependence of the concurrence and hence the entanglement between the qubits on various system parameters. Figure 5 shows the time evolution of the concurrence for $r_{12} = \lambda/8$ and $a = 0.6$ and for different values of the initial phase, χ. We observe from this figure that the initial entanglement between the qubits vanishes and exhibits revival. The amplitude of revival and the revival time (the time at which the entanglement revive in the system) are directly related to the initial coherence in the system. The higher the initial coherence the higher the amplitude of revival and the shorter the revival time is. Not surprisingly the magnitude of revival diminishes when the initial coherence decreases. Now keeping the initial coherence at its maximum value ($\chi = \pi/2$), we investigate the influence of the population distribution between the excited and ground states on the concurrence. Figure 6 shows the evolution of the concurrence for $\chi = \pi/2$ and for different values of a. This figure indicates that when the initial population in the excited state grows the transient entanglement falls sharply and even disappears for $a = 0.8$ ($\rho_{ee}(0) \approx 0.27$) in the short time window. The entanglement then shows revival and a slowly damping

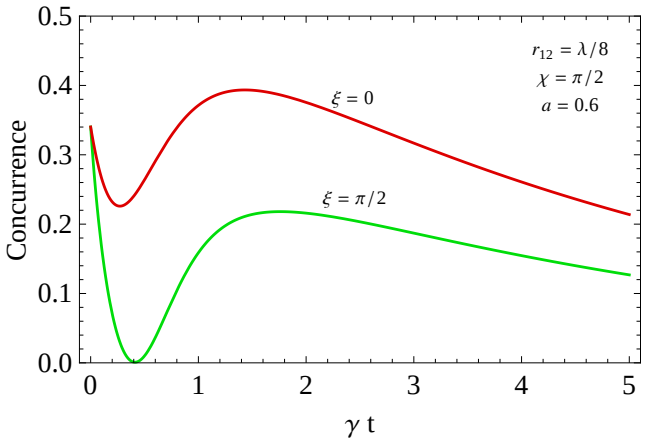

FIG. 7: (Color online)Plots of the evolution of concurrence with initial condition $b = c = |z| = 1$ and for $r_{12} = \lambda/8$ ($\gamma_{12}/\gamma = 0.9, \Omega_{12}/\gamma = -0.9$), $\chi = \pi/2$ and for the direction of propagation of the laser field perpendicular ($\xi = \pi/2$) and parallel ($\xi = 0$) to the line joining the two atoms.

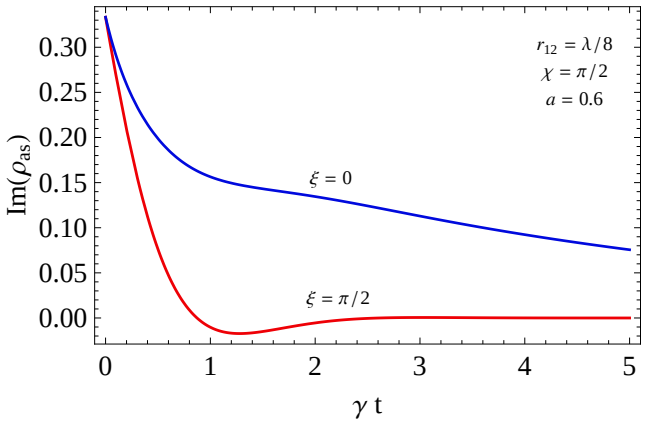

FIG. 8: (Color online)Plots of the imaginary part of the one photon coherence ρ_{as} with initial condition $b = c = |z| = 1$ and for $r_{12} = \lambda/8$ ($\gamma_{12}/\gamma = 0.9, \Omega_{12}/\gamma = -0.9$), $\chi = \pi/2$ and for the direction of propagation of the laser field perpendicular ($\xi = \pi/2$) and parallel ($\xi = 0$) to the line joining the two atoms.

behavior afterwards for all values of initial populations.

We next analyze the evolution of entanglement in the system by introducing the spatial excitation phase φ into the dynamics. By comparing the previous results for $\varphi = 0$ with the numerical plots for $\varphi \neq 0$, we discuss the effect of the excitation phase on the entanglement dynamics. Our results are summarized in Figs. 7 and 8. In Fig. 7, we present a comparison of concurrence taking into account the spatial excitation phase $\varphi = \pi/4$ ($\xi = 0$) and in the absence of excitation phase, $\varphi = 0$ ($\xi = \pi/2$) for interatomic distance less than the radiation wavelength, $r_{12} = \lambda/8$. Recall that $\varphi = (2\pi/\lambda) r_{12} \cos\xi$, where ξ is the angle between the laser propagation direction and the line joining the two atoms. These plots clearly show that the excitation phase effectively protects the initial entanglement from experiencing a sudden death and even enhances the entanglement from its initial value during the revival period. The amount of entanglement then drops gradually and approaches zero at $t \to \infty$. It is worth noting that the spatial excitation phase creates additional coherence and hence improves the revival magnitude over that observed for the case $\varphi = 0$. This enhanced coherence, as shown in Fig. 8, is a signature of stronger entanglement between the qubits

V. CONCLUSION

We have investigated the effect of quantum interference induced by position dependent excitation phase on the dynamical behavior of entanglement between two interacting qubits coupled to a common vacuum reservoir. We have considered both pure and mixed entangled states for our analysis. Our results show that for the atoms initially prepared in a symmetric state, the excitation phase induces quantum interference in the two-qubit system that leads to coherent population transfer between the symmetric and antisymmetric states. This thus creates a coherence which in effect slows down the otherwise fast decay of two-qubit entanglement considerably. Hence we find that the evolution of entanglement crucially depends on the coherence between the symmetric and antisymmetric states. Furthermore, when the qubits are prepared in a Werner type mixed entangled state the entanglement is known to suffer sudden death. However, if one takes into account the excitation phase into the dynamics the entanglement exhibits revival. This revival is attributed to strong coherence developed between the symmetric and antisymmetric states. A viable candidate for realization of our findings would be semiconductor quantum dots. Note that coupled quantum dots with interdot distance less than the radiation wavelength has already been investigated in context to quantum gates [45] and photoluminescence spectra [33]. As a future perspective, one can further study the effect of virtual processes on the dynamics of the system as these processes are known to influence the evolution of the symmetric and antisymmetric states.

Acknowledgments

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